Action Investment Energy Games*

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Abstract. We introduce the formalism of action investment energy games where we study the trade-off between investments limited by given budgets and resource constrained (energy) behavior of the underlying system. More specifically, we consider energy games extended with costs of enabling actions and fixed budgets for each player. We ask the question whether for any *Player 2* investment there exists a *Player 1* investment such that *Player 1* wins the resulting energy game. We study the action investment energy game for energy intervals with both upper and lower bounds, and with a lower bound only, and give a complexity results overview for the problem of deciding the winner in the game.

1 Introduction

Embedded systems are often executed on hardware with limited resources and they interact with uncontrollable or even hostile environments. By adding on top of this the ever increasing demand on software functionality and reliability for the lowest possible price, several interesting computational and optimization problems emerge. We introduce the formalism of action investment energy games where we study the trade-off between investments limited by given budgets and resource constrained (energy) behavior of the underlying system.

An action investment energy game (AIEG) is a two player game, played by *Player 1* and *Player 2*, each having a finite investment budget. The game consists of two independent phases: the *investment-configuration phase* and the *energy-game phase*. It is played on a finite multi-graph where transitions are labeled with action names and integers, representing energy changes. Furthermore each action has its own investment cost. In the investment-configuration phase, each player makes an investment by choosing a set of actions costing less than his/her budget. The result of these investments configures the board where an energy game is played on in the second energy-game phase. An energy game [3] is a turn based game, played on a finite multi-graph labeled with integer energy weights created in the investment-configuration phase. *Player 1* wins the energy game if she has a strategy such that the accumulated energy along any play according to the strategy is within a given interval.

Player 1 wins an AIEG if for any *Player 2* investment costing him less than his budget, it is possible for her to choose an investment costing less than her

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budget such that she wins the resulting energy game. Our focus is on the complexity of the decision problem asking whether *Player 1* has a winning strategy in the given AIEG.

Player 1 actions	*coffee*	*coco*
Action cost	2	4

Player 2 actions	*MEMICS*	*CAV*
Action cost	4	6

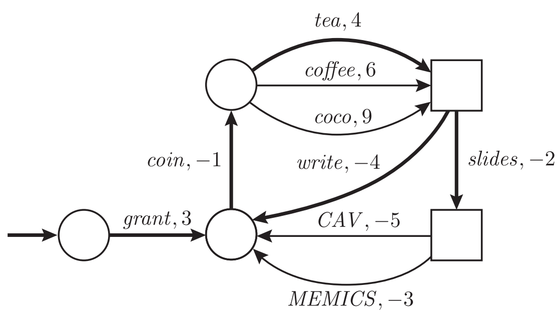


Fig. 1: Example of an action investment energy game

An example of an action investment energy game is given in Figure 1. Here the nodes that have a circle shape belong to *Player 1* (she, young PhD student) and the squares are nodes that belong to *Player 2* (he, PhD supervisor who likes to stress PhD students to their limits). Every edge (transition) is labelled with the action name and a weight (energy change). The student repeatedly buys a drink of her choice after which she asks her adviser if she should keep writing a paper or prepare slides and present them at a conference. Having a drink increases her energy level while inserting a coin, writing a paper, preparation of slides and a trip to *CAV* or *MEMICS* cost energy. In order to survive, the student must always have a nonnegative energy level, while increasing the energy above a given upper-bound makes the student to quit. The thick transitions in the graph are always present while the thin ones can be enabled by the players in the first phase of the game, depending on their budgets. The student's budget gives her the opportunity to rent a drink-machine and configure the available drinks as long as they are within her budget. The supervisor has a travel budget allowing him to send the student to different conferences, provided it does not exceed the financial limit. In our example this means that *Player 1* can enable the actions *coffee* and *coco*, if her budget B_1 is sufficient for this and *Player 2* is in control of the actions *CAV* and *MEMICS*, relative to his budget B_2.

The game starts by *Player 2* investing in actions within his budget B_2. Then *Player 1* buys her actions up to the total cost B_1. All transitions under the selected actions are then included into the final graph. Now the players start moving a pebble across the graph such that *Player 1* selects her next move (edge) from any circle and *Player 2* from any square node. Starting with energy level 0, the energy weight on the selected edge is added to the so far accumulated energy. The question is whether for a given interval $[a, b]$, $a \leq 0 \leq b$ where b can be an infinity, *Player 1* has a winning strategy so that the accumulated energy stays within the interval $[a, b]$.

For example if the budget of *Player 1* is $B_1 = 4$ and the budget of *Player 2* is $B_2 = 5$, we can see that *Player 1* wins in the interval $[0, 9]$. The reason is that *Player 2* can choose only to invest in the action *MEMICS* as *CAV* is beyond the travel budget. *Player 1* then responds by buying the action *coffee* staying within her budget. The board of the game is now configured and the players start the energy-game phase. After getting a *grant*, inserting a *coin* and choosing the *coffee* transition, the student ends up in the situation with the accumulated energy equal to 8. Now *Player 2* can force the student to prepare the *slides* for a presentation and travel to *MEMICS*, reaching the accumulated energy 3 and returning the student to the configuration that already appeared (and hence it is winning for *Player 1*), or he can decide that the student should first write a paper, ending up with energy level 4. Now *Player 1* can insert a coin and choose to drink *coffee* again, increasing her accumulated energy to 9. Should *Player 2* decide to ask her to prepare slides and go to *MEMICS* now, the energy drops to 4. This is again a winning situation for *Player 1* as it has been reached before. If on the other hand *Player 2* decides that she should continue to work on the paper, the energy level reaches 5. Now *Player 1* can decide to drink a cup of *tea*, giving her the accumulated energy 8 that we have already seen before. Hence *Player 1* has a winning strategy in the game.

In fact the reader can verify that *Player 1* has a different winning strategy even in the interval $[0, 7]$ but if her budget does not allow to invest into neither *coffee* or *coco*, then *Player 1* losses for any given interval.

The main contribution of our work is the definition of action investment energy games and a detailed complexity analysis of the decision problem determining the winner of the game. In a work [1] related to ours, a similar trade-off scenario was studied where a dual-price schema for modal transition systems was introduced. Here the authors study the trade-off between a long-run average execution cost and a hardware investment cost, but they do not consider constrained resources and do not model uncontrollable environments. Several problems related to energy games were recently studied in [6, 10, 5, 4], including extensions to real-time games [3, 2] and imperfect information [9] but none of these works considered the investment phase. Another related formalism of feature transition systems [8, 7] studies the problem of CTL/LTL model checking of the transition systems configured via a set of available features. However, features do not have any associated costs, the checked property is different from our energy condition and the game is restricted to 1-player only.

Proofs that are missing due to the space limitations can be found in the full version of the paper.

2 Definitions

We shall now present the definition of the action investment energy games. We start by recalling the notion of energy games.

2.1 Energy Game

Our notion of energy games is based on the definition from [3] where we consider general energy intervals $[a, b]$ instead of only $[0, b]$.

Definition 1 (Energy Game). *An* Energy Game *(EG) is defined as a tuple* $\mathcal{G} = (Q_1, Q_2, \Sigma, \rightarrow, q_0)$ *where*

- Q_1, Q_2 *are finite disjoint sets of states, we denote* $Q = Q_1 \uplus Q_2$*,*
- Σ *is a finite set of actions,*
- $\rightarrow \subseteq Q \times \Sigma \times \mathbb{Z} \times Q$ *is the successor relation where* $(q, \sigma, z, q') \in \rightarrow$ *is written as* $q \xrightarrow{\sigma,z} q'$*, and*
- $q_0 \in Q$ *is the initial state.*

An energy game can be depicted as a graph where each node represents a state such that circled nodes belong to Q_1 and squared ones to Q_2. Edges represent a transition between states and each edge is labeled with an action name and its weight. The energy game is played by moving a token around on the graph. The token starts in the initial state q_0. If the token is in a circle state, then *Player 1* moves the token to a successor state. Likewise, if the token is in a square state then *Player 2* moves the token to one of its successor states. The sequence on which the token moves is called a run, and it is defined as a finite or infinite sequence of transitions

$$r = q_0 \xrightarrow{\sigma_0,z_0} q_1 \xrightarrow{\sigma_1,z_1} q_2 \xrightarrow{\sigma_2,z_2} \dots$$

where $q_i \in Q$ and $q_i \xrightarrow{\sigma_i,z_i} q_{i+1} \in \rightarrow$ for all i. Given a finite run $r = q_0 \xrightarrow{\sigma_0,z_0} \dots \xrightarrow{\sigma_{n-1},z_{n-1}} q_n$, let the last state of the run be denoted by $Last(r) = q_n$. A run r is *maximal* if it is infinite, or finite and $Last(r)$ has no successors.

Definition 2 (Valid Run). *A maximal run* r *is valid in a given interval* $[a, b]$*,* $a \in \mathbb{Z}$*,* $b \in \mathbb{Z} \cup \{\infty\}$*,* $a \leq 0 \leq b$*, if* $a \leq \sum_{i=0}^{n} z_i \leq b$ *for all* n*.*

A play of the game can produce different runs depending on what strategy each player uses. A strategy δ_i for *Player i*, where $i = \{1, 2\}$, maps each finite non-maximal run r where $Last(r) = q_n \in Q_i$ to its successor $q_n \xrightarrow{\sigma_n,z_n} q_{n+1}$.

Problem 1 (Interval Bounded Energy Game). *Given an EG* $\mathcal{G}$ *and an interval* $[a, b]$*, does there exist a strategy* δ_1 *for Player 1 such that any play on* $\mathcal{G}$ *using the strategy* δ_1 *produces a run valid in the interval* $[a, b]$*?*

Player 1 wins and *Player 2* looses in the interval $[a, b]$ if the answer to Problem 1 in the interval $[a, b]$ is positive. There is also a more relaxed version of the interval bounded energy game called the *lower-bound problem* where the interval is of the form $[a, \infty]$.

2.2 Action Investment Energy Game

Let us split a given action alphabet Σ into three disjoint action sets $\Sigma_0 \uplus \Sigma_1 \uplus \Sigma_2 = \Sigma$ and call it the *action investment alphabet*. The action set Σ_i, $i = \{1, 2\}$, belongs to Player i, while Σ_0 is the set of default actions that are always present.

Definition 3 (Action Investment Energy Game). *An* action investment energy game *(AIEG) is a tuple,* $\mathcal{A} = (Q_1, Q_2, \Sigma, \rightarrow, q_0, actCost, B_1, B_2)$ *where,*

- $(Q_1, Q_2, \Sigma, \rightarrow, q_0)$ *is an energy game,*
- Σ *is an action investment alphabet,*
- $actCost : \Sigma_1 \cup \Sigma_2 \rightarrow \mathbb{N}$ *is a function assigning positive cost to the actions, and*
- $B_1, B_2 \in \mathbb{N}_0$ *are two nonnegative budgets.*

An *investment* is a subset of actions $I \subseteq \Sigma_1 \cup \Sigma_2$. The cost of an investment is the sum of the cost of the actions in the investment $invCost(I) = \sum_{\sigma \in I} actCost(\sigma)$. An investment for Player i, denoted by I_i, satisfies $I_i \subseteq \Sigma_i$.

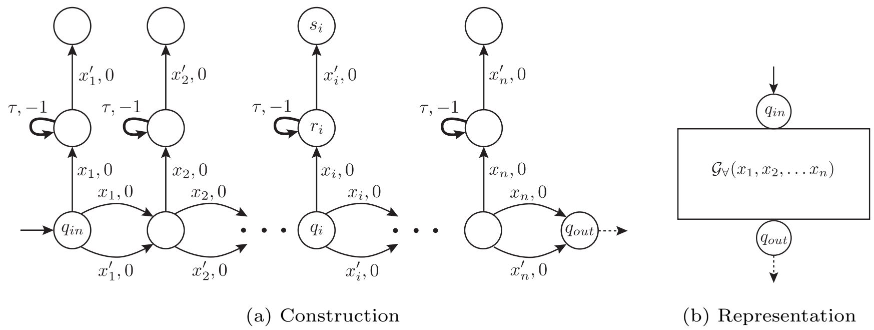
Problem 2 (AIEG Problem). *Given an AIEG* $\mathcal{A}$ *and an interval* $[a, b]$*, does there for any Player 2 investment* $I_2 \subseteq \Sigma_2$ *where* $invCost(I_2) \leq B_2$ *exist a Player 1 investment* $I_1 \subseteq \Sigma_1$ *where* $invCost(I_1) \leq B_1$*, such that* Player 1 *wins the energy game* $\mathcal{G}' = (Q, Q_1, Q_2, \Sigma', \rightarrow', q_0)$ *in the interval* $[a, b]$*, where* $\Sigma' = \Sigma_0 \cup I_1 \cup I_2$ *and* $\rightarrow' = \rightarrow \cap (Q \times \Sigma' \times \mathbb{Z} \times Q)$*?*

The AIEG problem can be understood as a two-phase game: an *investment-configuration phase* and an *energy-game phase*. In the *investment-configuration phase*, *Player 2* starts by choosing his investment $I_2 \subseteq \Sigma_2$ costing less than his budget $invCost(I_2) \leq B_2$, then *Player 1* chooses her investment $I_1 \subseteq \Sigma_1$ costing less than her budget $invCost(I_1) \leq B_1$. This ends the first phase and the *energy-game phase* starts. In the *energy-game phase* the energy game is played on the reconfigured board where only actions in the two investments I_1, I_2 and the default actions from Σ_0 are present.

It is clear that for an AIEG problem where $\Sigma_1 = \emptyset$ and $\Sigma_2 = \emptyset$, or where $B_1 = 0$ and $B_2 = 0$, the AIEG problem reduces to the classical energy game as none of the players can make any investment.

3 Gadgets for Complexity Bounds

Our main contribution is a detailed complexity analysis of the general action investment energy game problem and some of its prominent subclasses. For this reason, we start by establishing several gadgets (basically instances of AIEG) that will be used in the next section in order to prove complexity lower-bounds by reductions from different variants of quantified boolean formulae satisfiability problem. Let us assume a set of n boolean variables $x_1, \ldots, x_n$ for which we consider the action alphabet $\{x_j, x'_j \mid 1 \leq j \leq n\}$. We start by the definition of a valid investment.



(a) Construction (b) Representation

Fig. 2: Gadget $\mathcal{G}_\forall(\boldsymbol{x})$ where $\boldsymbol{x} = (x_1, \dots, x_n)$

Definition 4 (Valid Investment). *Let $i \in \{1,2\}$ and let $\Sigma_i = \{x_j, x'_j \mid 1 \leq j \leq n\}$. An investment $I_i \subseteq \Sigma_i$ is* valid *for Player i if for all j, $1 \leq j \leq n$, either $x_j \in I_i$ or $x'_j \in I_i$ and $\{x_j, x'_j\} \not\subseteq I_i$.*

Hence a valid investment represents an assignment of truth values to the boolean variables such that if $x_j \in I_i$ then the value of x_j is true and if $x'_j \in I_i$ then x_j takes the value false. It is clear that each player needs a sufficient budget to make a valid investment. This is defined in the next definition.

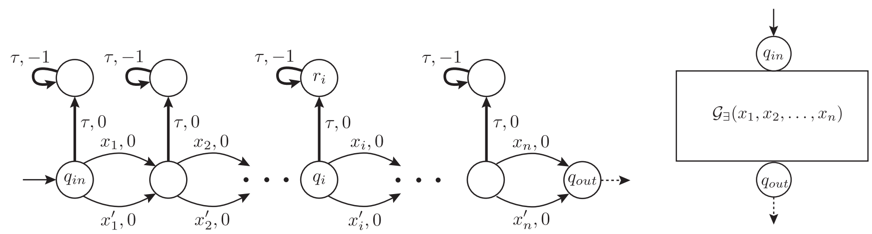
Definition 5 (Sufficient Budget). *A budget B_i is sufficient for Player i, where $i = \{1,2\}$, if $actCost(I_i) \leq B_i$ for any valid investments $I_i \subseteq \Sigma_i$.*

3.1 Gadget $\mathcal{G}_\forall(\boldsymbol{x})$

The gadget $\mathcal{G}_\forall$ is used by *Player 2* to enforce a valid truth assignment of the variables $\{x_1, \dots, x_n\}$ of his choice. In this gadget we let $\Sigma_2 = \{x_1, \dots x_n, x'_1 \dots x'_n\}$, and all states belong to *Player 1*. Let $\tau \in \Sigma_0$ be a default action that is always present. The construction of $\mathcal{G}_\forall$ and its graphical representation is given in Figure 2. The construction ensures that *Player 2* needs to make a valid investment otherwise *Player 1* has a strategy to win. In addition if *Player 2* chooses a valid investment then any play starting from q_{in} reaches q_{out} or it is loosing for *Player 1*. The following lemma formalizes this fact, assuming that *Player 2* has a sufficient budget to make a valid investment.

Lemma 1 (Properties of the Gadget $\mathcal{G}_\forall$).

(a) If I_2 is a valid investment for Player 2, then any play starting from q_{in} is either loosing for Player 1 or reaches q_{out} and Player 1 has moreover a strategy to ensure that any play from q_{in} reaches q_{out}.
(b) If I_2 is not a valid investment for Player 2, then Player 1 has a strategy to win any play starting from q_{in}.



(a) Construction

(b) Representation

Fig. 3: Gadget $\mathcal{G}_\exists(\boldsymbol{x})$ where $\boldsymbol{x} = (x_1, \ldots, x_n)$

3.2 Gadget $\mathcal{G}_\exists(\boldsymbol{x})$

The gadget $\mathcal{G}_\exists(\boldsymbol{x})$ is used by *Player 1* to fix her truth assignment of variables $\{x_1, \ldots, x_n\}$. We let $\Sigma_1 = \{x_1, \ldots x_n, x'_1 \ldots x'_n\}$ and fix the budget for *Player 1* to correspond to the number of variables, in other words $B_1 = n$. We also assume that the cost of any action in Σ_1 is equal to 1 so that it is possible for *Player 1* to make a valid investment. Let $\tau \in \Sigma_0$ be a default action that is always present. The gadget and its graphical representation is depicted in Figure 3. The point is that *Player 1* needs to choose a valid investment, otherwise *Player 2* has a winning strategy. Note that all nodes in the gadget belong to *Player 1*, so she is the only one that decides the moves in this gadget.

Lemma 2 (Properties of the Gadget $\mathcal{G}_\exists$).

(a) If I_1 is a valid investment for Player 1, then any play starting from q_{in} is either loosing for Player 1 or reaches q_{out} and Player 1 has moreover a strategy to ensure that any play from q_{in} reaches q_{out}.
(b) If I_1 is not a valid investment for Player 1, then any play from q_{in} is winning for Player 2.

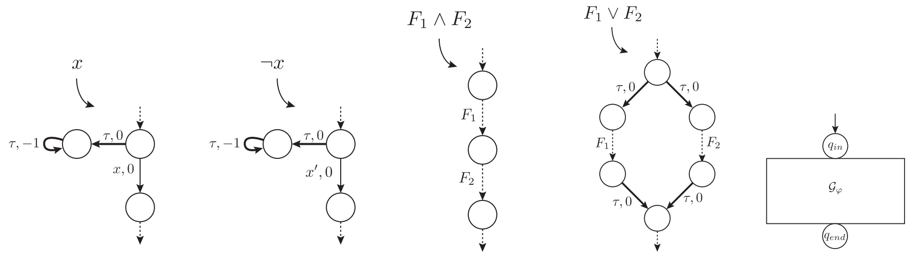
3.3 Gadget $\mathcal{G}_\varphi$

After introducing gadgets that allow the players of the AIEG game to choose valid investments that correspond to truth assignments of boolean variables, we need to make a gadget that will check if a boolean formula φ is true under the selected assignment. Such a gadget $\mathcal{G}_\varphi$ is defined inductively in Figure 4 where the dotted lines represents subgadgets already constructed for the subformulae.

For the proof of correctness, recall that $\Sigma_1 \cap \Sigma_2 = \emptyset$ and that $\tau \in \Sigma_0$ is a default action always present in the game.

Lemma 3 (Properties of the Gadget $\mathcal{G}_\varphi$).
Let I be a valid investment and let

$$\upsilon(x) = \begin{cases} \mathit{true} & \text{if } x \in I \\ \mathit{false} & \text{if } x' \in I \end{cases}$$



(a) Positive literal (b) Negative literal (c) Conjunction (d) Disjunction (e) Representation

Fig. 4: Gadget $\mathcal{G}_\varphi$

be the corresponding truth assignment.

(a) If φ is true under the assignment υ then Player 1 has a strategy to win any play starting from q_{in} in $\mathcal{G}_\varphi$.
(b) If φ is false under the assignment υ then Player 2 has a strategy to win any play starting from q_{in} in $\mathcal{G}_\varphi$.

3.4 Linking gadgets

For the purpose of our complexity proofs in the following section, gadgets can be linked together in a sequence and thereby create a combined AIEG. This is done via a transition labelled with $\tau, 0$ starting from q_{out} in one gadget and leading to the state q_{in} of the other gadget. The shorthand notation for linking gadgets is an arrow such that for example $\mathcal{G}_\forall(\boldsymbol{x}) \to \mathcal{G}_\exists(\boldsymbol{y}) \to \mathcal{G}_\varphi$ corresponds to an AIEG starting with the universal gadget over the vector of variables $\boldsymbol{x}$, followed by the existential gadget over the vector of variables $\boldsymbol{y}$ and finished by the gadget checking the validity of a formula under the generated truth assignment. It is necessary to rename the states of the linked gadgets to avoid name clashes and to update the successor relation accordingly. This can be done in the expected way, so we omit the details here.

4 Complexity results

We shall now present an overview of complexity results for different subclasses of the AIEG problem. The new results in Table 1 are listed in bold font and we consider the action investment energy game problem as well as its well-studied existential variant where all states belong to *Player 1*. We moreover study the variants of the game where one of the players has a zero budget (the case when both of them have zero budget corresponds to classical energy games) and we distinguish intervals that are closed or open to the right.

Some of the bounds in Table 1 use complexity classes from the polynomial hierarchy. We refer the reader to some classical textbook like [11] for more details

Budget restrictions	Interval	Energy game type	
		Existential	Game
$B_1 = 0$, $B_2 = 0$	$[a, \infty]$	$\in$ P [3]	$\in$ UP $\cap$ coUP [3]
	$[a, b]$	NP-Hard, $\in$ PSPACE [3]	EXPTIME-complete [3]
$B_2 = 0$	$[a, \infty]$	**NP-Complete**, Lem. 4	**NP-Complete** Lem. 10
	$[a, b]$	**NP-Hard,** $\in$ **PSPACE**, Lem. 5	**EXPTIME-complete** Lem. 13
$B_1 = 0$	$[a, \infty]$	**co-NP-complete**, Lem. 6	**co-NP-Complete** 11
	$[a, b]$	Π_2^P**-hard**, $\in$ **PSPACE**, Lem. 7	**EXPTIME-complete** Lem. 13
$-$	$[a, \infty]$	Π_2^P**-complete**, Lem. 8	Π_2^P**-complete** 12
	$[a, b]$	Π_2^P**-hard** $\in$ **PSPACE**, Lem. 9	**EXPTIME-complete** Lem. 13

Table 1: Complexity Overview for AIEG Problems

about the hierarchy. Most of the complexity bounds are a direct application of our gadgets presented in the previous section, apart from Lemma 7 that is considerably more involved as an additional binary encoding of multiple weights into a single integer is needed there.

Lemma 4. *The AIEG problem for the interval* $[a, \infty]$ *where* $Q_2 = \emptyset$ *and* $B_2 = 0$ *is NP-complete.*

Proof (sketch). For the lower bound let $\exists \boldsymbol{x} \varphi(\boldsymbol{x})$ be an instance of the NP-complete SAT problem over the vector of variables $\boldsymbol{x} = (x_1, x_2 \dots x_n)$. We construct in polynomial time the AIEG $\mathcal{A}$ given by $\mathcal{G}_\exists(\boldsymbol{x}) \to \mathcal{G}_\varphi$ and fix the budget for *Player 1* to $B_1 = n$. It is now clear that $\exists \boldsymbol{x} \varphi(\boldsymbol{x})$ is true iff *Player 1* is the winner of the AIEG problem $\mathcal{A}$. For the upper bound, an algorithm for solving the problem guesses a *Player 1* investment and solves in polynomial time [3] the interval bound problem of the resulting energy game. □

Lemma 5. *The AIEG problem for the interval* $[a, b]$ *where* $Q_2 = \emptyset$ *and* $B_2 = 0$ *is NP-Hard and in PSPACE.*

Lemma 6. *The AIEG problem for the interval* $[a, \infty]$ *where* $Q_2 = \emptyset$ *and* $B_1 = 0$ *is co-NP-complete.*

Proof (Sketch). For the lower bound let $\forall \boldsymbol{x} \varphi(\boldsymbol{x})$ be an instance of Π_1^P-SAT (co-NP complete problem). We construct in polynomial time the AIEG $\mathcal{A}$ given by $\mathcal{G}_\forall(\boldsymbol{x}) \to \mathcal{G}_\varphi$ and let B_2 be any sufficient budget. It is now easy to see that $\forall \boldsymbol{x} \varphi(\boldsymbol{x})$ is true iff *Player 1* is the winner in the AIEG problem $\mathcal{A}$. An algorithm for solving the problem in co-NP enumerates using universal branching all possible *Player 2* investments and for each investment solves in polynomial time [3] the interval bound problem of the resulting energy game. □

Lemma 7. *The AIEG problem for the interval* $[a, b]$ *where* $Q_2 = \emptyset$ *and* $B_1 = 0$ *is* Π_2^P*-hard and in PSPACE.*

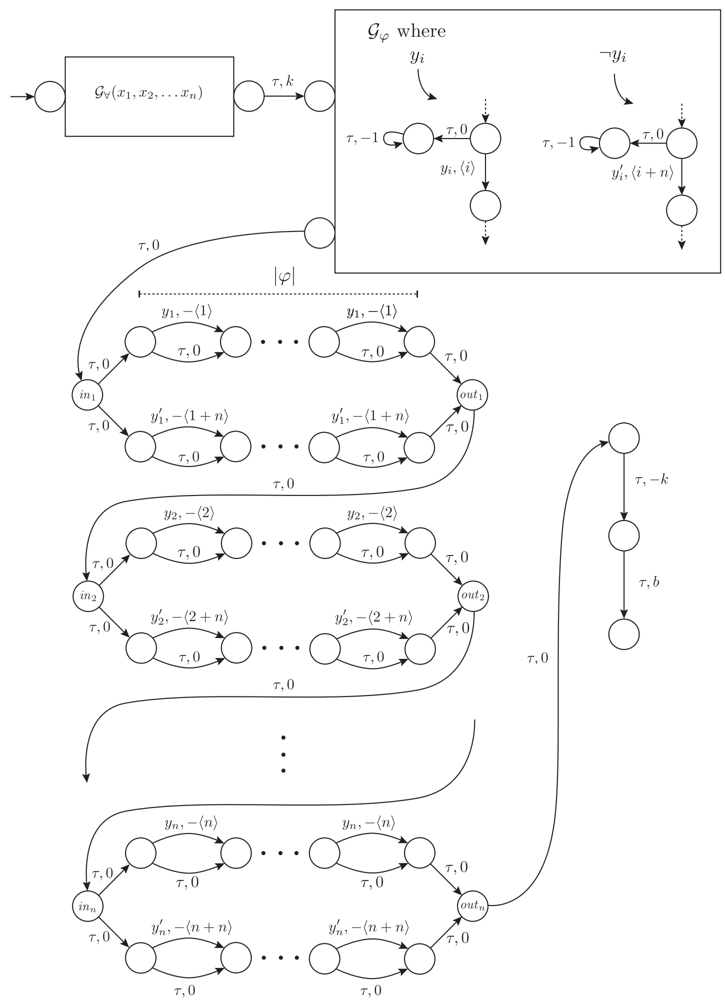


Fig. 5: AIEG used in proof of Lemma 7

Proof (sketch). For the lower bound let $\forall \boldsymbol{x} \exists \boldsymbol{y} \varphi(\boldsymbol{x}, \boldsymbol{y})$ be an instance of Π_2^P-SAT. We construct an AIEG $\mathcal{A}$, illustrated on Figure 5. The first part of the construction is the gadget $\mathcal{G}_\forall(\boldsymbol{x})$ that ensures that *Player 2* is forced to select a valid investment over the variables $\{x_1, x_1', \ldots, x_n, x_n'\}$ (assuming we select the budget B_2 as a sufficient one). It would be intuitive to link this gadget with the gadget $\mathcal{G}_\exists(\boldsymbol{y})$ to force *Player 1* to choose her valid investment over the variables $\{y_1, y_1', \ldots, y_n, y_n'\}$, however, $B_1 = 0$ and *Player 1* can not make any investment. Therefore we need to make an alternative construction. We shall use $2n$ counters in order to record how many times the actions y_i and y_i' were seen while traversing the modified gadget G_φ. The counters are encoded in binary, and the number of bits needed for each counter is given by $c = \lceil \log(|\varphi| + 1) \rceil$ where $|\varphi|$ is the number of literals that appear in the boolean formula φ. We need two counters for each boolean variable y_i such that the counter i counts the number of positive occurrences of the variable y_i and the counter $i+n$ counts the number of negative occurrences. All counters will be stored in a single integer. To ensure that counters do not get 'entangled' by under- or overflow, we add two separator bits 10 between any two neighboring counters, giving us that in total we need $2n(c+2)$ bits. We define $k = \sum_{i=1}^{2n} 2^{i(2+c)}$ as the initial counter value where all counter are 0 and with the separator bits between them. For an example if $n = 2$ and $c = 3$ then k is defined (in binary notation) as follows:

$$k = \overbrace{10\ \underbrace{000}_{\text{counter 4}}\ 10\ \underbrace{000}_{\text{counter 3}}\ 10\ \underbrace{000}_{\text{counter 2}}\ 10\ \underbrace{000}_{\text{counter 1}}}^{\text{20 bits}}.$$

Incrementing and decrementing of counters is done via weights on transitions, therefore we need a way to address each counter. Counter ℓ, $1 \leq \ell \leq 2n$, is addressed by $\langle \ell \rangle$ where $\langle \ell \rangle = 2^{(2+c)(\ell-1)}$. In this way if a transition with weight $\langle \ell \rangle$ is taken, the counter ℓ is incremented by one, and similarly when a transition with weight $-\langle \ell \rangle$ is taken, the counter ℓ is decremented by one. We set $b = 2^{2n(c+2)+1} - 1$ which is the highest possible number on $2n(c+2)$ bits (where all bits are 1) and will consider the resulted energy game in the interval $[0, b]$.

Now we can construct an alternative $\mathcal{G}_\varphi$ which is linked after the $\mathcal{G}_\forall(\boldsymbol{x})$ gadget as illustrated in Figure 5. The alternative $\mathcal{G}_\varphi$ gadget is as defined before, with the exception that every y_i literal gives rise to the $y_i, \langle i \rangle$ transition, and every $\neg y_i$ literal gives gives rise to $y'_i, \langle i+n \rangle$ transition, while assuming that $\{y_1, y'_1, \ldots y_n, y'_n\} \subseteq \Sigma_0$. By taking a path from the start to the end state of this gadget, we will record in the counters how many times each literal has been seen on such a path. The initial $\mathcal{G}_\forall(\boldsymbol{x})$ gadget is linked via a τ, k transition to this new $\mathcal{G}_\varphi$ gadget, ensuring that all counters are initialized to 0 by adding only the separator bits. At the end of the construction, we add a series of gadgets for each variable y_i, allowing *Player 1* to decrement at most $|\varphi|$-many times either the counter i or $n+i$ but not both at the same time. If the assignment recorded in the counters is a valid one, *Player 1* can decrease all counter values to zero, such that after removing the separator bits by the transition $\tau, -k$ we get the accumulated weight 0 and it is possible to add the upper bound b without violating the energy game interval $[0, b]$. It can now be shown that $\forall \boldsymbol{x} \exists \boldsymbol{y} \varphi(\boldsymbol{x}, \boldsymbol{y})$ is true iff *Player 1* is the winner of the AIEG problem $\mathcal{A}$ for the interval $[0, b]$.

For the upper bound, an algorithm for solving the problem in PSPACE enumerates all possible *Player 2* investments and for each investment solves the interval bound problem of the resulting energy game. The interval bound problem in the existential case, $Q_2 = \emptyset$, for an interval $[a, b]$ is in PSPACE, implying that the problem remains in PSPACE. □

Lemma 8. *The AIEG problem for the interval $[a, \infty]$ where $Q_2 = \emptyset$ is Π_2^P-complete.*

Proof (sketch). For the lower bound let $\forall \boldsymbol{x} \exists \boldsymbol{y} \varphi(\boldsymbol{x}, \boldsymbol{y})$ be a Π_2^P-complete instance of QSAT where $\boldsymbol{x} = (x_1, x_2 \ldots x_n)$ and $\boldsymbol{y} = (y_1, y_2 \ldots y_m)$. We construct the AIEG $\mathcal{A}$ given by $\mathcal{G}_\forall(\boldsymbol{x}) \to \mathcal{G}_\exists(\boldsymbol{y}) \to \mathcal{G}_\varphi(\varphi)$ and fix the budget $B_1 = n$ and let B_2 be any sufficient budget for *Player 2*. It is now easy to see that $\forall \boldsymbol{x} \exists \boldsymbol{y} \varphi(\boldsymbol{x}, \boldsymbol{y})$ is true iff *Player 1* is the winner of the AIEG problem $\mathcal{A}$. An algorithm for solving the problem enumerates via universal branching all possible *Player 2* investments and for each such investment guesses a *Player 1* investment and solves in polynomial time the resulting energy game. □

Lemma 9. *The AIEG problem for the interval* $[a, b]$ *where* $Q_2 = \emptyset$ *is* Π_2^P*-hard and in PSPACE.*

Lemma 10. *The AIEG problem for the interval* $[a, \infty]$ *where* $B_2 = 0$ *is NP-complete.*

Proof. The lower bound follows from Lemma 4. An algorithm for solving the problem guesses a *Player 1* investment and solves the interval bound problem of the resulting energy game which is in UP ∩ coUP, implying that the problem is in NP. □

Lemma 11. *The AIEG problem for the interval* $[a, \infty]$ *where* $B_1 = 0$ *is co-NP-complete.*

Proof. The lower bound follows from Lemma 6. An algorithm for solving the problem enumerates via universal branching all possible *Player 2* investments and for each investment solves the interval bound problem of the resulting energy game which is in UP ∩ coUP, implying that the problem is in co-NP. □

Lemma 12. *The AIEG problem for the interval* $[a, \infty]$ *is* Π_2^P*-complete.*

Proof. The lower bound follows from Lemma 8. An algorithm for solving the problem enumerates via universal branching all possible *Player 2* investments and for each such investment guesses a *Player 1* investment and solves the interval bound problem of the resulting energy game. The interval bound problem for an open interval $[a, \infty]$ is in UP ∩ coUP, implying that the problem is in Π_2^P. □

Lemma 13. *The AIEG problem for an interval* $[a, b]$ *is EXPTIME-complete.*

5 Conclusion

We have provided a complexity characterization of action investment energy games. The problem combines the action-investment phase with the energy-game phase and for many cases we proved matching complexity lower and upper bounds. Thanks to the general definition of our gadgets that can be combined using linking into different variants of the game, we were able to give intuitive constructions for most of the complexity results. A notable result is that for the interval problems with lower and upper bound, apart from the case where both budgets are zero, the complexity of the existential case and the general game setting remain the same. The few problems where we did not close the complexity bounds depend on the open problem of the existential energy game in the interval $[a, b]$, which is so far only known to be between NP and PSPACE.

We studied a version of AIEG where *Player 2* chooses his investment before *Player 1*. It is natural to consider also the opposite order in which the investment is established or even a turn-based investment phase. This will be studied in our future work.

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A Proofs from Section 3

A.1 Proof of Lemma 1

Proof. *(a)* Let I_2 be a valid investment for *Player 2*. We evaluate how a play can evolve in the first step from q_{in}. As I_2 is a valid investment, we have that either $x_1 \in I_2$ or $x'_1 \in I_2$. We evaluate the outcome for each case. If $x_1 \in I_2$, *Player 1* has two choices, an x_1 transition *up* or an x_1 transition *right*. If *Player 1* takes the transition *up* she loses, as the play ends in an infinite negative $\tau, -1$ loop, surely violating sooner or later her lower-bound. If *Player 1* takes the transition *right*, the play is one step closer to q_{end}. If on the other hand $x'_1 \in I_2$ the *Player 1* can only take the transition *right* and the play is one step closer to q_{end}. By repeating the arguments, any play can evolve from q_{in} to be either loosing for *Player 1* or to get one step closer to q_{end}. Hence any play from q_{in} is either loosing for *Player 1* or it reaches q_{out}.

To find a strategy for *Player 1* which ensures that any play from q_{in} reaches q_{out}, the *Player 1* will take either the transition x_j or x'_j *right* for $0 < j \leq n$; this is possible as I_2 is valid. Thus play from q_{in} where *Player 1* uses this strategy reaches q_{out}.

(b) Let I_2 be not a valid investment for *Player 2*. This means that there is a smallest i where $\{x_i, x'_i\} \subseteq I_2$ or $x_i, x'_i \notin I_2$. Now we need to find a winning strategy for *Player 1*. The *Player 1* strategy from q_{in} to q_i is to take transitions *right* until q_i is reached; this is possible as either $x_j \in I_2$ or $x'_j \in I_2$ for $0 < j < i$. From q_i there are two cases, if $\{x_i, x'_i\} \subseteq I_2$ or if $x_i, x'_i \notin I_2$. In the first case the strategy is to take two transition *up*, $q_i \xrightarrow{x_i,0} r_i$ followed by $r_i \xrightarrow{x'_i,0} s_i$ to s_i; this is a winning situation for *Player 1* as s_i does not have any successors and the run from q_{in} to s_i is maximal. In the second case *Player 1* wins as q_i does not have any successors at all and the run from q_{in} to q_i is maximal too. Hence any play from q_{in} where *Player 1* uses this strategy is winning for her. □

A.2 Proof of Lemma 2

Proof. *(a)* Let I_1 be a valid investment for *Player 1*. We evaluate how a play can evolve in the first step from q_{in}. As I_1 is a valid investment then either $x_1 \in I_1$ or $x'_1 \in I_1$. If *Player 1* takes one of theses transition *right*, the play one step closer to q_{end}. If *Player 1* takes the τ transition *up*, she loses as the play is stuck in an infinite negative loop. Be repeating the arguments, any play from q_{in} is either loosing for *Player 1* or reaches q_{out}. *Player 1* strategy which ensures that any play from q_{in} reaches q_{out} is clearly to always take a transition to the *right*.

(b) Let I_1 not be a valid investment for *Player 1*. This means that there is a smallest i such that $\{x_i, x'_i\} \subseteq I_1$ or $x_i, x'_i \notin I_1$. *Player 1* has a strategy to take transitions *right* from q_{in} to q_i and avoid loosing before q_i is reached; this is possible as $x_j \in I_1$ or $x'_j \in I_1$ for $1 \leq j < i$. From q_i there are two cases: either $x_i, x'_i \notin I_1$ or $\{x_i, x'_i\} \subseteq I_1$. If $x_i, x'_i \notin I_2$ then *Player 2* wins, as the only transition possible in q_i is to take a transition *up* to r_i where the play is stuck in an infinite negative loop. If $\{x_i, x'_i\} \subseteq I_2$ then as a consequence of the

fixed budget $B_1 = n$ for *Player 1*, there exists a smallest $m > i$, $m \leq n$ where $x_m, x'_m \notin I_1$ and following the argument above *Player 2* wins again. □

A.3 Proof of Lemma 3

Proof. ***(a)*** Let φ be true under the assignment υ and we prove by induction on the structure of $\mathcal{G}_\varphi$ that *Player 1* has a strategy to win any play starting from q_{in} in $\mathcal{G}_\varphi$.

Basis: We have two base cases, $\varphi = x$ and $\varphi = \neg x$. If $\varphi = x$, then $\mathcal{G}_\varphi$ is as in Figure 4a. *Player 1* strategy is to take the $x, 0$ transition; this transition is enabled as φ is true under υ. If $\varphi = \neg x$ then $\mathcal{G}_\varphi$ is as in Figure 4b. *Player 1* strategy is to take the $x', 0$ transition; this transition is enabled as φ is false under υ.

Induction Step: Assume by induction hypothesis (IH) that for $\varphi = F$ *Player 1* has a strategy to win any play on $\mathcal{G}_\varphi$ and reach the end state. We now prove that *Player 1* has a strategy to win any play where $\varphi = F_1 \wedge F_2$ and where $\varphi = F_1 \vee F_2$. Let $\varphi = F_1 \wedge F_2$ and we know that φ is true under υ. This implies that both F_1 and F_2 are true, and we know by the IH that *Player 1* has a strategy to win both F_1 and F_2, therefor he also has a strategy to win when $\varphi = F_1 \wedge F_2$ where the gadgets for F_1 and F_2 are put in series. Let $\varphi = F_1 \vee F_2$ and we know that φ is true under υ. This implies that either F_1 or F_2 is true. We know by the IH that if F_1 is true *Player 1* has a winning strategy for F_1 and if F_2 is true *Player 1* has a winning strategy for F_2. *Player 1* strategy is now to take the transition *Left* $\tau, 0$ if F_1 is true and take *Right* $\tau, 0$ if F_2 is true.

(b) Let φ be false under υ. By similar arguments as above we can prove that any play starting from q_{in} on $\mathcal{G}_\varphi$ is winning for *Player 2*.

□

B Proofs from Section 4

B.1 Proof of Lemma 4

Proof. For the lower bound let $\exists \boldsymbol{x}\varphi(\boldsymbol{x})$ be an instance of the NP-complete SAT problem over the vector of variables $\boldsymbol{x} = (x_1, x_2 \ldots x_n)$. We construct in polynomial time the AIEG $\mathcal{A}$ given by $\mathcal{G}_\exists(\boldsymbol{x}) \rightarrow \mathcal{G}_\varphi$ and fix the budget for *Player 1* to $B_1 = n$. We want to show the following.

(a) If $\exists \boldsymbol{x}\varphi(\boldsymbol{x})$ is true, then *Player 1* wins $\mathcal{A}$.
(b) If $\exists \boldsymbol{x}\varphi(\boldsymbol{x})$ is false, then *Player 2* wins $\mathcal{A}$.

(a) Suppose that $\exists \boldsymbol{x}\varphi(\boldsymbol{x})$ is true. Then there exists an assignment υ such that $\varphi({}^{\upsilon}/_{\boldsymbol{x}})$ is *true*. We now want to show that *Player 1* wins the AIEG $\mathcal{A}$. *Player 1* chooses the valid investment $I_1 = \{x \mid \upsilon(x) = \mathit{true}\} \cup \{x' \mid \upsilon(x) = \mathit{false}\}$. By Lemma 2 (a) we know that *Player 1* has a strategy such that any play from q_{in} reaches q_{out}. If the play reaches q_{out} then by the construction of linking the play continues in $\mathcal{G}_\varphi$. We know that $\varphi({}^{\upsilon}/_{\boldsymbol{x}})$ is *true* therefore by Lemma 3 (a) we know that *Player 1* wins any play starting from q_{in} in $\mathcal{G}_\varphi$. We have now shown that *Player 1* wins any play on $\mathcal{A}$.

(b) Suppose that $\exists \boldsymbol{x}\varphi(\boldsymbol{x})$ is false. We want to show that *Player 2* wins the AIEG $\mathcal{A}$. *Player 1* can pick either an invalid or a valid investment I_1. If I_1 is an invalid investment we know from Lemma 2 (b) that *Player 2* wins any play starting from q_{in}. If I_1 is a valid investment we know from Lemma 2 (a) that any play from q_{in} either reaches q_{out} or is loosing for *Player 1*. If the play reaches q_{out} in $\mathcal{G}_\exists(\boldsymbol{x})$ then by the construction of linking the play continues in the AIEG $\mathcal{G}_\varphi$. We know that $\exists \boldsymbol{x}\varphi(\boldsymbol{x})$ is false. Hence for every assignment υ this implies by Lemma 3 (b) that *Player 2* wins any play starting from q_{in} in $\mathcal{G}_\varphi$. Therefore *Player 2* wins the resulting energy game no matter what investment I_1 *Player 1* chooses. $\square$

B.2 Proof of Lemma 5

Proof. The lower bound follows from Lemma 4. An algorithm for solving the problem in PSPACE enumerates all possible *Player 1* investments and for each investment solves the interval bound problem of the resulting energy game. The interval bound problem in the existential case ($Q_2 = \emptyset$) for the interval $[a, b]$ is in PSPACE, implying that the problem remains in PSPACE. $\square$

B.3 Proof of Lemma 6

Proof. For the lower bound let $\forall \boldsymbol{x}\varphi(\boldsymbol{x})$ be an instance of Π_1^P-SAT (co-NP complete problem). We construct in polynomial time the AIEG $\mathcal{A}$ given by $\mathcal{G}_\forall(\boldsymbol{x}) \to \mathcal{G}_\varphi$ and let B_2 be any sufficient budget. We want to show the following.

(a) If $\forall \boldsymbol{x}\varphi(\boldsymbol{x})$ is true, then *Player 1* wins $\mathcal{A}$.
(b) If $\forall \boldsymbol{x}\varphi(\boldsymbol{x})$ is false, then *Player 2* wins $\mathcal{A}$.

(a) Suppose that $\forall \boldsymbol{x}\varphi(\boldsymbol{x})$ is true. We want to show that *Player 1* wins the AIEG $\mathcal{A}$. *Player 2* can pick either an invalid or a valid investment I_2. If I_2 is an invalid investment we know from Lemma 1 (b) that *Player 1* has a strategy to win any play starting from q_{in} in the AIEG $\mathcal{G}_\forall(\boldsymbol{x})$. If I_2 is a valid investment, we know from Lemma 1 (a) that *Player 1* has a strategy such that any play from q_{in} reaches q_{out}. If the play reaches q_{out} in $\mathcal{G}_\forall(\boldsymbol{x})$ then by the construction of linking the play continues in the AIEG $\mathcal{G}_\varphi$. We know that $\forall \boldsymbol{x}\varphi(\boldsymbol{x})$ is true. Hence for every assignment υ this implies by Lemma 3 (a) that *Player 1* wins any play starting from q_{in} in $\mathcal{G}_\varphi$. Therefore *Player 1* has a strategy to win the resulting energy game no matter what investment I_2 *Player 2* chooses.

(b) Suppose that $\forall \boldsymbol{x}\varphi(\boldsymbol{x})$ is false. Then there exists an assignment υ such that $\varphi({}^{\upsilon}/_{\boldsymbol{x}})$ evaluates to false. We want to show that *Player 2* wins the AIEG $\mathcal{A}$. *Player 2* chooses the valid investment $I_2 = \{x \mid \upsilon(x) = \mathit{true}\} \cup \{x' \mid \upsilon(x) = \mathit{false}\}$. By Lemma 1 (a) we know that any play starting in q_{in} gets to q_{out} in the AIEG $\mathcal{G}_\forall(\boldsymbol{x})$, or *Player 2* wins. If the play reaches q_{out} then by the construction of linking the play continues in $\mathcal{G}_\varphi$. We know that $\varphi({}^{\upsilon}/_{\boldsymbol{x}})$ evaluates to false therefore by Lemma 3 (b) we know that *Player 2* wins any play starting from q_{in} in $\mathcal{G}_\varphi$. We have now shown that *Player 2* wins any play on $\mathcal{A}$. $\square$

B.4 Proof of Lemma 7

Proof. Continuation of Proof of Lemma 7.

We want to show the following.

(a) If $\forall \boldsymbol{x} \exists \boldsymbol{y} \varphi(\boldsymbol{x}, \boldsymbol{y})$ is true, then *Player 1* wins $\mathcal{A}$ for the interval $[0, b]$.
(b) If $\forall \boldsymbol{x} \exists \boldsymbol{y} \varphi(\boldsymbol{x}, \boldsymbol{y}) 1$ is false, then *Player 2* wins $\mathcal{A}$ for the interval $[0, b]$.

(a) Suppose that $\forall \boldsymbol{x} \exists \boldsymbol{y} \varphi(\boldsymbol{x}, \boldsymbol{y})$ is true. Then for any assignment υ of $\boldsymbol{x}$ there exists an assignment υ' of $\boldsymbol{y}$ such that $\varphi({}^{\upsilon}/_{\boldsymbol{x}}, {}^{\upsilon'}/_{\boldsymbol{y}})$ evaluates to true. We want to show that *Player 1* wins the AIEG $\mathcal{A}$ for the interval $[0, b]$. *Player 2* can pick either an invalid or a valid investment I_2. If I_2 is an invalid investment we know from Lemma 1 (b) that *Player 1* has a strategy to win any play starting from q_{in} in the AIEG $\mathcal{G}_{\forall}(\boldsymbol{x})$. If I_2 is a valid investment we know from Lemma 1 (a) that *Player 1* has a strategy such that any play from q_{in} reaches q_{out}. If the play reaches q_{out} in $\mathcal{G}_{\forall}(\boldsymbol{x})$ then by the construction of linking the play continues in the alternative $\mathcal{G}_{\varphi}$ construction.

We know that for the chosen assignment υ there is an assignment υ' such that $\varphi({}^{\upsilon}/_{\boldsymbol{x}}, {}^{\upsilon'}/_{\boldsymbol{y}})$ evaluates to *true*. Also $\{y_1, \dots y_n, y'_1 \dots y'_n\} \subseteq \Sigma_0$ so all y_i and y'_i actions are present in $\mathcal{G}_{\varphi}$. Let $I_1 = \{y \mid \upsilon'(y) = \mathit{true}\} \cup \{y' \mid \upsilon'(y) = \mathit{false}\}$. The set I_0 is not an investment the budget of *Player 1* is zero, but the set will guide *Player 1* when traversing the gadget $\mathcal{G}_{\varphi}$. By Lemma 3 (a) and the fact how I_1 was constructed there is now a strategy for *Player 1* to move from the start to the end node of $\mathcal{G}_{\varphi}$ such that it uses only the edges under actions from I_1 and I_2. Moreover, every time an edge under y_i or y'_i is taken, the corresponding counter i or $i + n$ is increased by one. It is also clear that each counter can be increased at most $|\varphi|$-many times. Because of the construction above, we know that after traversing $\mathcal{G}_{\varphi}$, either the counter i or $i + n$ will be empty for each i.

Now the play passes through the $\tau, 0$ transition before it continues from the state in_1, where *Player 1* have the choice for each boolean variable y_i to take either the *upper* or *lower* $\tau, 0$ transition. The *Player 1* strategy is as follows: if $\upsilon'(y_i) = \mathit{true}$ then she takes the *upper* $\tau, 0$ and if $\upsilon'(y_1) = \mathit{false}$ she takes the *lower* $\tau, 0$ transition. After that *Player 1* has the choice to decrement at most $|\varphi|$ times the counter i or $i + n$, making sure that at the end of this step both counters end up zero. This repeats for all i until we reach out_n and the accumulated weight is exactly k (all counters are empty and only the separating bits are present). Now we can subtract k and add b and still stay within the interval $[0, b]$. As this is a maximum run, *Player 1* wins.

(b) Suppose that $\forall \boldsymbol{x} \exists \boldsymbol{y} \varphi(\boldsymbol{x}, \boldsymbol{y})$ is false. Then there exists an assignment υ of $\boldsymbol{x}$ such that for any assignment υ' of $\boldsymbol{y}$ the formula $\varphi({}^{\upsilon}/_{\boldsymbol{x}}, {}^{\upsilon'}/_{\boldsymbol{y}}) = 1$ evaluates to false. We want to show that *Player 2* wins the AIEG $\mathcal{A}$ for the interval $[0, b]$. *Player 2* first choose the valid investment $I_2 = \{x \mid \upsilon(x) = \mathit{true}\} \cup \{x' \mid \upsilon(x) = \mathit{false}\}$. Then we know from Lemma 1 (a) that any play starting from q_{in} is either loosing for *Player 1* or reaches q_{out}. If the play reaches q_{out} in $\mathcal{G}_{\forall}(\boldsymbol{x})$ then by the construction of linking the play continues in the alternative $\mathcal{G}_{\varphi}$ gadget.

Now *Player 1* can traverse the gadget $\mathcal{G}_{\varphi}$ but because there is no assignment of the variables $y_1, \dots, y_n$ where the formula φ is true, surely there will be some variable y_i such that both the counter i and $i+n$ is positive (and no more that $|\varphi|$). After traversing the rest of the gadget and reaching q_{out}, it is clear that either the counter i and $i+n$ will not change its value (due to the fact that incrementing/decrementing a counter at most $|\varphi|$ many times cannot influence the values of the neighboring counters). This means that after subtracting k and adding b, the accumulated weight exceeds the upper bound b and *Player 2* wins.

For the upper bound, an algorithm for solving the problem enumerates all possible *Player 2* investments and for each investment solves the interval bound problem of the resulting energy game. The interval bound problem in the existential case, $Q_2 = \emptyset$ for an interval $[a,b]$ is in PSPACE, implying that the problem is in PSPACE. □

B.5 Proof of Lemma 8

Proof. For the lower bound let $\forall \boldsymbol{x} \exists \boldsymbol{y} \varphi(\boldsymbol{x}, \boldsymbol{y})$ be a Π_2^P-complete instance of QSAT where $\boldsymbol{x} = (x_1, x_2 \dots x_n)$ and $\boldsymbol{y} = (y_1, y_2 \dots y_m)$. We construct the AIEG $\mathcal{A}$ given by $\mathcal{G}_\forall(\boldsymbol{x}) \to \mathcal{G}_\exists(\boldsymbol{y}) \to \mathcal{G}_\varphi(\varphi)$ and fix the budget $B_1 = n$ and let B_2 be any sufficient budget for *Player 2*.

We want to show the following.

(a) If $\forall \boldsymbol{x} \exists \boldsymbol{y} \varphi(\boldsymbol{x}, \boldsymbol{y})$ is true, then *Player 1* wins $\mathcal{A}$.
(b) If $\forall \boldsymbol{x} \exists \boldsymbol{y} \varphi(\boldsymbol{x}, \boldsymbol{y})$ is false, then *Player 2* wins $\mathcal{A}$.

(a) Suppose that $\forall \boldsymbol{x} \exists \boldsymbol{y} \varphi(\boldsymbol{x}, \boldsymbol{y})$ is true. Then there for any an assignment υ of $\boldsymbol{x}$ exists an assignment υ' of $\boldsymbol{y}$ such that $\varphi(^{\upsilon}/_{\boldsymbol{x}}, ^{\upsilon'}/_{\boldsymbol{y}})$ evaluates to true. We now want to show that *Player 1* wins the AIEG $\mathcal{A}$. *Player 2* can pick either an invalid or a valid investment I_2. If I_2 is an invalid investment we know from Lemma 1 (b) that *Player 1* has a strategy to win any play starting from q_{in} in the AIEG $\mathcal{G}_\forall(\boldsymbol{x})$. If I_2 is a valid investment we know from Lemma 1 (a) that *Player 1* has a strategy such that any play from q_{in} reaches q_{out}. If the play reaches q_{out} in $\mathcal{G}_\forall(\boldsymbol{x})$ then by the construction of linking the play continues in the AIEG $\mathcal{G}_\exists(\boldsymbol{y})$. *Player 1* chooses the valid investment $I_1 = \{y \mid \upsilon'(y) = \mathit{true}\} \cup \{y' \mid \upsilon'(y) = \mathit{false}\}$. By Lemma 2 (a) we know that *Player 1* has a strategy such that any play from q_{in} reaches q_{out}. If the play reaches q_{out} then by the construction of linking the play continues in $\mathcal{G}_\varphi$. We know that $\varphi(^{\upsilon}/_{\boldsymbol{x}}, ^{\upsilon'}/_{\boldsymbol{y}})$ evaluates to true, therefore by Lemma 3 (a) we know that *Player 1* wins any play starting from q_{in} in $\mathcal{G}_\varphi$. We have so shown that *Player 1* wins the AIEG $\mathcal{A}$.

(b) Suppose that $\forall \boldsymbol{x} \exists \boldsymbol{y} \varphi(\boldsymbol{x}, \boldsymbol{y})$ is false. Then there exists an assignment υ of $\boldsymbol{x}$ such that for any assignment υ' of $\boldsymbol{y}$ is $\varphi(^{\upsilon}/_{\boldsymbol{x}}, ^{\upsilon'}/_{\boldsymbol{y}})$ false. We now want to show that *Player 2* wins the AIEG $\mathcal{A}$. *Player 2* chooses the valid investment $I_2 = \{x \mid \upsilon(x) = true\} \cup \{x' \mid \upsilon(x) = false\}$. Then we know from Lemma 1 (a) that any play starting from q_{in} is either loosing for *Player 1* or reaches q_{out}. If the play reaches q_{out} in $\mathcal{G}_\forall(\boldsymbol{x})$ then by the construction of linking the play continues in the AIEG $\mathcal{G}_\exists(\boldsymbol{y})$. *Player 1* can pick either an invalid or a valid investment I_1. If I_1 is an invalid investment we know from Lemma 2 (b) that *Player 2* wins any play starting from q_{in}. If I_1 is a valid investment we know from Lemma 2 (a) that any play starting from q_{in} is either winning for *Player 2* or reaches q_{out}. If the play reaches q_{out} in $\mathcal{G}_{\exists}(\boldsymbol{x})$ then by the construction of linking the play continues in the AIEG $\mathcal{G}_{\varphi}$. We know that $\varphi({}^{\upsilon}/_{\boldsymbol{x}}, {}^{\upsilon'}/_{\boldsymbol{x}})$ evaluates to false for all υ'. This implies by Lemma 3 (b) that *Player 2* wins any play starting from q_{in} in $\mathcal{G}_{\varphi}$. We have now shown that *Player 2* wins the AIEG $\mathcal{A}$. □

B.6 Proof of Lemma 9

Proof. The lower bound follows from Lemma 8. An algorithm for solving the problem enumerates for all possible *Player 2* investments all possible *Player 1* investments and solves in polynomial space the interval bound problem of the resulting energy game. □

B.7 Proof of Lemma 13

Proof. EXPTIME-hardness follows from the complexity of the AIEG problem for the interval $[a, b]$ where $B_1 = 0$ and $B_2 = 0$. An algorithm for solving the problem enumerates all investment combinations and solves in EXPTIME the interval bound problem. □