Context-free Process Algebras Extended with Deadlocks

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Declaration

I declare that this thesis was composed by myself and all presented results are my own unless otherwise stated.

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Abstract

Recently it has been shown that the class of BPA (or context-free) processes can be widely used to describe sequential processes and to define their semantics. This class has been intensively studied. Bisimilarity and regularity appeared to be decidable within the BPA processes (see [CHS95], [BCS96]). We extend these processes with a deadlocking state into BPA_δ systems. We show that the BPA_δ class is more expressive w.r.t. bisimulation equivalence but it remains language equivalent to BPA . We prove that bisimilarity and regularity remain decidable in the BPA_δ class. Finally we give a characterisation of those BPA_δ processes which can be equivalently (up to bisimilarity) described within the 'pure' BPA syntax.

Keywords: context-free processes, BPA , deadlock, bisimilarity, regularity, language equivalence

Chapter 1

Introduction

The problem of determining whether a program satisfies a given specification is one of the main issues in theoretical computer science. Traditional input-output semantics declare two programs to be equivalent iff on the same input they produce the same output. We usually consider that the output is released only after the execution of such a program is completed. This approach, however, is not appropriate for reactive-like systems where coming to an end of such a system is rather tragedy than desired output. For such systems we need another notion of equivalence capturing the behaviour during the execution. Recently a labelled transition system as the abstract computational model, and the relation of bisimulation as the most suitable behavioural equivalence, have been generally accepted. It should be remarked that another equivalences have been explored. Probably the most exhaustive spectrum of these equivalences can be found in [vG90a] and [vG90b] but the bisimulation still appears as the finest one.

This thesis deals with BPA processes (Basic Process Algebra) extended with deadlocks. BPA represents the class of processes introduced by Bergstra and Klop (see [BK85]). This class corresponds to the transition systems associated with Greibach normal form (GNF) context-free grammars in which only left-most derivations are permitted. For detailed description of the relation between language and process theory we refer to [HM96]. We define the class BPA_δ of BPA processes extended with deadlocks in a new way and give two alternative definitions of bisimilarity and regularity within this class. The main results say that bisimilarity, regularity and other significant properties remain decidable in the class of BPA_δ systems.

Basic definitions used in this thesis can be found in Chapter 2. We give a definition of BPA_δ systems based on a special variable δ (we call it a deadlock). In an usual presentation every variable used in a BPA system is supposed to be defined but for the deadlock variable we allow no definition. This causes that if the system reaches a state where the first variable is δ, the system sticks at this state and no more actions can be performed. There are

two approaches to the deadlocking state. First, δ identifies only with the situation when the process gets into an inner state where it loops forever. However, no actions (for an observer of such a system) can be seen. Second, we identify the deadlock with a regularly finished execution of the process. A suitable example from operating systems can be given. We can consider δ as a process that cannot communicate with the surrounding world. First, we may say that what we cannot see does not exist and so we do not distinguish between a regularly finished process ϵ and the deadlocking process δ. Second, we may not want to identify these processes since δ still occupies some memory whereas ϵ does not. These are two different approaches to the problem and both will be discussed in this thesis.

Another topic this thesis deals with, is the issue of language equivalence. We show in Chapter 3 that extending BPA systems with deadlock does not yield any language extension. On the other hand the class of BPA_δ systems is larger with regard to bisimilarity – the behaviour equivalence. It is a standard fact that the input-output behaviour equivalences do not find any difference between processes $ab + ac$ and $a(b + c)$ but they are trivially nonbisimilar (see Figure 2.1).

It is known from [CHS95] that bisimilarity is decidable in the BPA systems. We show in Chapter 4 that this result extends to the BPA_δ systems too and bisimulation equivalence remains decidable in the BPA_δ systems. The trick used for this extention is based on an idea that the deadlocking state can be simulated by an unnormed variable. In the Chapter 5 we show that this principle can be applied for deciding regularity as well. Moreover we show that strict and nonstrict regularity coincide.

The last question explored in this thesis (Chapter 6) is concerned with deciding whether there exists an alternative description of a BPA_δ system in bisimilar BPA syntax. We prove that it is decidable for strict bisimilarity and we find a nice semantic characterisation of the situation in the nonstrict case. Moreover we show that the corresponding BPA syntax can be effectively constructed.

Chapter 2

Basic definitions

This chapter gives basic definitions of the formalism used in this thesis. For an experienced reader this chapter is rather standard introduction into the area of sequential process algebras. On the other hand some sections (2.4, 2.5 and 2.9) are worth reading. The notion of deadlock and two bisimilarity relations (strict and nonstrict) appear here. We also present the conversion into 3–GNF, which is heavily used in the whole thesis.

2.1 Transition systems

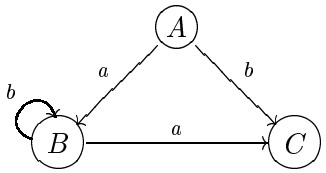
When dealing with processes we need some structure to describe their operational semantics. As the most suitable structure transition systems are widely used and in the rest of this thesis we will understand processes as nodes of a certain type transition system. We introduce the labelled transition system in the extended version with the set of final states as can be found e.g. in [Mol96].

Definition 2.1.1. (labelled transition system) *A* labelled transition system *is a tuple* $(S, \mathcal{A}ct, \Leftrightarrow\!\!\rightarrow, \alpha_0, F)$ *where*

- S *is a set of* states *(or* processes*)*
- $\mathcal{A}ct$ *is a set of* labels *(or* actions*)*
- $\Leftrightarrow\!\!\rightarrow \subseteq S \times \mathcal{A}ct \times S$ *is a* transition relation, *written* $\alpha \stackrel{a}{\Leftrightarrow\!\!\rightarrow} \beta$, *for* $(\alpha, a, \beta) \in \Leftrightarrow\!\!\rightarrow$
- $\alpha_0 \in S$ *is the* root *(or* start state*) of the transition system*
- $F \subseteq S$ *is the set of* final states *which are* terminal *: for each* $\alpha \in F$ *there is no* $a \in \mathcal{A}ct$ *and* $\beta \in S$ *such that* $\alpha \stackrel{a}{\Leftrightarrow\!\!\rightarrow} \beta$.

The transition relation $\Leftrightarrow\!\!\rightarrow$ can be alternatively understood as a set of binary relations $\{\overset{a}{\Leftrightarrow\!\!\rightarrow}\}_{a\in\mathcal{A}ct}$. As usual we extend the transition relation to the elements of $\mathcal{A}ct^*$ ($\alpha \overset{\epsilon}{\Leftrightarrow\!\!\rightarrow} \alpha$ and inductively $\alpha \overset{aw}{\Leftrightarrow\!\!\rightarrow} \beta$ iff $\exists\gamma : \alpha \overset{a}{\Leftrightarrow\!\!\rightarrow} \gamma$ and $\gamma \overset{w}{\Leftrightarrow\!\!\rightarrow} \beta$ where $\alpha, \beta, \gamma \in S$, $a \in \mathcal{A}ct$ and $w \in \mathcal{A}ct^*$). We also write $\alpha \Leftrightarrow\!\!\rightarrow^* \beta$ instead of $\alpha \overset{w}{\Leftrightarrow\!\!\rightarrow} \beta$ if $w \in \mathcal{A}ct^*$ is irrelevant. A state β is *reachable* from the state α, iff $\alpha \Leftrightarrow\!\!\rightarrow^* \beta$. *Reachable states* in a labelled transition system are the states reachable from the root. We also define the unary relation $\not\Leftrightarrow\!\!\rightarrow$ for $\alpha \in S$ as $\alpha \not\Leftrightarrow\!\!\rightarrow$ iff there is no $\beta \in S$ and no $a \in \mathcal{A}ct$ such that $\alpha \overset{a}{\Leftrightarrow\!\!\rightarrow} \beta$.

Example 2.1.2. For more intuitive description of labelled transition systems we can use a graphical representation. The graph consists of nodes that represent states of the labelled transition system. There is an edge between two nodes A, B labelled with a iff $(A, a, B) \in \Leftrightarrow\!\!\rightarrow$.
Let $\mathcal{T} = (\{A, B, C\}, \{a, b\}, \{(A, a, B), (A, b, C), (B, a, C), (B, b, B)\}, A, \{C\})$ be a labelled transition system. The corresponding graphical representation is following.



It can be easily seen from the picture that e.g. $A \overset{abba}{\Leftrightarrow\!\!\rightarrow} C$ and so $A \Leftrightarrow\!\!\rightarrow^* C$.

Definition 2.1.3. (language generation) *Let $(S, \mathcal{A}ct, \Leftrightarrow\!\!\rightarrow, \alpha_0, F)$ be a labelled transition system and suppose that $\alpha \in S$. The* language generated by the process α *is*

$$L(\alpha) \overset{\text{def}}{=} \{w \in \mathcal{A}ct^* \mid \exists\alpha' \in F : \alpha \overset{w}{\Leftrightarrow\!\!\rightarrow} \alpha'\}.$$

We say that two processes α and β are language equivalent *, written $\alpha =_L \beta$, iff $L(\alpha) = L(\beta)$. Two labelled transition systems are language equivalent iff their roots are language equivalent.*

Example 2.1.4. Let us consider the labelled transition system $\mathcal{T}$ from the Example 2.1.2. Since C is the only state such that $C \in F$ we have that $L(A) = \{a(b)^*a\} \cup \{b\}$.
Figure 2.1 shows an example of two language equivalent systems. We can see that $\alpha =_L \beta$ (assuming that the set F of final states consists of all terminal states) because $L(\alpha) = L(\beta) = \{ab, ac\}$.

2.2 Bisimilarity

In the concurrency theory, language equivalence is generally taken to be too coarse equivalence. For example, it equates the two transition systems

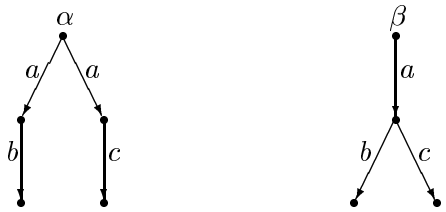


Figure 2.1: Example of two nonbisimilar but language equivalent processes

of Example 2.1 which generate the same language $\{ab, ac\}$ yet demonstrate different behavioural capabilities due to the nondeterministic behaviour exhibited by the first transition system. Many better equivalences have been introduced e.g. in [vG90b] and [vG90a] capturing more or less the reactivity of the systems, with *bisimulation equivalence* being perhaps the finest one. Bisimulation equivalence was defined by Park [Par81] and used with great effect by Milner [Mil89]. Its definition is following.

Definition 2.2.1. (bisimilarity) *Let $(S, \mathcal{A}ct, \Leftrightarrow\!\!\rightarrow, \alpha_0, F)$ be a labelled transition system. A binary relation $R \subseteq S \times S$ is a* bisimulation *iff whenever $(\alpha, \beta) \in R$ then for each $a \in \mathcal{A}ct$:*

- *if $\alpha \stackrel{a}{\Leftrightarrow\!\!\rightarrow} \alpha'$ then $\exists \beta' \in S : \beta \stackrel{a}{\Leftrightarrow\!\!\rightarrow} \beta' \ \wedge \ (\alpha', \beta') \in R$*
- *if $\beta \stackrel{a}{\Leftrightarrow\!\!\rightarrow} \beta'$ then $\exists \alpha' \in S : \alpha \stackrel{a}{\Leftrightarrow\!\!\rightarrow} \alpha' \ \wedge \ (\alpha', \beta') \in R$*
- *$\alpha \in F \ \Leftrightarrow \beta \in F$*

States $\alpha, \beta \in S$ are bisimulation equivalent *or* bisimilar *, written $\alpha \sim \beta$, iff $(\alpha, \beta) \in R$ for some bisimulation R.*

We can even ask whether two states α, β in a pair of different labelled transition systems $(S_1, \mathcal{A}ct_1, \Leftrightarrow\!\!\rightarrow_1, \alpha_1, F_1)$ and $(S_2, \mathcal{A}ct_2, \Leftrightarrow\!\!\rightarrow_2, \alpha_2, F_2)$ are bisimilar. In this case we can construct a new labelled transition system $(S_1 \cup S_2, \mathcal{A}ct_1 \cup \mathcal{A}ct_2, \Leftrightarrow\!\!\rightarrow_1 \cup \Leftrightarrow\!\!\rightarrow_2, \alpha_1, F_1 \cup F_2)$ and ask whether $\alpha \sim \beta$ in this system. All this can be done under the assumption that $S_1 \cap S_2 = \emptyset$. If the sets S_1 and S_2 are not disjoint we can e.g. rename the states of the system $(S_1, \mathcal{A}ct_1, \Leftrightarrow\!\!\rightarrow_1, \alpha_1, F_1)$.

Now we can state an obvious lemma.

Lemma 2.2.2. *Let $(S, \mathcal{A}ct, \Leftrightarrow\!\!\rightarrow, \alpha_0, F)$ be a labelled transition system. Then for all $\alpha, \beta \in S$ if $\alpha \sim \beta$ then $\alpha =_L \beta$.*

The converse does not hold. See Figure 2.1 where the processes α and β are language equivalent but not bisimilar.

It is an interesting property of the bisimulation relation that for some classes of labelled transition systems the language equivalence appears to be undecidable but bisimilarity remains decidable. An example of such a class are BPA systems which will be considered in the rest of this thesis.

2.3 Syntax and semantics of BPA systems

BPA represents the class of Basic Process Algebra processes of Bergstra and Klop [BK85], which describe the transition systems associated with Greibach normal form (GNF) context-free grammars in which only left-most derivations are permitted.
It is possible to define these context-free transition systems by algebraic equations using the operators '.' for *sequential composition* and '+' for for *nondeterministic choice*. First, we need to introduce the definition of BPA expressions.

Definition 2.3.1. *Assume that $\mathcal{V}ar$ and $\mathcal{A}ct$ are finite sets of* variables *and* actions *such that $\mathcal{V}ar \cap \mathcal{A}ct = \emptyset$. We define the class $\mathcal{E}_{BPA}$ of* BPA expressions *as the union of ϵ* (empty process) *and a set $\mathcal{E}^+_{BPA}$, which is the least fixed point of the following inductive definition:*

1. *if $X \in \mathcal{V}ar$ then $X \in \mathcal{E}^+_{BPA}$*
2. *if $a \in \mathcal{A}ct$ then $a \in \mathcal{E}^+_{BPA}$*
3. *if $E_1, E_2 \in \mathcal{E}^+_{BPA}$ then $E_1.E_2 \in \mathcal{E}^+_{BPA}$ and $E_1 + E_2 \in \mathcal{E}^+_{BPA}$*

We state $\mathcal{E}_{BPA} \stackrel{\text{def}}{=} \{\epsilon\} \cup \mathcal{E}^+_{BPA}$.

We will call the BPA expressions as processes and later on, when speaking about a BPA expression, we will assume fixed sets $\mathcal{V}ar$ and $\mathcal{A}ct$ if no confusion is caused.

Definition 2.3.2. (guarded BPA expression) *The class of BPA expressions can be divided into two disjoint sets of* guarded *resp.* unguarded BPA expressions *according to the following inductive definition. The set $g\mathcal{E}_{BPA} \subseteq \mathcal{E}_{BPA}$ will denote the set of guarded expressions.*
For all $a \in \mathcal{A}ct$ and $E_1, E_2 \in \mathcal{E}_{BPA}$:

$a \in g\mathcal{E}_{BPA}$

$E_1 + E_2 \in g\mathcal{E}_{BPA}$ *iff* $E_1, E_2 \in g\mathcal{E}_{BPA}$

$E_1.E_2 \in g\mathcal{E}_{BPA}$ *iff* $E_1 \in g\mathcal{E}_{BPA}$

Example 2.3.3. The expressions $a.X,\ a.(b + X),\ (a + b).X.(Y + Z)$ are guarded whereas $X,\ a + X,\ (a + b + X).c,\ \epsilon$ are not guarded.

$$\frac{}{a \overset{a}{\Leftrightarrow\!\!\rightarrow} \epsilon} \qquad \frac{E \overset{a}{\Leftrightarrow\!\!\rightarrow} E'}{E.F \overset{a}{\Leftrightarrow\!\!\rightarrow} E'.F} \text{ if } E' \neq \epsilon \qquad \frac{E \overset{a}{\Leftrightarrow\!\!\rightarrow} \epsilon}{E.F \overset{a}{\Leftrightarrow\!\!\rightarrow} F}$$

$$\frac{E \overset{a}{\Leftrightarrow\!\!\rightarrow} E'}{E + F \overset{a}{\Leftrightarrow\!\!\rightarrow} E'} \qquad \frac{F \overset{a}{\Leftrightarrow\!\!\rightarrow} F'}{E + F \overset{a}{\Leftrightarrow\!\!\rightarrow} F'} \qquad \frac{E \overset{a}{\Leftrightarrow\!\!\rightarrow} E'}{X \overset{a}{\Leftrightarrow\!\!\rightarrow} E'} \text{ if } X \overset{\text{def}}{=} E \in \Delta$$

Figure 2.2: SOS rules for BPA systems

Definition 2.3.4. (guarded BPA system) *A* guarded BPA system *is a quadruple* $(\mathcal{V}ar, \mathcal{A}ct, \Delta, X_1)$. $\mathcal{V}ar$ *and* $\mathcal{A}ct$ *are finite sets of distinct variables* $(\mathcal{V}ar = \{X_1, X_2, \ldots, X_n\})$ *resp. actions,* $X_1 \in \mathcal{V}ar$ *is the* leading variable, *and* Δ *is a finite set of recursive equations* $\Delta = \{X_i \overset{\text{def}}{=} E_i \mid i = 1, \ldots, n\}$ *where each* $E_i \in g\mathcal{E}_{BPA}$ *is a guarded BPA expression (notice that* $E_i \neq \epsilon$*) with variables drawn from the set* $\{X_1, \ldots, X_n\}$ *and actions from the set* $\mathcal{A}ct$ *.*

When speaking about variables and actions used in the system $(\mathcal{V}ar, \mathcal{A}ct, \Delta, X_1)$ we use the notation $\mathcal{V}ar(\Delta)$ and $\mathcal{A}ct(\Delta)$ and for shorter referring to the BPA system we often identify the system $(\mathcal{V}ar, \mathcal{A}ct, \Delta, X_1)$ with Δ. In what follows we restrict our attention to guarded BPA systems and often omit the word 'guarded'. We also use the notation X^n where $X \in \mathcal{V}ar$, meaning sequential composition $\underbrace{X.X...X}_{n\times}$.

Definition 2.3.5. (BPA labelled transition system)
Assume that we have a guarded BPA system $(\mathcal{V}ar, \mathcal{A}ct, \Delta, X_1)$. *This system determines a labelled transition system* $(S, \mathcal{A}ct, \{\overset{a}{\Leftrightarrow\!\!\rightarrow}\}_{a \in \mathcal{A}ct}, X_1, \{\epsilon\})$ *whose states are BPA expressions built over* $\mathcal{V}ar$ *and* $\mathcal{A}ct$ *,* $\mathcal{A}ct$ *is the set of labels, the transition relations are the least relations satisfying the SOS rules of Figure 2.2,* X_1 *is the root and* ϵ *is the only final state.*

We may assume that the operator '.' for sequential composition is associative and the operator '$+$' for nondeterministic choice is associative and commutative.
States of the labelled transition system generated by a BPA system $\mathcal{T} = (\mathcal{V}ar, \mathcal{A}ct, \Delta, X_1)$ are BPA expressions and are often called *processes* or *states* of $\mathcal{T}$. Speaking about *reachable states* (or *processes*) of the BPA system $\mathcal{T}$, we mean the set of all states (or processes) reachable from the leading variable of $\mathcal{T}$.

Example 2.3.6. In Example 2.1.2 we illustrated a simple transition system. The system (only its reachable states) can be described by the following BPA system $(\mathcal{V}ar, \mathcal{A}ct, \Delta, X_1)$ where:

- $\mathcal{V}ar = \{X_1, X_2\}$
- $\mathcal{A}ct = \{a, b\}$
- $\Delta = \{X_1 \stackrel{\text{def}}{=} a.X_2 + b, X_2 \stackrel{\text{def}}{=} b.X_2 + a\}$

We often use more synoptical labelling of variables as $A, B, X, Y, \ldots$ for better lucidity. Hence Δ could also contain following definitions.

$$A \stackrel{\text{def}}{=} a.B + b$$

$$B \stackrel{\text{def}}{=} b.B + a$$

2.4 Syntax and semantics of BPA$_\delta$ systems

We now define the class BPA$_\delta$ of BPA systems with deadlock. The definition is very similar to the definition of BPA systems except for a new distinct variable δ. This variable δ has no definition in the BPA$_\delta$ system.

Definition 2.4.1. (guarded BPA$_\delta$ system) *A* guarded BPA$_\delta$ system *is a quadruple* $(\mathcal{V}ar, \mathcal{A}ct, \Delta, X_1)$. $\mathcal{V}ar$ *and* $\mathcal{A}ct$ *are finite sets of distinct variables* $(\mathcal{V}ar = \{X_1, X_2, \ldots, X_n, \delta\})$ *resp. actions,* $X_1 \in \mathcal{V}ar$ *is the* leading variable, δ *is a special variable called* deadlock, *and* Δ *is a finite set of recursive equations* $\Delta = \{X_i \stackrel{\text{def}}{=} E_i \mid i = 1, \ldots, n\}$ *where each* $E_i \in g\mathcal{E}_{BPA}$ *is a guarded BPA expression (notice that* $E_i \neq \epsilon$*) with variables drawn from the set* $\{X_1, \ldots, X_n, \delta\}$ *and actions from* $\mathcal{A}ct$ *.*

Notice the difference between BPA and BPA$_\delta$ systems. In BPA$_\delta$ there is a special variable δ such that it can occur in any BPA expression E_i but there is no definition for this variable. Moreover it is obvious that any BPA system is trivially a BPA$_\delta$ system (we simply add δ into variables but we do not use it).

Definition 2.4.2. (BPA$_\delta$ labelled (strict or nonstrict) transition system) *Assume that we have a guarded BPA*$_\delta$ *system* $(\mathcal{V}ar, \mathcal{A}ct, \Delta, X_1)$. *This system determines a labelled transition system* $(S, \mathcal{A}ct, \{\stackrel{a}{\leftrightarrow}\}_{a \in \mathcal{A}ct}, X_1, F)$ *whose states are BPA expressions built over* $\mathcal{V}ar$ *and* $\mathcal{A}ct$ *,* $\mathcal{A}ct$ *is the set of labels, the transition relations are the least relations satisfying the SOS rules of Figure 2.2,* X_1 *is the root. If* $F = \{\epsilon\}$ *is the only final state we call the labelled transition system* strict *and if the final states are* $F = \{\epsilon, \delta\} \cup \{\delta.E | E \in \mathcal{E}^+_{BPA}\}$ *we call it* nonstrict.

Remark. As there is no defining equation for the variable δ it holds that $\delta.E \not\leftrightarrow$ for any $E \in \mathcal{E}^+_{\text{BPA}}$.

2.5 Strict and nonstrict bisimilarity

We have already introduced the notion of bisimilarity for labelled transition systems on page 5. It is easy to determine the set of final states F for the case of BPA systems ($F = \{\epsilon\}$) – see Definition 2.3.5. In the case of BPA$_\delta$ systems we have two possibilities, both of them are evincible. We can state $F = \{\epsilon\}$ or $F = \{\epsilon, \delta\} \cup \{\delta.E | E \in \mathcal{E}^+_{\mathrm{BPA}}\}$ and this brings two definitions of corresponding labelled transition system – strict and nonstrict – and so the relation of bisimulation differs for both these approaches.

Definition 2.5.1. *We call the bisimulation* strict *resp.* nonstrict *(and write $\overset{s}{\sim}$ resp. $\overset{n}{\sim}$) according to the type of labelled transition system we take into account ($F = \{\epsilon\}$ resp. $F = \{\epsilon, \delta\} \cup \{\delta.E | E \in \mathcal{E}^+_{BPA}\}$).*

Remark. These two notions of bisimilarity imply that $\delta \overset{n}{\sim} \epsilon$ but $\delta \overset{s}{\not\sim} \epsilon$. In the nonstrict case we identify the process δ that cannot emit any action with a finished process ϵ, in the strict case we do not.

We say that a pair of BPA$_\delta$ systems $\mathcal{T} = (\mathcal{V}ar, \mathcal{A}ct, \Delta, X_1)$ and $\mathcal{T}' = (\mathcal{V}ar', \mathcal{A}ct', \Delta', X'_1)$ is (strictly resp. nonstrictly) bisimilar (and we write $\Delta \overset{s}{\sim} \Delta'$ or $\mathcal{T} \overset{s}{\sim} \mathcal{T}'$ resp. $\Delta \overset{n}{\sim} \Delta'$ or $\mathcal{T} \overset{n}{\sim} \mathcal{T}'$) iff their corresponding (strict resp. nonstrict) labelled transition systems are bisimilar. Notice that in the case of BPA systems the relations $\overset{s}{\sim}$ and $\overset{n}{\sim}$ coincide as there is no δ in defining equation of any variable. Hence in this case we can simply write $\sim$ instead of $\overset{s}{\sim}$ and $\overset{n}{\sim}$. Following lemma is an easy consequence of the definition of $\overset{s}{\sim}$ and $\overset{n}{\sim}$.

Lemma 2.5.2. $\overset{s}{\sim} \subseteq \overset{n}{\sim}$

Remark. It can be easily seen that in the class of BPA and BPA$_\delta$ systems both strict and nonstrict bisimulation equivalence is an equivalence relation and moreover the strict bisimulation is a congruence w.r.t. BPA operators '.' and '+'. In the case of nonstrict bisimilarity (in BPA$_\delta$ systems) we have $a \overset{n}{\sim} a.\delta$ and using the property of congruence we would get $a.b \overset{n}{\sim} a.\delta.b$ which is obviously not true. That is the reason why, if using the nonstrict version of bisimilarity, we loose the property of congruence for $\overset{n}{\sim}$. Nevertheless $\overset{n}{\sim}$ remains to be an equivalence and left congruence (i.e. if $E \overset{n}{\sim} F$ then $G.E \overset{n}{\sim} G.F$).

2.6 Normed processes

An important subclass of BPA (resp. BPA$_\delta$) systems can be obtained by an extra restriction on the involved processes – *normedness*. The condition of normedness says that the BPA process E can generate some nonempty

language. In other words, there is $w \in \mathcal{A}ct$ such that $E \stackrel{w}{\Leftrightarrow} . \not\Rightarrow$. It can be easily seen that unnormed variables in a BPA system cannot generate any nonempty language and so they are uninteresting. On the other hand if we are dealing with bisimilarity, we cannot simply remove unnormed variables since it can change radically the behaviour of the system.

Definition 2.6.1. *Let $E \in \mathcal{E}_{BPA}$. We define the* norm *of E as:*

$$\|E\| \stackrel{\text{def}}{=} \begin{cases} \min\{\ \textit{length}(w) \mid \exists G : E \stackrel{w}{\Leftrightarrow} G \not\Leftrightarrow\}, \textit{if such } w \textit{ exists} \\ \infty, \quad \textit{otherwise} \end{cases}$$

We call the expression E normed iff $\| E \| < \infty$. A process Δ is normed iff its leading variable is normed.

We remind the fact that the norm of E can be effectively computed in BPA_δ systems.

2.7 Regular BPA systems

An interesting property of processes is *regularity*. A process is regular if it is bisimilar to some finite-state one. The theory of finite-state processes is well established today and there are several applications of it. If we want to examine some properties of a process with infinitely many states it is worth asking whether the process is bisimilar to some finite-state process and then analyze this process.
Regularity has been intensively studied and there are several positive results in some classes of process algebras. Jančar and Esparza proved in [JE96] that regularity is decidable for labelled Petri nets. Consequently, it is also decidable for BPP processes. Regularity appeared to be decidable in the class of normed PA processes even in polynomial time – result achieved by Kučera in [Kuč96]. A recent result [Jan97] due to Jančar says that regularity is decidable for one-counter processes. Burkart, Caucal and Steffen demonstrated in [BCS96] that regularity is decidable in the class we are interested in – the class of BPA systems (even unnormed).
In this section we give the definition of regular BPA systems. The definition of BPA_δ regularity we delay to the Chapter 5 where we also show that decidability of regularity extends to BPA_δ systems.

Definition 2.7.1. *A BPA system Δ is* regular *iff there is a BPA system Δ' with finitely many reachable states such that $\Delta \sim \Delta'$.*

It is obvious that a process is regular iff it can reach only finitely many states up to bisimilarity.

Example 2.7.2. Let us consider the following BPA system.

$$X \stackrel{\text{def}}{=} aXX + aXXX$$

$$
\begin{array}{rcll}
E+F & = & F+E & \text{(A1)} \\
E+(F+G) & = & (E+F)+G & \text{(A2)} \\
E+E & = & E & \text{(A3)} \\
(E+F).G & = & E.G+F.G & \text{(A4)} \\
E.(F.G) & = & (E.F).G & \text{(A5)}
\end{array}
$$

$$
\begin{array}{rcll}
\delta.E & = & \delta & \text{(B1)} \\
\delta+E & = & E & \text{(B2)}
\end{array}
$$

Figure 2.3: BPA and BPA$_\delta$ laws

The labelled transition system of this process is infinite since infinitely many different states are reachable. However all these states are bisimilar and thus the system is regular. Corresponding bisimilar and finite state system is e.g.:

$$X \stackrel{\text{def}}{=} aX$$

2.8 Axiomatisation of bisimulation equivalence

In the usual presentation of BPA (see e.g. [BK88]) much effort is usually paid to so–called BPA laws. These laws together with BPA$_\delta$ laws can be seen in Figure 2.3. The BPA and BPA$_\delta$ laws are easily shown to be sound w.r.t. bisimilarity, irrespective of any restrictions on the involved processes.

Lemma 2.8.1. *[BK88] For any BPA expressions E, F and G we have that $E+F \sim F+E$, $E+(F+G) \sim (E+F)+G$, $E+E \sim E$, $(E+F).G \sim E.G+F.G$ and $E.(F.G) \sim (E.F).G$.*

The BPA laws do not form a complete axiomatisation of BPA systems. Some notion of fixed–point induction must be added to prove equations of recursively defined systems. Details can be found in [Hüt91].

Lemma 2.8.2. *For any BPA$_\delta$ expression E we have that $\delta.E \stackrel{s}{\sim} \delta$, $\delta.E \stackrel{n}{\sim} \delta$, $\delta+E \stackrel{s}{\sim} E$ and $\delta+E \stackrel{n}{\sim} E$.*

Proof. The proof is immediate. $\{(\delta.E, \delta)\}$ is both strict and nonstrict bisimulation relation and thus we have $\delta.E \stackrel{s}{\sim} \delta$ and $\delta.E \stackrel{n}{\sim} \delta$. Similarly we know that $E \stackrel{s}{\sim} E$ and $E \stackrel{n}{\sim} E$ so there is both strict and nonstrict bisimulation relation R such that $(E, E) \in R$. The union $R \cup \{(\delta+E, E)\}$ remains to be bisimulation which implies that $\delta+E \stackrel{s}{\sim} E$ and $\delta+E \stackrel{n}{\sim} E$.

□

2.9 Greibach normal form

In what follows we will need to manipulate formally with defining equations of variables and so a normal form of equations will be heavily used

to simplify our considerations.

Definition 2.9.1. *A BPA (resp. BPA$_\delta$) system Δ is said to be in* Greibach Normal Form (GNF) *iff all its defining equations are of the form*

$$X \stackrel{\text{def}}{=} \sum_{j=1}^{m} a_j \alpha_j$$

where m is a natural number ($m > 0$), $a_j \in \mathcal{A}ct(\Delta)$ and $\alpha_j \in \mathcal{V}ar(\Delta)^*$. If length$(\alpha_j) < k$ for each j, $1 \leq j \leq m$, then Δ is said to be in k–GNF.

The normal form is called Greibach normal form by analogy with context-free grammars in Greibach normal form. The proof of the next theorem is based on the proof of 3–GNF for BPA systems that can be found e.g. in [Hüt91, BBK93, BBK87, HM96]. The construction shown in [Hüt91] had to be modified to capture the behaviour of deadlocks.

Theorem 2.9.2. *Let Δ be a guarded BPA$_\delta$ system. We can effectively find a BPA$_\delta$ system Δ' in 3–GNF such that $\Delta' \stackrel{s}{\sim} \Delta$ and $\Delta' \stackrel{n}{\sim} \Delta$. Moreover, if Δ is normed then so is Δ'.*

Proof. An effective algorithm for rewriting Δ into 3–GNF consists from transforming Δ into GNF and then from rewriting the system into 3–GNF.

First, apply the laws (A4), (B1) and (B2) in Figure 2.3 (from left to right) as far as possible. Notice that these reductions are strongly normalising.

Now we replace all internal occurences of atomic actions by equations. Let us define inductively two mappings $f, g : \mathcal{E}_{\text{BPA}} \leftrightarrows \mathcal{E}_{\text{BPA}}$, according to the following prescription.

	$f(\epsilon) = g(\epsilon) = \epsilon$		
for $a \in \mathcal{A}ct(\Delta)$:	$f(a) = a$	$g(a) = X_a$	where X_a is a fresh variable
for $X \in \mathcal{V}ar(\Delta)$:	$f(X) = g(X) = X$		
for $E, F \in \mathcal{E}_{\text{BPA}}$:	$f(E + F) = f(E) + f(F)$	$g(E + F) = g(E) + g(F)$	
for $E, F \in \mathcal{E}_{\text{BPA}}$:	$f(E.F) = f(E).g(F)$	$g(E.F) = g(E).g(F)$	

Let us transform every equation $X \stackrel{\text{def}}{=} E \in \Delta$ into $X \stackrel{\text{def}}{=} f(E)$ and add $X_a \stackrel{\text{def}}{=} a$ for each fresh variable X_a introduced by f. Notice that the system remains guarded and both strictly and nonstrictly bisimilar to the previous one. Moreover, due to applying the axiom (A4), every equation in Δ is now of the form:

$$X \stackrel{\text{def}}{=} \sum_i a_i \alpha_i (E_i + F_i) + \sum_j a_j \alpha_j$$

where $a_i, a_j \in \mathcal{A}ct$ are actions, $\alpha_i, \alpha_j \in \mathcal{V}ar^*$ are sequences of variables and $E_i, F_i \in \mathcal{E}_{\text{BPA}}$. Moreover E_i, F_i are built only from variables, i.e. $g(E_i) = E_i$ and $g(F_i) = F_i$.
In what follows let the symbols $a \in Act, \alpha \in \mathcal{V}ar^*$ and $E, F, E' \in \mathcal{E}_{\text{BPA}}$ range over their appropriate domains.

Example 2.9.3. Let us have the following system.

$$\{X \stackrel{\text{def}}{=} (acX + b + \delta).(a + b\delta X)\}$$

After the application of (A4), (B1) and (B2) we get

$$\{X \stackrel{\text{def}}{=} acX.(a + b\delta) + b.(a + b\delta)\}$$

and finally using the mapping f we transform the system into

$$\{X \stackrel{\text{def}}{=} aX_cX.(X_a + X_b\delta) + b.(X_a + X_b\delta)$$
$$X_a \stackrel{\text{def}}{=} a \quad X_b \stackrel{\text{def}}{=} b \quad X_c \stackrel{\text{def}}{=} c\}.$$

In each equation and for all the summands of the form $a\alpha(E + F)$ (we call the sum $(E + F)$ as an *unresolved sum*) introduce a fresh variable X_{E+F}. Replace this summand with $a\alpha X_{E+F}$ and introduce a new equation $X_{E+F} \stackrel{\text{def}}{=} E + F$.

Now all the 'old' equations are in GNF (i.e. every summand in such an equation is of the form $a\alpha$). However, in the definition of some 'new' variable X_{E+F} an unguarded summand of the form YE', $Y \in \mathcal{V}ar$ could have been introduced. At this state the defining equation for Y must be in GNF and assuming that the axioms (B1) and (B2) were applied as far as possible, we get $Y \neq \delta$. That means that

$$Y \stackrel{\text{def}}{=} \sum_i b_i\beta_i$$

and using the axiom (A4) we can

$$\text{replace the summand } YE' \text{ with } \textstyle\sum_i b_i\beta_iE'.$$

Again the system is (strictly and nonstrictly) bisimilar to the former one, all the equations are guarded and the number of different unresolved sums decreased. Repeat this procedure until there are no unresolved sums.

The resulting system is now in k–GNF for some $k > 0$. We will transform the system into 3–GNF. Each summand with even variable sequence length of the form

$$aX_1X_2X_3X_4 \ldots X_{2m}$$

replace with

$$aU_{X_1X_2}U_{X_3X_4}\ldots U_{X_{2m-1}X_{2m}} \text{ for } m>0.$$

Each summand with odd variable sequence length of the form

$$aX_1X_2X_3X_4\ldots X_{2m+1}$$

replace with

$$aU_{X_1X_2}U_{X_3X_4}\ldots U_{X_{2m-1}X_{2m}}X_{2m+1} \text{ for } m>0.$$

The variables $U_{X_iX_j}$ are fresh variables with defining equations $U_{X_iX_j} \stackrel{\text{def}}{=} X_iX_j$. These equations are unguarded and we use again the same trick as before. We know that X_i is of the form $X_i \stackrel{\text{def}}{=} \sum_p a_p\alpha_p$. The application of the axiom (B1) ensures $X_i \neq \delta$. So we can

$$\text{replace } X_iX_j \text{ in definition of } U_{X_iX_j} \text{ with } \textstyle\sum_p a_p\alpha_pX_j.$$

If some summand of $U_{X_iX_j}$ is of the form $a_p\alpha_pX_j$ with $\alpha_p = U_{\ldots}\ldots U_{\ldots}X_{2m+1}$ we need to introduce a fresh variable $U_{X_{2m+1}X_j} \stackrel{\text{def}}{=} X_{2m+1}X_j$ and make the equation guarded. Notice that a new variable $U_{X_{i'}X_{j'}}$ comprises just a pair of 'old' variables (i.e. variables different from $U_{\ldots}$). There are only finitely many 'old' variables so the procedure must finish.

Observe, the system is in $\lceil k/2\rceil+1$ – GNF. Repeat this procedure until the resulting system is in 3–GNF.

We have constructed a BPA$_\delta$ system Δ' such that $\Delta' \stackrel{s}{\sim} \Delta$ and $\Delta' \stackrel{n}{\sim} \Delta$. Moreover, if Δ is normed, so is Δ' – from the construction. □

We may assume that we are working only with BPA$_\delta$ systems in GNF since it has been proved that any BPA$_\delta$ (and also BPA) system can be effectively presented in 3–GNF and this construction preserves bisimilarity. This justifies also the assumption that all reachable states of the given BPA or BPA$_\delta$ system are elements of $\mathcal{V}ar^*$.

Chapter 3

Expressibility of BPA$_\delta$ systems

In this chapter we justify the importance of introducing a deadlocking state into the BPA systems. We show that deadlocks enlarge the descriptive power of BPA systems w.r.t. both strict and nonstrict bisimilarity. On the other hand introducing deadlocks does not allow to generate more languages than in the case of BPA .

3.1 Expressibility w.r.t. bisimilarity

Theorem 3.1.1. *There exists a BPA$_\delta$ system such that no BPA system is strictly bisimilar to it.*

Proof. No BPA system can be strictly bisimilar to the system $\Delta = \{X \stackrel{\text{def}}{=} a\delta\}$ since the state δ is reachable in this system and there is no match for δ in any BPA system. □

Theorem 3.1.2. *There exists a BPA$_\delta$ system such that no BPA system is nonstrictly bisimilar to it.*

Proof. We define a BPA$_\delta$ system Δ and show that there is no BPA system Δ' such that $\Delta \stackrel{n}{\sim} \Delta'$.
Consider $\Delta = \{X \stackrel{\text{def}}{=} aXX + b + c\delta\}$ (see Figure 3.1) and w.l.o.g. suppose that there is a BPA system Δ' in 3–GNF, $\Delta' = \{Y_i \stackrel{\text{def}}{=} E_i \mid i = 1, \ldots, n\}$, such that $\Delta \stackrel{n}{\sim} \Delta'$. Then there are infinitely many states reachable from the leading variable X of the system Δ. They are of the form X^n for $n \geq 1$ and for each such state there must be a reachable state E from Δ' such that $X^n \stackrel{n}{\sim} E$. This state can be composed only of one variable Y_i for some $1 \leq i \leq n$ since the norms of X^n and E must be equal and $\|X^n\| = 1$.
This implies that Δ is nonstrictly bisimilar to a system with finitely many reachable states. It is easy to see that Δ is a system where infinitely many nonstrictly nonbisimilar states are reachable, which is contradiction. We

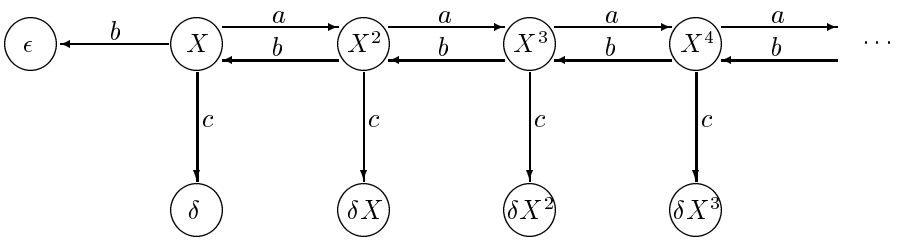


Figure 3.1: Labelled transition system for $X \stackrel{\text{def}}{=} aXX + b + c\delta$

remind that the states X^n and X^m are nonstrictly nonbisimilar for $n > m$ since $X^n \stackrel{b^n}{\Leftrightarrow} \epsilon$ but $X^m \stackrel{b^n}{\not\Leftrightarrow}$. □

3.2 Expressibility w.r.t. language generation

In this section we show that the classes of BPA systems and BPA$_\delta$ systems are equivalent w.r.t. language generation. We will consider just the non-strict case ($F = \{\epsilon, \delta\} \cup \{\delta.E | E \in \mathcal{E}^+_{\text{BPA}}\}$) since it is obvious that the strict case does not yield any language extention.

Definition 3.2.1. *Let $(\mathcal{V}ar, \mathcal{A}ct, \Delta, X_1)$ be a guarded BPA$_\delta$ system. We define the* language generated by Δ *as $L(\Delta) \stackrel{\text{def}}{=} L(X_1)$. (For the definition of $L(X_1)$ see page 4.)*

Definition 3.2.2. *We define classes of languages generated by BPA and BPA$_\delta$ systems as following:*

$$\mathcal{L}(\text{BPA}) = \{L(\Delta) \mid \Delta \text{ is a guarded BPA system } \}$$

$$\mathcal{L}(\text{BPA}_\delta) = \{L(\Delta_\delta) \mid \Delta_\delta \text{ is a guarded BPA}_\delta \text{ system } \}$$

Theorem 3.2.3. *The classes of languages generated by BPA and BPA$_\delta$ systems are equal, i.e. $\mathcal{L}(\text{BPA}) = \mathcal{L}(\text{BPA}_\delta)$.*

Proof. We show that for a BPA$_\delta$ system Δ_δ there exists a BPA system Δ such that $L(\Delta_\delta) = L(\Delta)$. The other direction is obvious because a BPA system is a BPA$_\delta$ system as well.
Our proof will be constructive. For each variable $X \in \Delta_\delta$ we define a couple of new variables X^ϵ, X^δ. The first one will simulate the language behaviour of X when reaching the state ϵ, the second one will simulate ending in suffix of the form $\delta\alpha$.
We use notation $a\alpha \in Y$ meaning that $a\alpha$ is a summand of the defining equation of the variable Y (e.g. if $Y \stackrel{\text{def}}{=} aY + aXZ$ then $aY, aXZ \in Y$).

W.l.o.g. let Δ_δ be a BPA$_\delta$ system in 3–GNF. The variables of the system Δ will be $\mathcal{V}ar(\Delta) \stackrel{\text{def}}{=} \cup_{X \in \mathcal{V}ar(\Delta_\delta)-\{\delta\}}\{X^\epsilon, X^\delta\} \cup \{X_1^{\epsilon\delta}\}$ where X^ϵ and X^δ are distinct fresh variables and $X_1^{\epsilon\delta}$ is the leading variable supposing that X_1 was the leading variable of the system Δ_δ. Next we realize that the summands of the defining equation for $X \in \mathcal{V}ar(\Delta_\delta) \Leftrightarrow\{\delta\}$ are exactly of one of the following form (because of the 3–GNF):

$$\begin{array}{ll} \textbf{(a)} & aAB \\ \textbf{(b)} & bC \\ \textbf{(c)} & c \\ \textbf{(d)} & dD\delta \\ \textbf{(e)} & e\delta \end{array} \tag{3.1}$$

where $a, b, c, d, e \in \mathcal{A}ct(\Delta_\delta)$ and $A, B, C, D \in \mathcal{V}ar(\Delta_\delta)$ such that $A, B, C, D \neq \delta$. Notice that we can suppose that there is no summand of the form $a\delta A$ because the variable A cannot have any effect on the generated language and thus this summand can be replaced with $a\delta$.
We now define the variables from Δ. For each $X \in \mathcal{V}ar(\Delta_\delta) \Leftrightarrow\{\delta\}$ and for the summands of the variables X^ϵ andX^δ will hold:

$$\begin{array}{llrlr} \text{if} & aAB \in X & \text{then} & aA^\epsilon B^\epsilon \in X^\epsilon & \text{and} \quad aA^\epsilon B^\delta + aA^\delta \in X^\delta \\ \text{if} & bC \in X & \text{then} & bC^\epsilon \in X^\epsilon & \text{and} \quad bC^\delta \in X^\delta \\ \text{if} & c \in X & \text{then} & c \in X^\epsilon & \\ \text{if} & dD\delta \in X & \text{then} & & dD^\epsilon + dD^\delta \in X^\delta \\ \text{if} & e\delta \in X & \text{then} & & e \in X^\delta \end{array}$$

if $X_1^\epsilon \stackrel{\text{def}}{=} E$ and $X_1^\delta \stackrel{\text{def}}{=} F$ then $X_1^{\epsilon\delta} \stackrel{\text{def}}{=} E + F$

If it is the case that there is a variable $Y \in \mathcal{V}ar(\Delta)$ such that Y does not have any summand we define $Y \stackrel{\text{def}}{=} aY$. (This variable cannot generate any nonempty language because it is unnormed). Finally we state $X_1^{\epsilon\delta}$ to be the leading variable of the system Δ.

Example 3.2.4. Let us have a BPA$_\delta$ system

$$\Delta_\delta = \{X \stackrel{\text{def}}{=} aXX + b + c\delta + bY,\ Y \stackrel{\text{def}}{=} b\}.$$

The corresponding language equivalent BPA system Δ looks as following.

$$\Delta = \{X^\epsilon \stackrel{\text{def}}{=} aX^\epsilon X^\epsilon + b + bY^\epsilon,\ X^\delta \stackrel{\text{def}}{=} aX^\epsilon X^\delta + aX^\delta + c + bY^\delta,$$

$$Y^\epsilon \stackrel{\text{def}}{=} b,\ Y^\delta \stackrel{\text{def}}{=} a.Y^\delta,$$

$$X^{\epsilon\delta} \stackrel{\text{def}}{=} aX^\epsilon X^\epsilon + b + bY^\epsilon + aX^\epsilon X^\delta + aX^\delta + c + bY^\delta\}$$

It is not difficult to see that the newly defined system Δ is in 3–GNF and we show that $L(\Delta_\delta) = L(\Delta)$. For this we need one lemma using following notation.

Definition 3.2.5. *Let Δ' be a BPA (resp. BPA$_\delta$) system in 3–GNF, $n \geq 1$ and $Y \in \mathcal{V}ar(\Delta')$. We define $L_n^\epsilon(Y)$ and $L_n^\delta(Y)$ as following.*

$$L_n^\epsilon(Y) \stackrel{\text{def}}{=} \{w \in \mathcal{A}ct(\Delta')^* \mid Y \stackrel{w}{\Leftrightarrow} \epsilon \wedge \textit{length}(w) \leq n\}$$
$$L_n^\delta(Y) \stackrel{\text{def}}{=} \{w \in \mathcal{A}ct(\Delta')^* \mid \exists \alpha \in \mathcal{V}ar(\Delta')^* : Y \stackrel{w}{\Leftrightarrow} \delta\alpha \wedge \textit{length}(w) \leq n\}$$

Lemma 3.2.6. *For all $n \geq 1$ and $X \in \mathcal{V}ar(\Delta_\delta) \Leftrightarrow \{\delta\}$ holds that $L_n^\epsilon(X) = L_n^\epsilon(X^\epsilon)$ and $L_n^\delta(X) = L_n^\epsilon(X^\delta)$.*

Proof. By induction on n following the cases from 3.1.

- $\mathbf{n = 1}$: There are only two cases to be considered. If the process X terminates in ϵ we have to consider the case (c) and if it terminates in $\delta\alpha$ we have to consider the case (e). Both these cases are obvious.
- **induction step**: Let us suppose that the assertion is true for all $i \leq n$. Let us prove it for $n + 1$.
  1. $L_{n+1}^\epsilon(X) \subseteq L_{n+1}^\epsilon(X^\epsilon)$
     Let $w \in L_{n+1}^\epsilon(X)$ and $\textit{length}(w) = n + 1$. There are several cases according to the first action being performed:
     - **case (a)**: Suppose $w = aw_1w_2$ where $a \in \mathcal{A}ct$ and $w_1, w_2 \in \mathcal{A}ct^+$ such that $X \stackrel{a}{\Leftrightarrow} AB \stackrel{w_1}{\Leftrightarrow} B \stackrel{w_2}{\Leftrightarrow} \epsilon$. According to the definition of X^ϵ and using the IH applied to variables A and B for the words w_1 and w_2 (that are sharply shorter than w) we get that $X^\epsilon \stackrel{a}{\Leftrightarrow} A^\epsilon B^\epsilon \stackrel{w_1}{\Leftrightarrow} B^\epsilon \stackrel{w_2}{\Leftrightarrow} \epsilon$ and so $w \in L_{n+1}^\epsilon(X^\epsilon)$. For the rest of this proof all the conditions clear from the context will be omitted.
     - **case (b)**: $w = bw_1$ and $X \stackrel{b}{\Leftrightarrow} C \stackrel{w_1}{\Leftrightarrow} \epsilon$ but then $X^\epsilon \stackrel{b}{\Leftrightarrow} C^\epsilon \stackrel{w_1}{\Leftrightarrow} \epsilon$.
     - **case (c)**: This case is impossible because we suppose that $\textit{length}(w) \geq 2$.
     - **cases (d), (e)**: These cases are impossible because in this inclusion we consider only $X \stackrel{w}{\Leftrightarrow} \epsilon$ and it is clear that both $dD\delta$ and $e\delta$ are not able to reach the state ϵ.
  2. $L_{n+1}^\epsilon(X) \supseteq L_{n+1}^\epsilon(X^\epsilon)$
     Let $w \in L_{n+1}^\epsilon(X^\epsilon)$ and $\textit{length}(w) = n + 1$. There are several cases according to the first action being performed:
     - **case (a)**: Suppose $w = aw_1w_2$ and $X^\epsilon \stackrel{a}{\Leftrightarrow} A^\epsilon B^\epsilon \stackrel{w_1}{\Leftrightarrow} B^\epsilon \stackrel{w_2}{\Leftrightarrow} \epsilon$ but then $X \stackrel{a}{\Leftrightarrow} AB \stackrel{w_1}{\Leftrightarrow} B \stackrel{w_2}{\Leftrightarrow} \epsilon$. So $w \in L_{n+1}^\epsilon(X)$.

- **case (b)**: $w = bw_1$ and $X^\epsilon \stackrel{b}{\Leftrightarrow} C^\epsilon \stackrel{w_1}{\Leftrightarrow} \epsilon$ but then $X \stackrel{b}{\Leftrightarrow} C \stackrel{w_1}{\Leftrightarrow} \epsilon$. So $w \in L^\epsilon_{n+1}(X)$.
- **cases (c), (d), (e)**: These cases are impossible.

3. $L^\delta_{n+1}(X) \subseteq L^\epsilon_{n+1}(X^\delta)$
Let $w \in L^\delta_{n+1}(X)$ and $\mathit{length}(w) = n+1$. There are several cases according to the first action being performed:
   - **case (a)**: Suppose $w = aw_1$ and $X \stackrel{a}{\Leftrightarrow} AB \stackrel{w_1}{\Leftrightarrow} \delta\alpha$. Then $A \stackrel{w_1}{\Leftrightarrow} \delta\alpha'$ with $\alpha = \alpha' B$ or $w_1 = w_2 w_3$ with $AB \stackrel{w_2}{\Leftrightarrow} B \stackrel{w_3}{\Leftrightarrow} \delta\alpha$. For the first subcase we have $X^\delta \stackrel{a}{\Leftrightarrow} A^\delta \stackrel{w_1}{\Leftrightarrow} \epsilon$ according to the definition of X^δ and because of the IH used on A^δ. For the second subcase we get $X^\delta \stackrel{a}{\Leftrightarrow} A^\epsilon B^\delta \stackrel{w_2}{\Leftrightarrow} B^\delta \stackrel{w_3}{\Leftrightarrow} \epsilon$ because $A \stackrel{w_2}{\Leftrightarrow} \epsilon$ and using the IH we deduce that $A^\epsilon \stackrel{w_2}{\Leftrightarrow} \epsilon$ and similarly for B^δ. So we have that $w \in L^\epsilon_{n+1}(X^\delta)$.
   - **case (b)**: $w = bw_1$ and $X \stackrel{b}{\Leftrightarrow} C \stackrel{w_1}{\Leftrightarrow} \delta\alpha$ but then $X^\delta \stackrel{b}{\Leftrightarrow} C^\delta \stackrel{w_1}{\Leftrightarrow} \epsilon$ and so $w \in L^\epsilon_{n+1}(X^\delta)$.
   - **case (d)**: $w = dw_1$ and $X \stackrel{d}{\Leftrightarrow} D\delta \stackrel{w_1}{\Leftrightarrow} \delta\alpha$. Then $D \stackrel{w_1}{\Leftrightarrow} \epsilon$ with $\alpha = \epsilon$ or $D \stackrel{w_1}{\Leftrightarrow} \delta\alpha'$ such that $\alpha = \alpha'\delta$. It is easy to see that for both these subcases we get $X^\delta \stackrel{d}{\Leftrightarrow} D^\epsilon \stackrel{w_1}{\Leftrightarrow} \epsilon$ or $X^\delta \stackrel{d}{\Leftrightarrow} D^\delta \stackrel{w_1}{\Leftrightarrow} \epsilon$. This implies that $w \in L^\epsilon_{n+1}(X^\delta)$.
   - **cases (c), (e)**: These cases are impossible.

4. $L^\delta_{n+1}(X) \supseteq L^\epsilon_{n+1}(X^\delta)$
Let $w \in L^\epsilon_{n+1}(X^\delta)$ and $\mathit{length}(w) = n+1$. There are several cases according to the first action being performed:
   - **case (a)** : First suppose $w = aw_1w_2$ and $X^\delta \stackrel{a}{\Leftrightarrow} A^\epsilon B^\delta \stackrel{w_1}{\Leftrightarrow} B^\delta \stackrel{w_2}{\Leftrightarrow} \epsilon$. Then $X \stackrel{a}{\Leftrightarrow} AB \stackrel{w_1}{\Leftrightarrow} B \stackrel{w_1}{\Leftrightarrow} \delta\alpha$ and so $w \in L^\delta_{n+1}(X)$. Second suppose $w = aw_1$ and $X^\delta \stackrel{a}{\Leftrightarrow} A^\delta \stackrel{w_1}{\Leftrightarrow} \epsilon$ but then $X \stackrel{a}{\Leftrightarrow} AB \stackrel{w_1}{\Leftrightarrow} \delta\alpha B$ and $w \in L^\delta_{n+1}(X)$ as well.
   - **case (b)**: $w = bw_1$ and $X^\delta \stackrel{b}{\Leftrightarrow} C^\delta \stackrel{w_1}{\Leftrightarrow} \epsilon$ but then $X \stackrel{b}{\Leftrightarrow} C \stackrel{w_1}{\Leftrightarrow} \delta\alpha$ and so $w \in L^\delta_{n+1}(X)$.
   - **case (d)**: First suppose $w = dw_1$ and $X^\delta \stackrel{d}{\Leftrightarrow} D^\epsilon \stackrel{w_1}{\Leftrightarrow} \epsilon$ but then $X \stackrel{d}{\Leftrightarrow} D\delta \stackrel{w_1}{\Leftrightarrow} \delta$ and second suppose $w = dw_1$ and $X^\delta \stackrel{d}{\Leftrightarrow} D^\delta \stackrel{w_1}{\Leftrightarrow} \epsilon$ but then $X \stackrel{d}{\Leftrightarrow} D\delta \stackrel{w_1}{\Leftrightarrow} \delta\alpha\delta$ and so $w \in L^\delta_{n+1}(X)$.
   - **cases (c), (e)**: These cases are impossible.

□

To finish the proof of our theorem let us define for $n \geq 1$ the set $L_n(Y) \stackrel{\text{def}}{=} \{w \in L(Y) \mid \mathit{length}(w) \leq n\}$. Notice that because of the Lemma 3.2.6 we get $L_n(X_1) = L^\epsilon_n(X_1) \cup L^\delta_n(X_1) = L^\epsilon_n(X^\epsilon_1) \cup L^\epsilon_n(X^\delta_1) = L_n(X^{\epsilon\delta}_1)$ for all $n \geq 1$.

Now it is clear that $L(X_1) = L(X_1^{\epsilon\delta})$ since if $w \in L(X_1)$ then $\exists n : w \in L_n(X_1)$ and so $w \in L_n(X_1^{\epsilon\delta})$ which implies that $w \in L(X_1^{\epsilon\delta})$. The other direction is similar. We have shown that $L(\Delta_\delta) = L(\Delta)$ and our proof is complete. □

Chapter 4

Bisimilarity in BPA$_\delta$ systems

The first result indicating that decidability issues for bisimilarity are rather different from the ones for language equivalence is due to Baeten, Bergstra and Klop. They proved in [BBK87, BBK93] that bisimilarity is decidable for normed BPA . Much simpler proofs of this were later given in [Cau88],[HS91] and [Gro92].
It is well known result by Christensen, Hüttel and Stirling that the bisimulation equivalence is decidable in the class of all BPA systems – [CHS92]. The proof consists of two semidecidable procedures running in parallel. Burkart, Caucal and Steffen demonstrated in [BCS95] also an elementary decision procedure for BPA bisimilarity.
On the contrary the language equivalence of BPA processes is undecidable. The negative result for BPA [BHPS61] follows from the fact that BPA effectively defines the class of context-free languages. This argument can be shown to hold for the class of normed BPA systems as well. This undecidability result extends also to all equivalences which lie in Glabbeek's spectrum [vG90b] between bisimilarity and language equivalence [GH94, HT95]. Another result [Jan95] due to Jančar says that bisimilarity is undecidable for Petri Nets.
We show that the decidability of bisimilarity in BPA systems can be extended to BPA$_\delta$ systems. In the proof we exploit the result in [CHS92] and transform the examined BPA$_\delta$ systems into BPA systems, interpreting δ as a new unnormed variable. In this chapter w.l.o.g. we implicitly assume that all considered systems are in 3–GNF.

4.1 Decidability of nonstrict bisimilarity

Theorem 4.1.1. *Given two BPA$_\delta$ systems $\mathcal{T} = (\mathcal{V}ar, \mathcal{A}ct, \Delta, X_1)$ and $\overline{\mathcal{T}} = (\overline{\mathcal{V}ar}, \overline{\mathcal{A}ct}, \overline{\Delta}, \overline{X_1})$ it is decidable whether $\mathcal{T} \overset{n}{\sim} \overline{\mathcal{T}}$.*

Proof. We reduce this problem to the problem of decidability of bisimilar-

ity in BPA systems. We simply substitute the deadlock δ with a fresh unnormed variable.
Let us fix a fresh variable D such that $D \notin \mathcal{V}ar \cup \overline{\mathcal{V}ar}$ and an action d such that $d \notin \mathcal{A}ct \cup \overline{\mathcal{A}ct}$. We define a homomorphism $f : \mathcal{E}_{\mathrm{BPA}} \leftrightarrows \mathcal{E}_{\mathrm{BPA}}$ as follows:

$f(a) = a \quad$ for $a \in \mathcal{A}ct \cup \overline{\mathcal{A}ct}$
$f(X) = X \quad$ for $X \in (\mathcal{V}ar \cup \overline{\mathcal{V}ar}) \leftrightarrows \{\delta\}$
$f(\delta) = D$
$f(E + F) = f(E) + f(F), f(E.F) = f(E).f(F) \quad$ for $E, F \in \mathcal{E}^+_{\mathrm{BPA}}$

Let us define the systems $\mathcal{T}'$ and $\overline{\mathcal{T}'}$ as

$$\mathcal{T}' = (\mathcal{V}ar \cup \{D, X'_1\}, \mathcal{A}ct \cup \{d\}, \Delta', X'_1)$$
$$\overline{\mathcal{T}'} = (\overline{\mathcal{V}ar} \cup \{D, \overline{X'_1}\}, \overline{\mathcal{A}ct} \cup \{d\}, \overline{\Delta'}, \overline{X'_1})$$

where, assuming that $(X_1 \stackrel{\mathrm{def}}{=} E_1) \in \Delta$ and $(\overline{X_1} \stackrel{\mathrm{def}}{=} \overline{E_1}) \in \overline{\Delta}$, we state

$$\Delta' = \{X_i \stackrel{\mathrm{def}}{=} f(E_i) | X_i \stackrel{\mathrm{def}}{=} E_i \in \Delta\} \cup \{X'_1 \stackrel{\mathrm{def}}{=} f(E_1).D, D \stackrel{\mathrm{def}}{=} d.D\}$$
$$\overline{\Delta'} = \{\overline{X_i} \stackrel{\mathrm{def}}{=} f(\overline{E_i}) | \overline{X_i} \stackrel{\mathrm{def}}{=} \overline{E_i} \in \overline{\Delta}\} \cup \{\overline{X'_1} \stackrel{\mathrm{def}}{=} f(\overline{E_1}).D, D \stackrel{\mathrm{def}}{=} d.D\}.$$

The systems $\mathcal{T}'$ and $\overline{\mathcal{T}'}$ are now very similar to the previous ones except for the case when the systems reach the empty process (ϵ) or the deadlock (δ or $\delta.G$ where $G \in \mathcal{E}^+_{\mathrm{BPA}}$). The behaviour in these states is changed to capture the property that the empty process is nonstrict bisimilar to the deadlock. A new unnormed variable D is added to simulate these states.
It is easy to see that $\mathcal{T} \stackrel{n}{\sim} \overline{\mathcal{T}}$ if and only if $\mathcal{T}' \sim \overline{\mathcal{T}'}$. Moreover the systems $\mathcal{T}'$ and $\overline{\mathcal{T}'}$ are guarded BPA systems and bisimulation is decidable in the class BPA (see [CHS92]). Thus we can also decide whether $\mathcal{T} \stackrel{n}{\sim} \overline{\mathcal{T}}$.

□

Example 4.1.2. Let $\Delta = \{X \stackrel{\mathrm{def}}{=} aXX + b + c\delta\}$. The system Δ' from the proof above is following.

$$\Delta' = \{X' \stackrel{\mathrm{def}}{=} (aXX + b + cD).D,\ X \stackrel{\mathrm{def}}{=} aXX + b + cD,\ D \stackrel{\mathrm{def}}{=} d.D\}$$

4.2 Decidability of strict bisimilarity

Theorem 4.2.1. *Given two BPA$_\delta$ systems $\mathcal{T} = (\mathcal{V}ar, \mathcal{A}ct, \Delta, X_1)$ and $\overline{\mathcal{T}} = (\overline{\mathcal{V}ar}, \overline{\mathcal{A}ct}, \overline{\Delta}, \overline{X_1})$ it is decidable whether $\mathcal{T} \stackrel{s}{\sim} \overline{\mathcal{T}}$.*

Proof. The proof is quite easy because for the strict bisimilarity we have that $\delta \stackrel{s}{\not\sim} \epsilon$ and we can use the slightly modified trick from the proof above. We construct the same systems $\mathcal{T}'$ and $\overline{\mathcal{T}'}$ as before with one difference. The leading variables of the systems $\mathcal{T}'$ and $\overline{\mathcal{T}'}$ will remain X_1 and $\overline{X_1}$, however

we do not add the new equations $X_1' \stackrel{\text{def}}{=} f(E_1).D$ and $\overline{X_1'} \stackrel{\text{def}}{=} f(\overline{E_1}).D$. This ensures that in the newly defined systems (which are BPA systems) we can possibly reach the empty process. This empty process is not bisimilar to the state D (nor $D.G$ for $G \in \mathcal{E}^+_{\text{BPA}}$) simulating deadlocking. □

In Chapter 3 we have shown that BPA$_\delta$ systems have more descriptive power (up to bisimilarity) than BPA ones. This chapter answers positively to the question whether the bisimulation equivalence is decidable in BPA$_\delta$ class. This enables to construct systems with deadlocks and examine algorithmically their behaviour equivalence.

Chapter 5

Regularity in BPA$_\delta$ systems

Regularity of a transition system means in fact finiteness of the number of states up to bisimilarity. If we prove that a transition system can be expressed (up to bisimilarity) as a finite state system and that the construction is effective, we can decide all the interesting properties within such a regular system. Burkart, Caucal and Steffen demonstrated in [BCS96] that regularity is decidable for BPA processes and we exploit this result, thus extending the decidability to the BPA$_\delta$ systems.

5.1 BPA$_\delta$ regularity

Defining regularity of a BPA system is not difficult. We state a BPA system Δ to be *regular* iff it is bisimilar to a BPA system with finitely many reachable states. But in the case of BPA$_\delta$ we introduced two notions of bisimilarity (strict and nonstrict) and moreover we may consider regularity with regard to finite state BPA and BPA$_\delta$ system. There is no sence to consider strict bisimilarity w.r.t. finite state BPA . The nonstrict case is solved by the following lemma.

Lemma 5.1.1. *Let Δ be a BPA$_\delta$ system with finitely many reachable states. Then there exists a BPA system Δ' with finitely many reachable states such that $\Delta \overset{n}{\sim} \Delta'$.*

Proof. We can assume that the process Δ is in normal form, i.e. every equation is of the form:

$$X_i \overset{\text{def}}{=} \sum_j a_j X_j + \sum_k a_k,$$

where X_j can possibly be δ. This can be done because if there are only finitely reachable states, we give a special new name to every such a state. The set of variables will be formed with the names of these states and we add corresponding transitions. This trivially preserves nonstrict bisimilarity (the resulting transition systems are even isomorphic). We construct the

system Δ' from Δ by deleting all occurences of δ in each defining equation. The systems Δ and Δ' are easily seen to be nonstrictly bisimilar. □

When dealing with regularity we give two definitions for the case of strict and nonstrict bisimilarity. The second one is motivated by the Lemma 5.1.1 above.

Definition 5.1.2. *A BPA$_\delta$ system Δ is* strictly regular *iff there exists a BPA$_\delta$ system Δ' with finitely many reachable states such that $\Delta \overset{s}{\sim} \Delta'$.*

Definition 5.1.3. *A BPA$_\delta$ system Δ is* nonstrictly regular *iff there exists a BPA system Δ' with finitely many reachable states such that $\Delta \overset{n}{\sim} \Delta'$.*

We show that both strict and nonstrict regularity is decidable in the class BPA$_\delta$ and thus extend the result from [BCS96].

5.2 Decidability of strict regularity

Theorem 5.2.1. *Let Δ be a BPA$_\delta$ system. It is decidable whether Δ is strictly regular. If it is the case, the corresponding finite state BPA$_\delta$ system can be effectively constructed.*

Proof. We use again the trick from the proof of the Theorem 4.2.1. We reduce the problem to the problem of decidability of regularity in the BPA class. As in the proof above we transform the system Δ into Δ' such that all occurences of δ are replaced with a fresh variable D and a new defining equation for D, $D \overset{\text{def}}{=} d.D$, is added where $d \in \mathcal{A}ct$ is a fresh action. Now it is obvious that Δ' is regular (in the sense of BPA systems) if and only if Δ is strictly regular. Since regularity in the class of BPA systems is decidable (see [BCS96]), the strict regularity in the BPA$_\delta$ systems is also decidable. Moreover the corresponding finite state BPA$_\delta$ system can be easily constructed as we can find a finite state BPA system Δ'' in normal form, such that $\Delta'' \sim \Delta'$. It is enough to replace all occurences of each variable bisimilar to D with δ and remove definitions of such variables. □

Example 5.2.2. Let us have a BPA$_\delta$ system

$$\Delta = \{A \overset{\text{def}}{=} aBA\delta + a\delta,\ B \overset{\text{def}}{=} bAB + b\delta\}.$$

After the transformation we get

$$\Delta' = \{A \overset{\text{def}}{=} aBAD + aD,\ B \overset{\text{def}}{=} bAB + bD,\ D \overset{\text{def}}{=} aD\}.$$

This BPA system is regular and the bisimilar finite state system in normal form is e.g.

$$\Delta'' = \{A' \overset{\text{def}}{=} aB' + aD',\ B' \overset{\text{def}}{=} bA' + bD',\ D' \overset{\text{def}}{=} aD'\}.$$

By replacing D' with δ ($D' \sim D$) we get

$$\Delta''' = \{A' \stackrel{\text{def}}{=} aB' + a\delta,\ B' \stackrel{\text{def}}{=} bA' + b\delta\}$$

such that $\Delta \stackrel{s}{\sim} \Delta'''$.

5.3 Decidability of nonstrict regularity

For the proof of the nonstrict case we use the following lemma where we show that there is no difference between strict and nonstrict regularity.

Lemma 5.3.1. *A BPA$_\delta$ system Δ is strictly regular iff Δ is nonstrictly regular.*

Proof. We prove the implication from left to right. Suppose that Δ is strictly regular, i.e. there exists a BPA$_\delta$ system Δ' with finitely many reachable states such that $\Delta \stackrel{s}{\sim} \Delta'$. Because of the Lemma 2.5.2 we know that $\Delta \stackrel{n}{\sim} \Delta'$ and using the Lemma 5.1.1 we can see that there exists a BPA system Δ'' with finitely many reachable states such that $\Delta' \stackrel{n}{\sim} \Delta''$. Thus we have shown that $\Delta \stackrel{n}{\sim} \Delta''$ which implies that Δ is nonstrictly regular.
The implication from right to left is a bit more complicated. Suppose that Δ is nonstrictly regular, i.e. there exists a BPA system Δ' with finitely many reachable states such that $\Delta \stackrel{n}{\sim} \Delta'$. W.l.o.g. we may assume that Δ' is in normal form introduced in the proof of the Lemma 5.1.1. Let X_1 and X_1' be leading variables of the systems Δ resp. Δ'. Then we know that there exists some relation of nonstrict bisimulation R such that $(X_1, X_1') \in R$. Let us modify the system Δ' into Δ'' following the rules below. For each $X \in \mathcal{V}ar(\Delta')$ and $a \in \mathcal{A}ct(\Delta')$:

- Remove all the summands of the form a from the definition of X.
- If $(E, X) \in R$ such that $E \stackrel{a}{\Leftrightarrow} \delta$ or $E \stackrel{a}{\Leftrightarrow} \delta.G$ for some $G \in \mathcal{E}^+_{\text{BPA}}$ then add the summand $a\delta$ into the definition of X.
- If $(E, X) \in R$ such that $E \stackrel{a}{\Leftrightarrow} \epsilon$ then add the summand a into the definition of X.

Let us define the relation S as following.

$$S \stackrel{\text{def}}{=} (R \Leftrightarrow \{(\delta, \epsilon)\} \Leftrightarrow \{(\delta.G, \epsilon) \mid G \in \mathcal{E}^+_{\text{BPA}}\}) \cup \{(\delta, \delta)\} \cup \{(\delta.G, \delta) \mid G \in \mathcal{E}^+_{\text{BPA}}\}$$

Then obviously $(X_1, X_1') \in S$ and moreover we show that S is the relation of strict bisimulation. This implies that $\Delta \stackrel{s}{\sim} \Delta''$.
In fact we have removed all the inconvenient pairs from R and added all the deadlocking pairs. It is an easy observation that if $(\alpha, \beta) \in S$ then $\alpha \in F$ iff $\beta \in F$. This means that there is no collision between ϵ and δ any more.

- Let $(E, X) \in S$ and $a \in \mathcal{A}ct(\Delta'')$.
  - If $E \stackrel{a}{\Leftrightarrow} E'$ such that $E' \neq \epsilon$ and $E' \neq \delta$ and $E' \neq \delta.G$ for all $G \in \mathcal{E}^+_{\mathrm{BPA}}$ then $X \stackrel{a}{\Leftrightarrow} X'$ such that $(E', X') \in R$ which implies that $(E', X') \in S$.
  - If $E \stackrel{a}{\Leftrightarrow} \delta$ or $E \stackrel{a}{\Leftrightarrow} \delta.G$ for some $G \in \mathcal{E}^+_{\mathrm{BPA}}$ then $X \stackrel{a}{\Leftrightarrow} \delta$ and $(\delta, \delta) \in S$ resp. $(\delta.G, \delta) \in S$.
  - If $E \stackrel{a}{\Leftrightarrow} \epsilon$ then $X \stackrel{a}{\Leftrightarrow} \epsilon$ and obviously $(\epsilon, \epsilon) \in S$.
- Let $(E, X) \in S$ and $a \in \mathcal{A}ct(\Delta'')$.
  - If $X \stackrel{a}{\Leftrightarrow} X'$ such that $X' \neq \epsilon$ and $X' \neq \delta$ then $E \stackrel{a}{\Leftrightarrow} E'$ such that $(E', X') \in R$ which implies that $(E', X') \in S$.
  - If $X \stackrel{a}{\Leftrightarrow} \delta$ then $E \stackrel{a}{\Leftrightarrow} \delta$ or $E \stackrel{a}{\Leftrightarrow} \delta.G$ for some $G \in \mathcal{E}^+_{\mathrm{BPA}}$ and we can see that $(\delta, \delta) \in S$ resp. $(\delta.G, \delta) \in S$.
  - If $X \stackrel{a}{\Leftrightarrow} \epsilon$ then $E \stackrel{a}{\Leftrightarrow} \epsilon$ and $(\epsilon, \epsilon) \in S$.

□

Theorem 5.3.2. *Let Δ be a BPA_δ system. It is decidable whether Δ is nonstrictly regular. If it is the case, the corresponding finite state BPA system can be effectively constructed.*

Proof. Using the Lemma 5.3.1 and the Theorem 5.2.1 we can decide whether Δ is nonstrictly regular since Δ is nonstrictly regular iff Δ is strictly regular. Moreover the first part in the proof of the Lemma 5.3.1 gives directions how to construct the corresponding finite state BPA system. □

Chapter 6

Describing BPA$_\delta$ in BPA syntax

In the Section 3.1 we have shown that the class of BPA$_\delta$ systems is sharply larger (w.r.t. bisimilarity) than that of BPA . This challenges the question whether given a BPA$_\delta$ system this system can be equivalently described in BPA syntax. The answer for both strict and nonstrict bisimilarity taken as the equivalence relation is the topic of this chapter. The characterisation for the strict bisimulation is given by Theorem 6.1.1 and Theorem 6.2.15 demonstrates the corresponding result for the nonstrict bisimulation.

6.1 Strict case

Theorem 6.1.1. *Let $(\mathcal{V}ar, \mathcal{A}ct, \Delta, X_1)$ be a BPA$_\delta$ system. It is decidable whether there exists a BPA system Δ' such that $\Delta \overset{s}{\sim} \Delta'$. Moreover if the answer is positive, the system Δ' can be effectively constructed.*

Proof. The proof is rather technical and is based on the fact that $\delta \overset{s}{\not\sim} \epsilon$. Consider the system Δ. If a state of the form δ or $\delta.E$ for $E \in \mathcal{E}^+_{\mathrm{BPA}}$ is reachable from the leading variable then there cannot be any BPA system bisimilar to Δ. If the deadlocking state is not reachable the system Δ can be easily transformed into a BPA system.
Suppose w.lo.g. that the system Δ is in 3–GNF. We construct the sets $M_0, M_1, \ldots$ of variables from which the deadlock is reachable as following. The notation $\alpha \in E$ means again that α is a summand in the expression E.

$$M_0 \overset{\mathrm{def}}{=} \{\delta\}$$

And for $i \geq 0$ the sets M_{i+1} are defined as:

$$M_{i+1} \overset{\mathrm{def}}{=} M_i \cup \{X \in \mathcal{V}ar \mid \exists a \in \mathcal{A}ct, \exists Y \in \mathcal{V}ar, \exists D \in M_i :$$

$$(X \overset{\mathrm{def}}{=} E) \in \Delta,\ \ a.D \in E \ \vee\ a.D.Y \in E \ \vee\ (a.Y.D \in E \text{ and } \|Y\| < \infty)\}$$

We remind that the norm of a variable can be effectively computed. Since there are only finitely many variables used in the system Δ then for some $k \geq 0$ the set M_k is a fixed point of this construction, i.e. $M_k = M_{k+l}$ for each $l > 0$. Let us denote the set M_k simply as M.
Now we get an easy consequence clear from the construction of the sets M_i. For each $X \in \mathcal{V}ar$:

$$X \Leftrightarrow\!\!\rightarrow^* \delta.\alpha \text{ for some } \alpha \in \mathcal{V}ar^* \iff X \in M$$

If $X_1 \in M$ then Δ cannot be expressed by a BPA syntax since the deadlocking state is reachable from X_1. If $X_1 \notin M$ we can transform Δ into a BPA system. For this case realize that if $Y \in M$ then $X_1 \not\Leftrightarrow\!\!\rightarrow^* Y.\alpha$ for any $\alpha \in \mathcal{V}ar^*$. Let us define $(\mathcal{V}ar \Leftrightarrow M, \mathcal{A}ct, \Delta', X_1)$ where for each $(X \stackrel{\text{def}}{=} E) \in \Delta$ we have that $(X \stackrel{\text{def}}{=} E') \in \Delta'$ whenever $X \notin M$ and E' is the same as E except for the summand of the type $a.YD$ where $Y \in \mathcal{V}ar$ and $D \in M$, which is replaced with $a.Y$. This can be done because Y must be an unnormed variable, otherwise $X \in M$.
It is clear that Δ' is strictly bisimilar to Δ (only irredundant variables were disposed) and moreover Δ' is a BPA system – from the construction. □

6.2 Nonstrict case

In this section we focus on those BPA_δ systems which can be described in corresponding BPA syntax w.r.t. nonstrict bisimilarity. The situation, when allowing deadlocks can bring more descriptive power, is nicely characterised by the Theorem 6.2.15.
We can simply observe that in a BPA_δ labelled transition system there are only finitely many successors of each state. In such case we call the system as *image-finite*.

Definition 6.2.1. *A labelled transition system* $(S, \mathcal{A}ct, \Leftrightarrow\!\!\rightarrow, \alpha_0, F)$ *is* image-finite *if the set* $\{\beta \mid \alpha \stackrel{a}{\Leftrightarrow\!\!\rightarrow} \beta\}$ *is finite for each* $\alpha \in S$ *and* $a \in \mathcal{A}ct$.

Bisimilarity in such image-finite systems is characterisable using the following sequence of approximations.

Definition 6.2.2. *Let* $(S, \mathcal{A}ct, \Leftrightarrow\!\!\rightarrow, \alpha_0, F)$ *be a labelled transition system. The* stratified bisimulation relations *[Mil89]* $\sim_k$ *are defined as follows.*

- $\alpha \sim_0 \beta$ *for all* $\alpha, \beta \in S$
- $\alpha \sim_{k+1} \beta$ *iff for each* $a \in \mathcal{A}ct$*:*
  - *if* $\alpha \stackrel{a}{\Leftrightarrow\!\!\rightarrow} \alpha'$ *then* $\beta \stackrel{a}{\Leftrightarrow\!\!\rightarrow} \beta'$ *for some* β' *such that* $\alpha' \sim_k \beta'$
  - *if* $\beta \stackrel{a}{\Leftrightarrow\!\!\rightarrow} \beta'$ *then* $\alpha \stackrel{a}{\Leftrightarrow\!\!\rightarrow} \alpha'$ *for some* α' *such that* $\alpha' \sim_k \beta'$

– $\alpha \in F$ *iff* $\beta \in F$

The following lemma is standard.

Lemma 6.2.3. *Let $(S, \mathcal{A}ct, \Leftrightarrow\rightarrow, \alpha_0, F)$ be an image-finite labelled transition system and $\alpha, \beta \in S$. Then $\alpha \sim \beta$ iff $\alpha \sim_k \beta$ for all $k \geq 0$.*

Remark. In the case of BPA$_\delta$ systems and considering the nonstrict bisimilarity, the third condition $\alpha \in F$ iff $\beta \in F$ in the Definition 6.2.2 is always true since all the terminal states are included in F.

In what follows, the set of variables from which the deadlock is reachable will be of great importance. Hence we define the set $\mathcal{V}ar_\delta$ of such variables.

Definition 6.2.4. *Let $(\mathcal{V}ar, \mathcal{A}ct, \Delta, X_1)$ be a BPA$_\delta$ system. Let us define the sets*

$$\mathcal{V}ar_\delta \stackrel{\text{def}}{=} \{X \in \mathcal{V}ar \mid X \Leftrightarrow\rightarrow^* \delta \text{ or } \exists E \in \mathcal{E}^+_{BPA} : X \Leftrightarrow\rightarrow^* \delta.E\} \Leftrightarrow\{\delta\}$$

$$\mathcal{V}ar_\epsilon \stackrel{\text{def}}{=} \mathcal{V}ar \Leftrightarrow\{\delta\} \Leftrightarrow\mathcal{V}ar_\delta.$$

This separates the variables from $\mathcal{V}ar$ into two sets $\mathcal{V}ar_\delta$ and $\mathcal{V}ar_\epsilon$ (i.e. $\mathcal{V}ar = \mathcal{V}ar_\delta \cup \mathcal{V}ar_\epsilon \cup \{\delta\}$). For the purpose of this section let the variables U, V, X, Y, Z range over $\mathcal{V}ar_\delta$ and A, B over $\mathcal{V}ar_\epsilon$.

Remark. We remind that the sets $\mathcal{V}ar_\delta$ and $\mathcal{V}ar_\epsilon$ can be effectively constructed as we have demonstrated in the proof of the Theorem 6.1.1.

Theorem 6.2.5. *Let $(\mathcal{V}ar, \mathcal{A}ct, \Delta, X_1)$ be a BPA$_\delta$ system in 3–GNF. Suppose that there are only finitely many pairwise nonstrictly nonbisimilar $Y\alpha \in \mathcal{V}ar_\delta.\mathcal{V}ar^*$ such that $X_1 \Leftrightarrow\rightarrow^* Y\alpha$. Then there exists a BPA system $(\mathcal{V}ar', \mathcal{A}ct', \Delta', X_1')$ such that $\Delta \stackrel{n}{\sim} \Delta'$.*

Proof. Let us assume that $X_1 \in \mathcal{V}ar_\epsilon$. Then the system Δ can be trivially transformed into bisimilar BPA system Δ'. In what follows let us assume that $X_1 \in \mathcal{V}ar_\delta$.
We may suppose w.l.o.g. that each summand of every defining equation in Δ does not contain an unnormed variable (resp. δ) followed by another variable. In other words, every occurrence of an unnormed variable (resp. δ) is always at the rear of the appropriate summand. For example if the summand of the form aNM where N is unnormed is presented we replace it with aN and the summand $a\delta M$ we replace with $a\delta$.
Let us define functions f_α for each $\alpha \in \mathcal{V}ar^*$. These functions take an expression from $\mathcal{E}^+_{\text{BPA}}$ in 3–GNF and transform it into another expression (possibly adding some new variables of the form X^β). Our goal is following. We want to achieve $f_\alpha(E) \stackrel{n}{\sim} E\alpha$ and there should be no deadlock in $f_\alpha(E)$. We remind that $X, Y, U \in \mathcal{V}ar_\delta$ and $A, B, C \in \mathcal{V}ar_\epsilon$ where moreover $\|C\| = \infty$.

$$f_\alpha(\sum_{i=1}^{n} a_i\alpha_i) = \sum_{i=1}^{n} f_\alpha(a_i\alpha_i)$$

$f_\alpha(aXY)$	=	$aX^{Y\alpha}$	
$f_\alpha(aX\delta)$	=	aX^{ϵ}	
$f_\alpha(a\delta)$	=	a	
$f_\alpha(aX)$	=	aX^{α}	
$f_\alpha(aAB)$	=	$aAB\beta U^{\gamma}$	if $\alpha = \beta U\gamma$, where $\beta \in \mathcal{V}ar^*_\epsilon$ such that $\|\beta\| < \infty$, $\gamma \in \mathcal{V}ar^*$
	=	$aAB\beta C$	if $\alpha = \beta C\gamma$, where $\beta \in \mathcal{V}ar^*_\epsilon$ such that $\|\beta\| < \infty$, $\gamma \in \mathcal{V}ar^*$
	=	$aAB\alpha$	otherwise ($\alpha \in \mathcal{V}ar^*_\epsilon$)
$f_\alpha(a)$	=	$a\beta U^{\gamma}$	if $\alpha = \beta U\gamma$, where $\beta \in \mathcal{V}ar^*_\epsilon$ such that $\|\beta\| < \infty$, $\gamma \in \mathcal{V}ar^*$
	=	$a\beta C$	if $\alpha = \beta C\gamma$, where $\beta \in \mathcal{V}ar^*_\epsilon$ such that $\|\beta\| < \infty$, $\gamma \in \mathcal{V}ar^*$
	=	$a\alpha$	otherwise ($\alpha \in \mathcal{V}ar^*_\epsilon$)
$f_\alpha(aA\delta)$	=	aA	
$f_\alpha(aA)$	=	$aA\beta U^{\gamma}$	if $\alpha = \beta U\gamma$, where $\beta \in \mathcal{V}ar^*_\epsilon$ such that $\|\beta\| < \infty$, $\gamma \in \mathcal{V}ar^*$
	=	$aA\beta C$	if $\alpha = \beta C\gamma$, where $\beta \in \mathcal{V}ar^*_\epsilon$ such that $\|\beta\| < \infty$, $\gamma \in \mathcal{V}ar^*$
	=	$aA\alpha$	otherwise ($\alpha \in \mathcal{V}ar^*_\epsilon$)
$f_\alpha(aXA)$	=	$aX^{A\alpha}$	
$f_\alpha(aAX)$	=	aAX^{α}	

For each $\alpha \in \mathcal{V}ar^*$ let us define a function r_α which returns the set of the new variables added by the function f_α.

$$r_\alpha(\sum_{i=1}^{n} a_i\alpha_i) = \bigcup_{i=1}^{n} r_\alpha(a_i\alpha_i)$$

$$
\begin{array}{lcll}
r_\alpha(aXY) & = & \{X^{Y\alpha}\} & \\
r_\alpha(aX\delta) & = & \{X^{\epsilon}\} & \\
r_\alpha(a\delta) & = & \emptyset & \\
r_\alpha(aX) & = & \{X^{\alpha}\} & \\
r_\alpha(aAB) & = & \{U^{\gamma}\} & \text{if } \alpha = \beta U\gamma, \text{ where } \beta \in \mathcal{V}ar_\epsilon^* \text{ such that } \|\beta\| < \infty, \\
 & & & \gamma \in \mathcal{V}ar^* \\
 & = & \emptyset & \text{if } \alpha = \beta C\gamma, \text{ where } \beta \in \mathcal{V}ar_\epsilon^* \text{ such that } \|\beta\| < \infty, \\
 & & & \gamma \in \mathcal{V}ar^* \\
 & = & \emptyset & \text{otherwise } (\alpha \in \mathcal{V}ar_\epsilon^*) \\
r_\alpha(a) & = & \{U^{\gamma}\} & \text{if } \alpha = \beta U\gamma, \text{ where } \beta \in \mathcal{V}ar_\epsilon^* \text{ such that } \|\beta\| < \infty, \\
 & & & \gamma \in \mathcal{V}ar^* \\
 & = & \emptyset & \text{if } \alpha = \beta C\gamma, \text{ where } \beta \in \mathcal{V}ar_\epsilon^* \text{ such that } \|\beta\| < \infty, \\
 & & & \gamma \in \mathcal{V}ar^* \\
 & = & \emptyset & \text{otherwise } (\alpha \in \mathcal{V}ar_\epsilon^*) \\
r_\alpha(aA\delta) & = & \emptyset & \\
r_\alpha(aA) & = & \{U^{\gamma}\} & \text{if } \alpha = \beta U\gamma, \text{ where } \beta \in \mathcal{V}ar_\epsilon^* \text{ such that } \|\beta\| < \infty, \\
 & & & \gamma \in \mathcal{V}ar^* \\
 & = & \emptyset & \text{if } \alpha = \beta C\gamma, \text{ where } \beta \in \mathcal{V}ar_\epsilon^* \text{ such that } \|\beta\| < \infty, \\
 & & & \gamma \in \mathcal{V}ar^* \\
 & = & \emptyset & \text{otherwise } (\alpha \in \mathcal{V}ar_\epsilon^*) \\
r_\alpha(aXA) & = & \{X^{A\alpha}\} & \\
r_\alpha(aAX) & = & \{X^{\alpha}\} &
\end{array}
$$

Let us now construct the nonstrictly bisimilar BPA system Δ' where

$$\mathcal{V}ar' \stackrel{\text{def}}{=} \mathcal{V}ar_\epsilon \cup \mathsf{Added},$$

$$\mathcal{A}ct' \stackrel{\text{def}}{=} \mathcal{A}ct,$$

$$\Delta' \stackrel{\text{def}}{=} \Delta_\epsilon \cup \Gamma,$$

$$X_1' \stackrel{\text{def}}{=} X_1^\epsilon.$$

The sets Added and Γ are outputs of the following algorithm and $\Delta_\epsilon \subseteq \Delta$ contains exactly the defining equations for variables from $\mathcal{V}ar_\epsilon$.

The transformation of the defining equations of the variables from $\mathcal{V}ar_\delta$ is the goal of the Algorithm 6.2.6. The set Solve contains the variables that need to be defined; Added is the set of variables that have been already defined or are in the set Solve ; Γ is the set of the current definitions; Add is the set of variables born in each repetition of the main loop.

Algorithm 6.2.6.

1 Solve:=$\{X_1^\epsilon\}$
2 Added:=$\{X_1^\epsilon\}$

3 $\Gamma := \emptyset$
4 **while** Solve $\neq \emptyset$ **do**
5 *Let us fix* $X^\alpha \in$ Solve *with* $(X \stackrel{\text{def}}{=} E) \in \Delta$
6 $\Gamma := \Gamma \cup \{X^\alpha \stackrel{\text{def}}{=} f_\alpha(E)\}$
7 Add:=$\{Y^\beta \in r_\alpha(E) \mid \forall Z^\omega \in$ Added $: Y\beta \stackrel{n}{\not\sim} Z\omega\}$
8 **while** $\exists Y^\beta, Z^\omega \in$ Add $: Y^\beta \neq Z^\omega \wedge Y\beta \stackrel{n}{\sim} Z\omega$ **do**
9 Add:= Add $\Leftrightarrow \{Y^\beta\}$
10 **endwhile**
11 Solve:= (Solve $\Leftrightarrow \{X^\alpha\}) \cup$ Add
12 Added:= Added $\cup$ Add
13 **for** $\forall Y^\beta \in r_\alpha(E) \Leftrightarrow$ Add **do**
14 *replace all occurences of* Y^β *in* Γ *with* Z^ω
15 *where* $Z^\omega \in$ Added $: Y\beta \stackrel{n}{\sim} Z\omega$
16 **endfor**
17 **endwhile**

In the following lemmas we demonstrate that the algorithm is correct and produces a BPA system Δ' such that $\Delta \stackrel{n}{\sim} \Delta'$.

Lemma 6.2.7. *For the loop 4–17 of the Algorithm 6.2.6 holds the following invariant* $\mathcal{I}$.

$$\forall Y^\beta, Z^\omega \in \text{Added} : Y^\beta \neq Z^\omega \Rightarrow Y\beta \stackrel{n}{\not\sim} Z\omega$$

Proof. The invariant $\mathcal{I}$ holds at line 3, because the set Added contains just one variable. Some new variables can potentially be added to the set Added at line 12. Because of the loop 8–10 the variables in Add are pairwise non-strictly nonbisimilar. Line 7 ensures that $\mathcal{I}$ will hold for Added:=Added$\cup$Add also. □

Lemma 6.2.8. *Whenever during the execution of the Algorithm 6.2.6 we have* $Y^\alpha \in$ Added *then* $Y \in \mathcal{V}ar_\delta$.

Proof. All variables in Added had to be produced by the function r_α (see line 7 and 12). It is an easy observation that $\{Y | Y^\beta \in r_\alpha(E)\} \subseteq \mathcal{V}ar_\delta$ for any $\alpha \in \mathcal{V}ar^*$ and $E \in \mathcal{E}^+_{\text{BPA}}$ such that E is in 3–GNF. □

Lemma 6.2.9. *Whenever during the execution of the Algorithm 6.2.6 we have* $Y^\beta \in$ Added *then* $X_1 \Leftrightarrow\!\!\rightarrow^* Y\beta$.

Proof. By induction on the number of repetitions of the loop 4–17.
Basic step: The only variable in the set Added before the execution of the loop 4–17 started is X_1^ϵ. But $X_1\epsilon = X_1$ and so $X_1 \Leftrightarrow\!\!\rightarrow^* X_1\epsilon$.
Induction step: Suppose that at line 12 we have added a new variable Y^β into Added. So at line 7 we had to have $Y^\beta \in r_\alpha(E)$ for some $X^\alpha \in$ Solve and $(X \stackrel{\text{def}}{=} E) \in \Delta$. The induction hypothesis says that $X_1 \Leftrightarrow\!\!\rightarrow^* X\alpha$

(X^α had to be added in some previous repetition of the main loop). It must hold that $a\gamma Y^\beta \in f_\alpha(E)$ where $\gamma \in \mathcal{V}ar^*_\epsilon$ and $\|\gamma\| < \infty$. From the construction of the function f_α we can also see that $X\alpha \Leftrightarrow\!\!\rightarrow^* Y\beta$. So we get $X_1 \Leftrightarrow\!\!\rightarrow^* X\alpha \Leftrightarrow\!\!\rightarrow^* Y\beta$.

□

Lemma 6.2.10. *Under the assumptions of the Theorem 6.2.5 the Algorithm 6.2.6 cannot loop forever.*

Proof. Suppose that the algorithm loops forever which means that the set Solve is never empty. But in every loop we remove exactly one element from the set Solve (line 11). This implies that the set Added will grow arbitrarily because the set Add is infinitely often unempty (otherwise the algorithm would stop). From the Lemma 6.2.9 and Lemma 6.2.8 we know that $\forall Y^\beta \in$ Added : $Y \in \mathcal{V}ar_\delta \ \wedge \ X_1 \Leftrightarrow\!\!\rightarrow^* Y\beta$. Moreover from the Lemma 6.2.7 follows that these states are pairwise nonstrictly nonbisimilar. The contradiction is immediate as we have shown that if the algorithm loops then there is no upper bound on the cardinality of the set Added. □

From the previous lemma we know that the Algorithm 6.2.6 will stop after finitely many repetitions of the main loop and thus the set Added will also be finite. The following lemma is crucial for the proof of our theorem.

Lemma 6.2.11. *After the execution of the Algorithm 6.2.6 we have $V^\alpha \overset{n}{\sim} V\alpha$ for all $V^\alpha \in$ Added.*

Proof. By induction on k we show that $V^\alpha \sim_k V\alpha$ for all $k \geq 0$. This implies that $V^\alpha \overset{n}{\sim} V\alpha$.
Basic step: We receive $V^\alpha \sim_0 V\alpha$ from the definition.
Induction step: We show that $V^\alpha \sim_{k+1} V\alpha$.
Suppose that $V^\alpha \overset{a}{\Leftrightarrow\!\!\rightarrow} V'$. Then one of the following cases applies (according to the definition of f_α):

- Let us consider the summand aXY. Then one of the following cases will hold:
  - $V^\alpha \overset{a}{\Leftrightarrow\!\!\rightarrow} X^{Y\alpha}$ but then $V\alpha \overset{a}{\Leftrightarrow\!\!\rightarrow} XY\alpha$. Using the induction hypothesis we get $X^{Y\alpha} \sim_k XY\alpha$, because $X^{Y\alpha} \in$ Added.
  - $V^\alpha \overset{a}{\Leftrightarrow\!\!\rightarrow} Z^\omega$ where $Z^\omega \in$ Added and $X^{Y\alpha}$ was at lines 14,15 replaced with Z^ω. Then $Z^\omega \sim_k Z\omega$ (induction hypothesis) and $Z\omega \overset{n}{\sim} XY\alpha$. This implies that $V\alpha \overset{a}{\Leftrightarrow\!\!\rightarrow} XY\alpha$ and $XY\alpha \sim_k Z\omega \sim_k Z^\omega$.
- Let us consider the summand $aX\delta$:

  - $V^\alpha \stackrel{a}{\Leftrightarrow} X^\epsilon$ but then $V\alpha \stackrel{a}{\Leftrightarrow} X\delta\alpha$. We know that $X\delta\alpha \stackrel{n}{\sim} X$ and using the induction hypothesis we get $X^\epsilon \sim_k X$ because $X^\epsilon \in$ Added. Thus we get $X\delta\alpha \sim_k X^\epsilon$.
  - $V^\alpha \stackrel{a}{\Leftrightarrow} Z^\omega$ where $Z^\omega \in$ Added and X^ϵ was at lines 14,15 replaced with Z^ω. Then $Z^\omega \sim_k Z\omega$ (induction hypothesis) and $Z\omega \stackrel{n}{\sim} X\delta\alpha$. This implies that $V\alpha \stackrel{a}{\Leftrightarrow} X\delta\alpha$ and $X\delta\alpha \sim_k Z\omega \sim_k Z^\omega$.

- Let us consider the summand $a\delta$:
  - $V^\alpha \stackrel{a}{\Leftrightarrow} \epsilon$ but then $V\alpha \stackrel{a}{\Leftrightarrow} \delta\alpha$ and $\epsilon \sim_k \delta\alpha$ because trivially $\epsilon \stackrel{n}{\sim} \delta\alpha$.

- Let us consider the summand aX:
  - this is very similar to aXY

- Let us consider the summand aAB:
  - $V^\alpha \stackrel{a}{\Leftrightarrow} AB\beta U^\gamma$ but then $V\alpha \stackrel{a}{\Leftrightarrow} AB\beta U\gamma$. Using the induction hypothesis we know $U^\gamma \sim_k U\gamma$ because $U^\gamma \in$ Added and we get $AB\beta U^\gamma \sim_k AB\beta U\gamma$.
  - $V^\alpha \stackrel{a}{\Leftrightarrow} AB\beta Z^\omega$ where $Z^\omega \in$ Added and U^γ was at lines 14,15 replaced with Z^ω. Then $Z^\omega \sim_k Z\omega$ (induction hypothesis) and $Z\omega \stackrel{n}{\sim} U\gamma$. This implies that $V\alpha \stackrel{a}{\Leftrightarrow} AB\beta U\gamma$ and $AB\beta U\gamma \sim_k AB\beta Z^\omega$.
  - $V^\alpha \stackrel{a}{\Leftrightarrow} AB\beta C$ such that $\|C\| = \infty$ but then $V\alpha \stackrel{a}{\Leftrightarrow} AB\beta C\gamma$ and easily $AB\beta C \sim_k AB\beta C\gamma$.
  - $V^\alpha \stackrel{a}{\Leftrightarrow} AB\alpha$ but then $V\alpha \stackrel{a}{\Leftrightarrow} AB\alpha$ and obviously $AB\alpha \sim_k AB\alpha$.

- Let us consider the summands a and aA:
  - these are very similar to aAB

- Let us consider the summand $aA\delta$:
  - this is very similar to $a\delta$

- Let us consider the summand aXA:
  - this is very similar to aXY

- Let us consider the summand aAX:
  - $V^\alpha \stackrel{a}{\Leftrightarrow} AX^\alpha$ but then $V\alpha \stackrel{a}{\Leftrightarrow} AX\alpha$. Using the induction hypothesis we get $X^\alpha \sim_k X\alpha$ because $X^\alpha \in$ Added and so $AX^\alpha \sim_k AX\alpha$.

– $V^\alpha \stackrel{a}{\Leftrightarrow} AZ^\omega$ where $Z^\omega \in$ Added and X^α was at lines 14,15 replaced with Z^ω. Then $Z^\omega \sim_k Z\omega$ (induction hypothesis) and $Z\omega \stackrel{n}{\sim} X\alpha$. This implies that $V\alpha \stackrel{a}{\Leftrightarrow} AX\alpha$ and $AX\alpha \sim_k AZ\omega \sim_k AZ^\omega$.

Suppose that $V\alpha \stackrel{a}{\Leftrightarrow} V'$.Then one of the following cases applies (according to the definition of f_α):

- Let us consider the summand aXY. If $V\alpha \stackrel{a}{\Leftrightarrow} XY\alpha$ then one of the following cases will hold:
  - $V^\alpha \stackrel{a}{\Leftrightarrow} X^{Y\alpha}$, where $X^{Y\alpha} \in$ Added and using the induction hypothesis we get $X^{Y\alpha} \sim_k XY\alpha$.
  - $V^\alpha \stackrel{a}{\Leftrightarrow} Z^\omega$, where $Z^\omega \in$ Added and $X^{Y\alpha}$ was at lines 14,15 replaced with Z^ω. Then $Z^\omega \sim_k Z\omega$ (induction hypothesis) and $Z\omega \stackrel{n}{\sim} XY\alpha$. This means that $Z^\omega \sim_k XY\alpha$.
- Let us consider the summand $aX\delta$. If $V\alpha \stackrel{a}{\Leftrightarrow} X\delta\alpha$ then one of the following cases will hold:
  - $V^\alpha \stackrel{a}{\Leftrightarrow} X^\epsilon$, where $X^\epsilon \in$ Added and using the induction hypothesis we get $X^\epsilon \sim_k X$ and so $X^\epsilon \sim_k X\delta\alpha$.
  - $V^\alpha \stackrel{a}{\Leftrightarrow} Z^\omega$, where $Z^\omega \in$ Added and X^ϵ was at lines 14,15 replaced with Z^ω. Then $Z^\omega \sim_k Z\omega$ (induction hypothesis) and $Z\omega \stackrel{n}{\sim} X$. This means that $Z^\omega \sim_k X\delta\alpha$.
- Let us consider the summand $a\delta$. If $V\alpha \stackrel{a}{\Leftrightarrow} \delta\alpha$ then
  - $V^\alpha \stackrel{a}{\Leftrightarrow} \epsilon$ and $\epsilon \sim_k \delta\alpha$.
- Let us consider the summand aX:
  - this case is very similar to aXY
- Let us consider the summand aAB. If $V\alpha \stackrel{a}{\Leftrightarrow} AB\alpha$ then one of the following cases will hold:
  - $V^\alpha \stackrel{a}{\Leftrightarrow} AB\beta U^\gamma$, where $\alpha = \beta U\gamma$ and $U^\gamma \in$ Added. Using the induction hypothesis we get $U^\gamma \sim_k U\gamma$ and so $AB\beta U^\gamma \sim_k AB\alpha$.
  - $V^\alpha \stackrel{a}{\Leftrightarrow} AB\beta Z^\omega$, where $\alpha = \beta U\gamma$, $Z^\omega \in$ Added and U^γ was at lines 14,15 replaced with Z^ω. Then $Z^\omega \sim_k Z\omega$ (induction hypothesis) and $Z\omega \stackrel{n}{\sim} U\gamma$. This means that $AB\beta Z^\omega \sim_k AB\alpha$.
  - $V^\alpha \stackrel{a}{\Leftrightarrow} AB\beta C$ such that $\| C \| = \infty$ and $\alpha = \beta C\gamma$ but then $AB\beta C \sim_k AB\alpha$.
  - $V^\alpha \stackrel{a}{\Leftrightarrow} AB\alpha$ but then $AB\alpha \sim_k AB\alpha$.

- Let us consider the summands a and aA:
  - these cases are very similar to aAB
- Let us consider the summands $aA\delta$:
  - this case is very similar to $a\delta$
- Let us consider the summands aXA:
  - this case is very similar to aXY
- Let us consider the summand aAX. If $V\alpha \stackrel{a}{\Leftrightarrow\!\!\rightarrow} AX\alpha$ then one of the following cases will hold:
  - $V^\alpha \stackrel{a}{\Leftrightarrow\!\!\rightarrow} AX^\alpha$, where $X^\alpha \in$ Added and using the induction hypothesis we get $X^\alpha \sim_k X\alpha$ and so $AX^\alpha \sim_k AX\alpha$.
  - $V^\alpha \stackrel{a}{\Leftrightarrow\!\!\rightarrow} AZ^\omega$, where $Z^\omega \in$ Added and X^α was at lines 14,15 replaced with Z^ω. Then $Z^\omega \sim_k Z\omega$ (induction hypothesis) and $Z\omega \stackrel{n}{\sim} X\alpha$. This means that $AZ^\omega \sim_k AX\alpha$.

□

Lemma 6.2.12. *The system Δ' is a BPA system and moreover $X_1 \stackrel{n}{\sim} X_1^\epsilon$.*

Proof. There are no undefined variables in Δ', which follows from the fact that each variable added into the set Added (line 12) had to be put into Solve (line 11) and so had to be expanded (line 6). Moreover observe that all δ's were removed by the function f_α. The fact $X_1 \stackrel{n}{\sim} X_1^\epsilon$ follows from the Lemma 6.2.11. □

Under the condition of our theorem (and for the given BPA$_\delta$ system Δ) we have constructed a BPA system Δ' such that $\Delta \stackrel{n}{\sim} \Delta'$.

□

Theorem 6.2.13. *Let $(\mathcal{V}ar, \mathcal{A}ct, \Delta, X_1)$ be a BPA$_\delta$ system. Suppose that there are infinitely many pairwise nonstrictly nonbisimilar $Y\alpha \in \mathcal{V}ar_\delta.\mathcal{V}ar^*$ such that $X_1 \Leftrightarrow\!\!\rightarrow^* Y\alpha$. Then there is no BPA system Δ' such that $\Delta \stackrel{n}{\sim} \Delta'$.*

Proof. The proof of this theorem is based on an immediate lemma.

Lemma 6.2.14. *Suppose that α and β are states of some BPA$_\delta$ system. Then*

$$\alpha \stackrel{n}{\sim} \beta \;\Rightarrow\; \|\alpha\|=\|\beta\| \,.$$

Proof. Suppose that the lemma is not true. Then w.l.o.g. let us have $\alpha \stackrel{n}{\sim} \beta$ such that $\|\alpha\|<\|\beta\|$. But then β cannot follow α's norm-reducing sequence w since the norm of β is strictly greater than that of α. This implies that there is no state γ such that $\beta \stackrel{w}{\Leftrightarrow\!\!\rightarrow} \gamma$ and $\gamma \not\Leftrightarrow\!\!\rightarrow$. Hence α cannot be nonstrictly bisimilar to β. □

Let us assume that there exists Δ' (w.l.o.g. we may suppose that Δ' is in 3–GNF) such that $\Delta \overset{n}{\sim} \Delta'$. We show that this is not possible. Since there are infinitely many reachable states $Y_1\alpha_1, Y_2\alpha_2, \ldots$ of Δ which are pairwise nonstrictly nonbisimilar there must be corresponding states $\beta_1, \beta_2, \ldots$ of the system Δ' such that $Y_i\alpha_i \overset{n}{\sim} \beta_i$ for $i = 1, 2, \ldots$. Let us now define a constant N_{max} as $N_{max} \overset{\text{def}}{=} \max\{\| Y_i \|_\delta \mid i = 1, 2, \ldots\}$ where $\| Y \|_\delta \overset{\text{def}}{=} \min\{\mathit{length}(w) \mid Y \overset{w}{\Longrightarrow} \delta \text{ or } \exists E \in \mathcal{E}^+_{\text{BPA}} : Y \overset{w}{\Longrightarrow} \delta.E\}$. Notice that the definition of N_{max} is correct since for all i $\|Y_i\|_\delta < \infty$ (because $Y_i \in \mathcal{V}ar_\delta$) and there are only finitely many different Y_i's.
Clearly $\|Y_i\alpha_i\| \leq N_{max}$ for all i. This implies that the norm of β_i is also less or equal N_{max} for all i (Lemma 6.2.14). However, Δ' is a BPA system and all variables in Δ' are guarded. This means that there are only finitely many states of Δ' such that their norm is less or equal N_{max}. Hence there must be two states β_k and β_l with $k \neq l$ such that $\beta_k = \beta_l$. This implies that $\beta_k \overset{n}{\sim} \beta_l$. Then also $Y_k\alpha_k \overset{n}{\sim} Y_l\alpha_l$, which is contradiction. □

Theorems above give us more intuitive image of what is the power of deadlocks. We are able to shorten rapidly the norm of a potentially large BPA process $Y\alpha$, whenever we introduce a new deadlocking variable, reachable from the head Y of this process. Suppose now that we have a BPA$_\delta$ system and that there are infinitely many nonbisimilar states from which, after some 'short' sequence of actions, a deadlocking state is reachable. Then the corresponding (nonstrictly bisimilar) BPA system does not exist. This condition appears to be both necessary and sufficient as is illustrated by the following theorem.

Theorem 6.2.15. *Let $(\mathcal{V}ar, \mathcal{A}ct, \Delta, X_1)$ be a BPA$_\delta$ system. There are only finitely many pairwise nonstrictly nonbisimilar $Y\alpha \in \mathcal{V}ar_\delta.\mathcal{V}ar^*$ such that $X_1 \Leftrightarrow^* Y\alpha$ if and only if there exists a BPA system $(\mathcal{V}ar', \mathcal{A}ct', \Delta', X_1')$ such that $\Delta \overset{n}{\sim} \Delta'$.*

Proof. The implication from left to right follows from the Theorem 6.2.5 and from the fact that a BPA$_\delta$ system can be bisimilarly described in 3–GNF, which has been proved in the Theorem 2.9.2. The other implication is an immediate consequence of the Theorem 6.2.13. □

Chapter 7

Conclusion

In this thesis we focused on the class of BPA processes extended with deadlocks. It has been shown that for input-output semantics the extention is no acquisition. On the other hand the BPA_δ class is larger with regard to the relation of bisimulation. We introduce two notions of bisimilarity to capture the different understanding of deadlock behaviour. If we do not distinguish between the state ϵ and δ we speak about nonstrict bisimilarity and if we do, we call the appropriate bisimulation equivalence as strict. We have shown that some decidable properties of BPA systems remain decidable in the BPA_δ class, e.g. decidability of bisimulation equivalence and regularity extends to BPA_δ systems.

Finally we have solved the question whether, given a BPA_δ system Δ, there is an equivalent description (with regard to bisimilarity) of Δ in terms of BPA syntax. The solution for strict bisimilarity is rather technical. However, the answer to the problem dealing with nonstrict bisimilarity exploited a nice semantic characterisation of the subclass of BPA_δ processes bisimilarly describable in BPA syntax: a BPA_δ system can be transformed into a BPA system (preserving nonstrict bisimilarity) if and only if finitely many nonbisimilar states starting with some in δ-ending variable are reachable. There is still an open problem whether this semantic characterisation is syntactically checkable. Future research could answer to this problem and there still remain many issues to examine such as extending the classes BPP and PA with deadlocks.

I think that this thesis offers a rather complete introduction into context-free process algebras extended with deadlocking states. Already the definition of deadlocks is interesting. What is important is that these BPA_δ systems are more powerful than the BPA ones (w.r.t. bisimilarity), in spite of the fact that they do not bring any language extention. Moreover BPA_δ systems appeared to be a suitable tool for specification of sequential processes since all the important properties remain decidable in this class.

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