Tradeoffs in Infinite Games

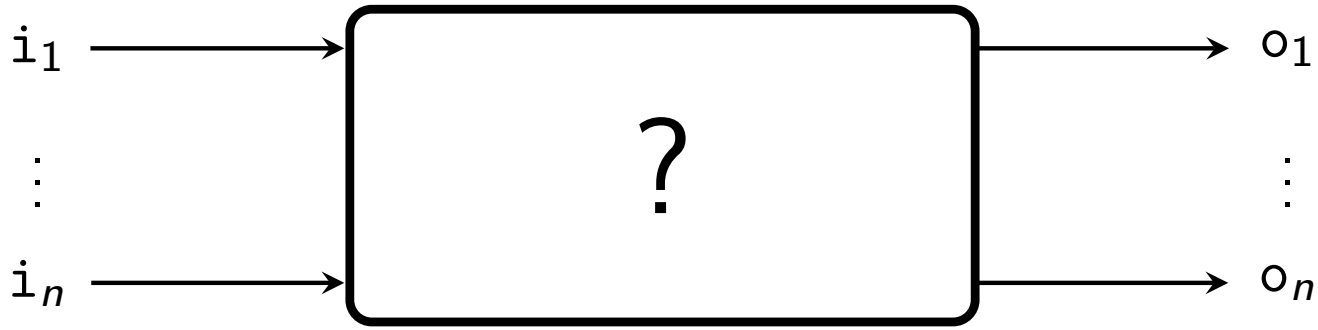
Martin Zimmermann

Saarland University

May 15th, 2018

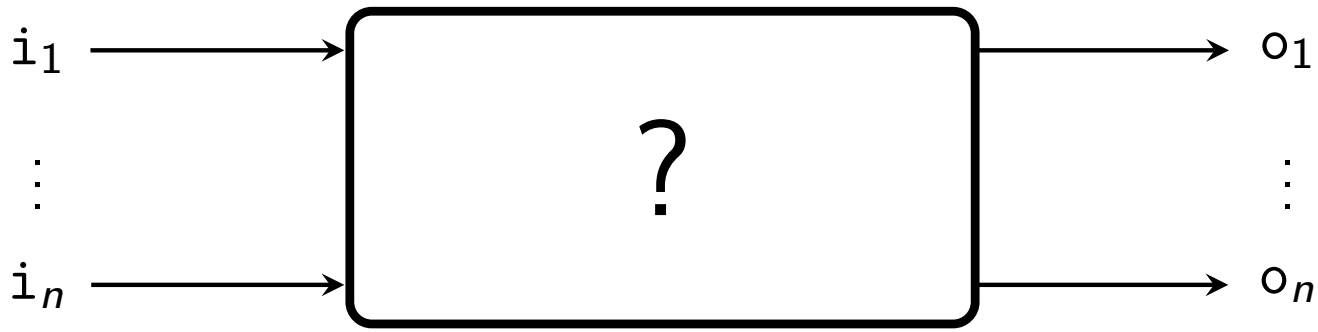
Scientific Talk
in the Habilitation Process

Church's Synthesis Problem



Church 1957: Given a specification on the input/output behavior of a circuit (in some suitable logical language), decide whether such a circuit exists, and, if yes, compute one.

Church's Synthesis Problem



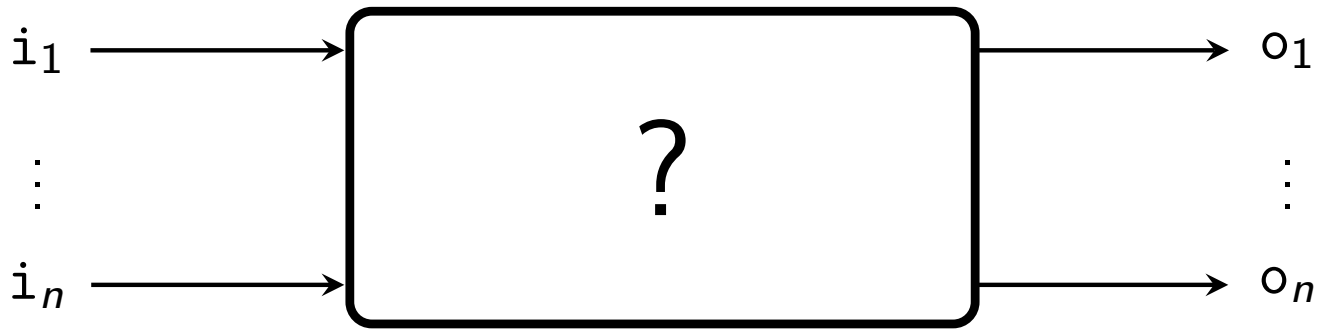
Example

Interpret input $\mathtt{i}_j = 1$ as client j requesting a shared resource and output $\mathtt{o}_j = 1$ as the corresponding grant to client j.

Typical properties:

1. If there are infinitely many requests of client j, then also infinitely many grants for client j.
2. At most one grant at a time (mutual exclusion).
3. No spurious grants.

Church's Synthesis Problem



Solved by Büchi & Landweber in 1969.

Insight: Problem can be expressed as two-player game of infinite duration between the environment (producing inputs) and the circuit (producing outputs).

Back to the Example

Consider the one-client case!

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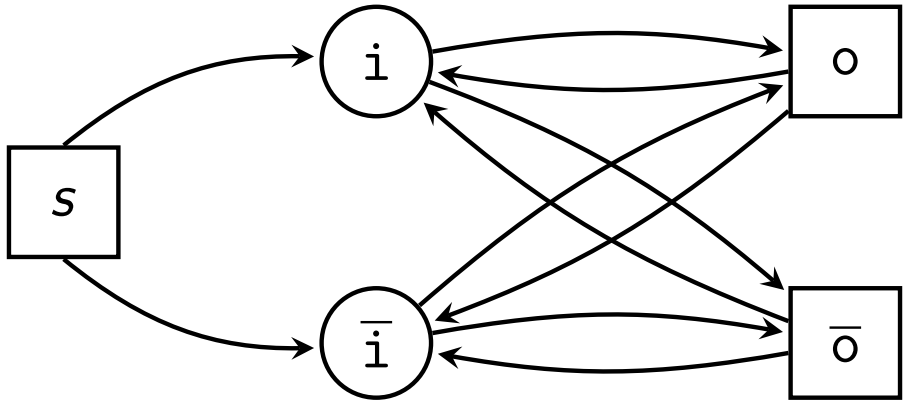

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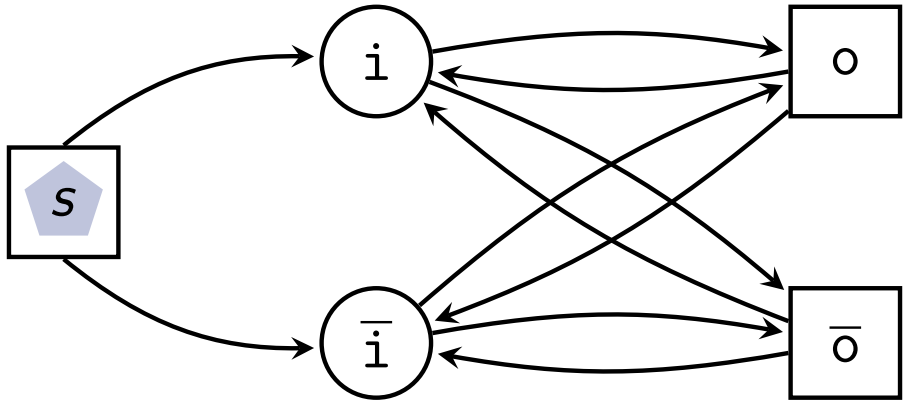

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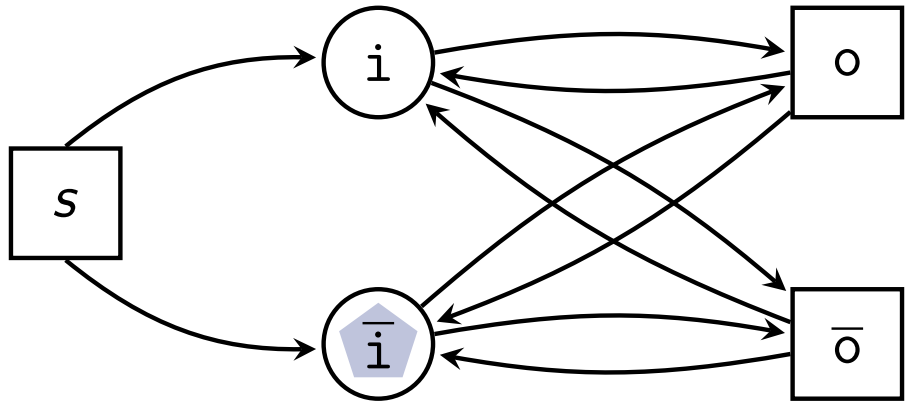

Input: 0
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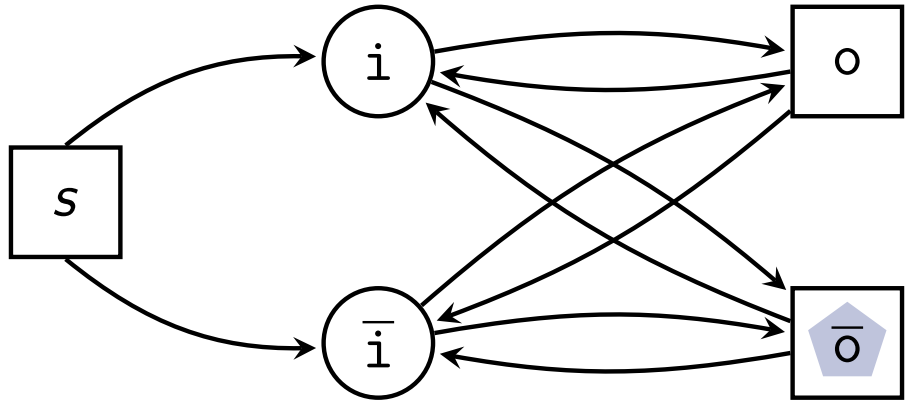

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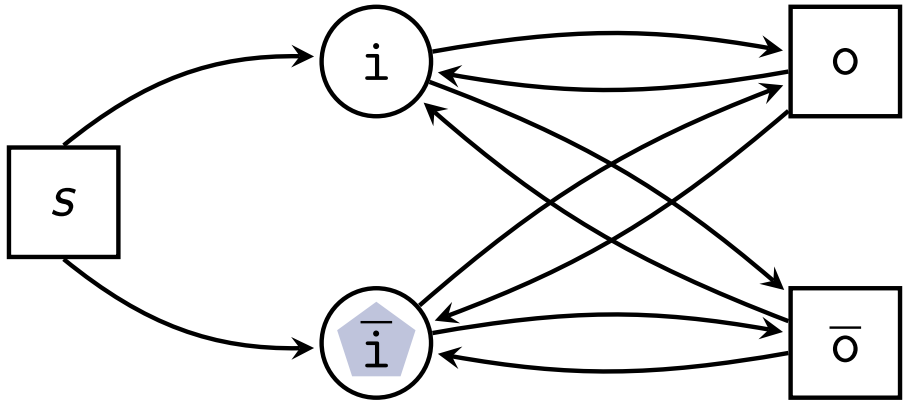

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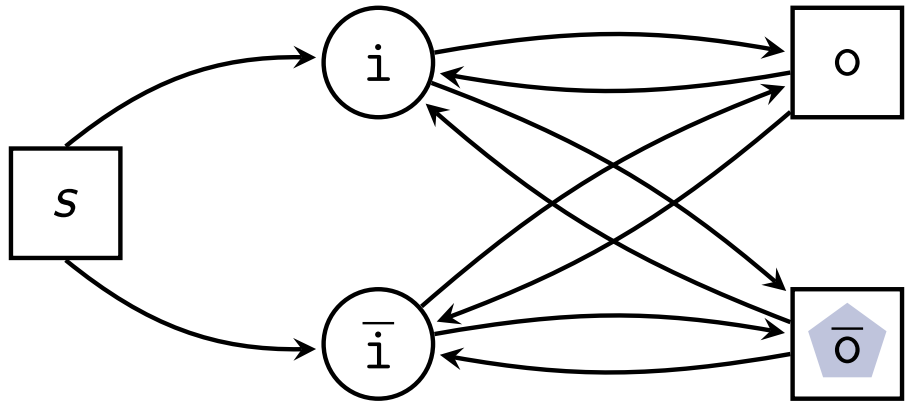

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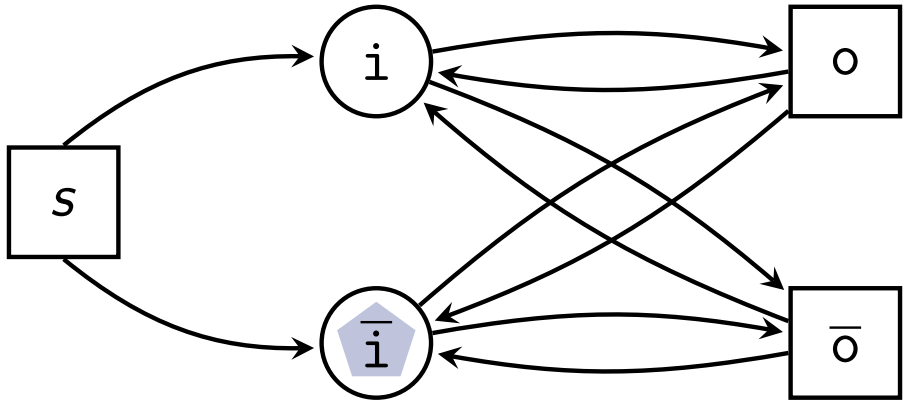

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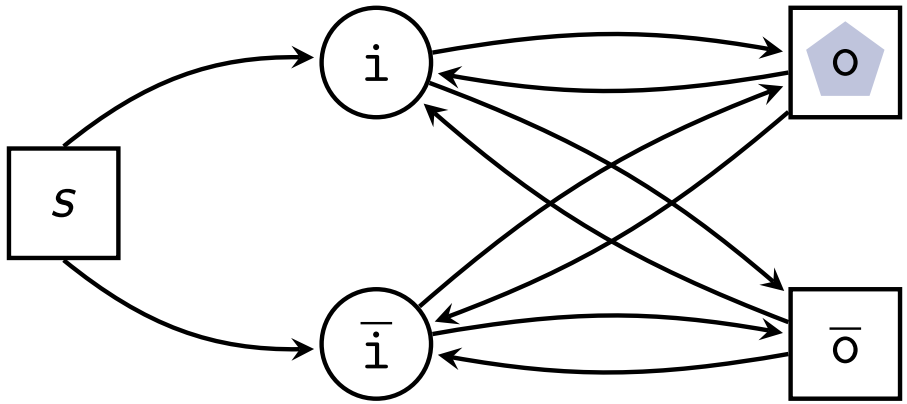

Input: 0 0 0
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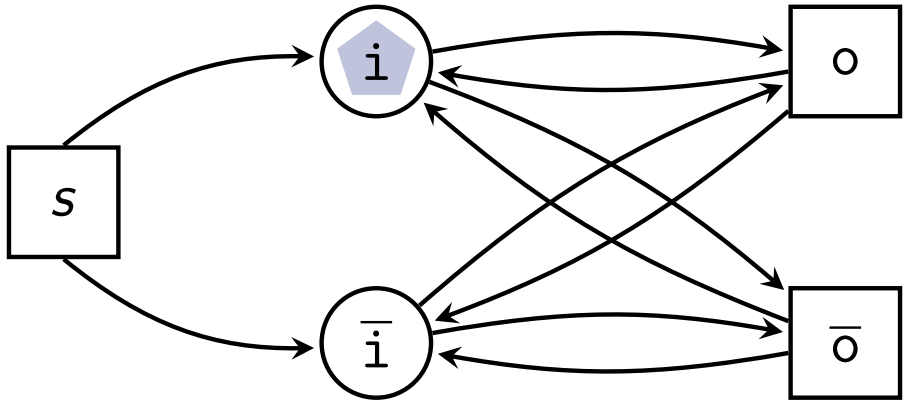

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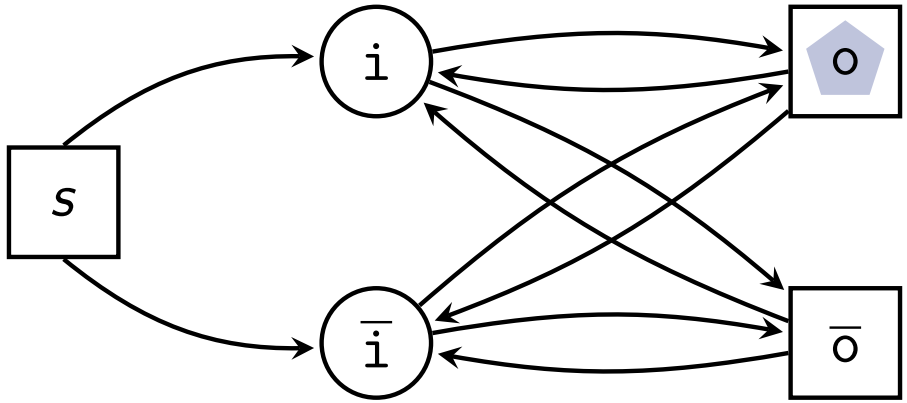

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Output:	0	0	1	1

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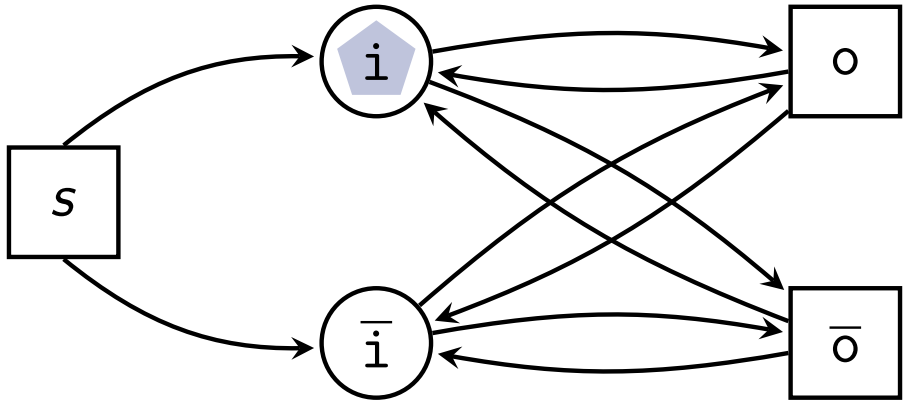

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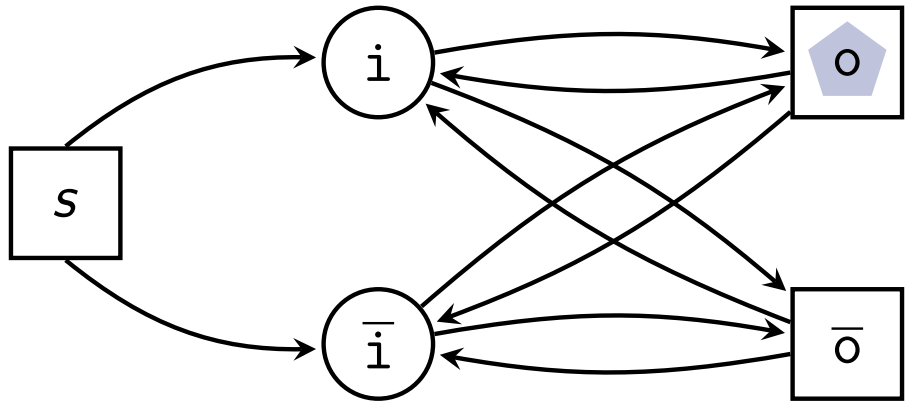

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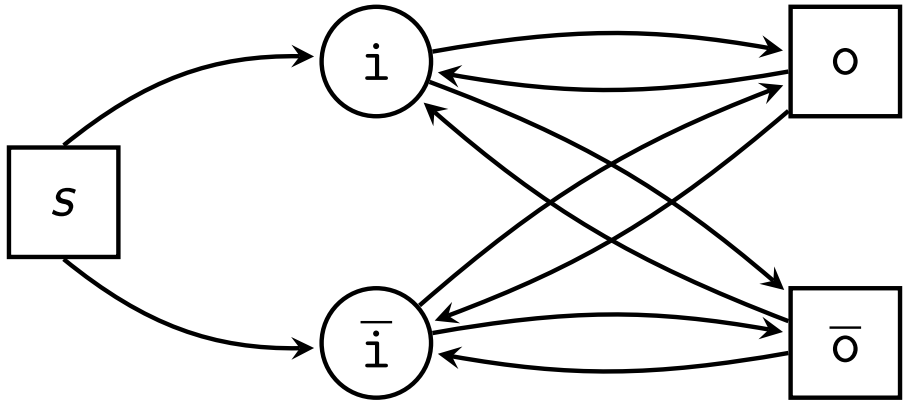

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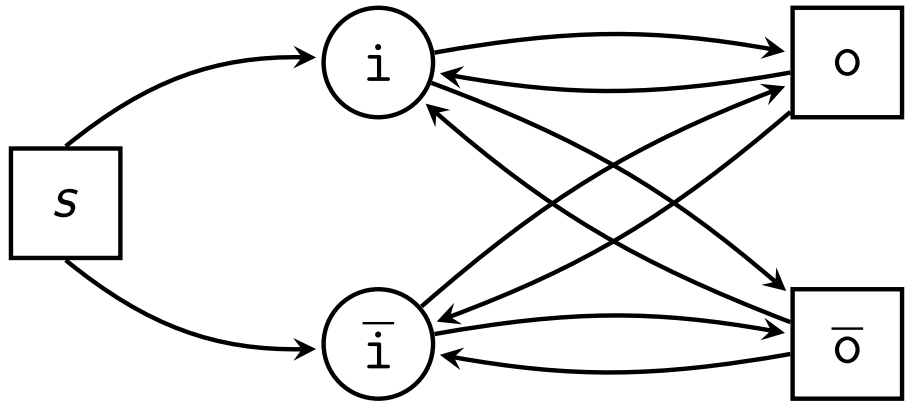

Winning plays for circuit player have to satisfy

1. if i is visited infinitely often, then o as well, and
2. if o is visited, then it has not been visited since the last visit of i.

This requires an expressive specification language, e.g., Linear Temporal Logic (LTL).

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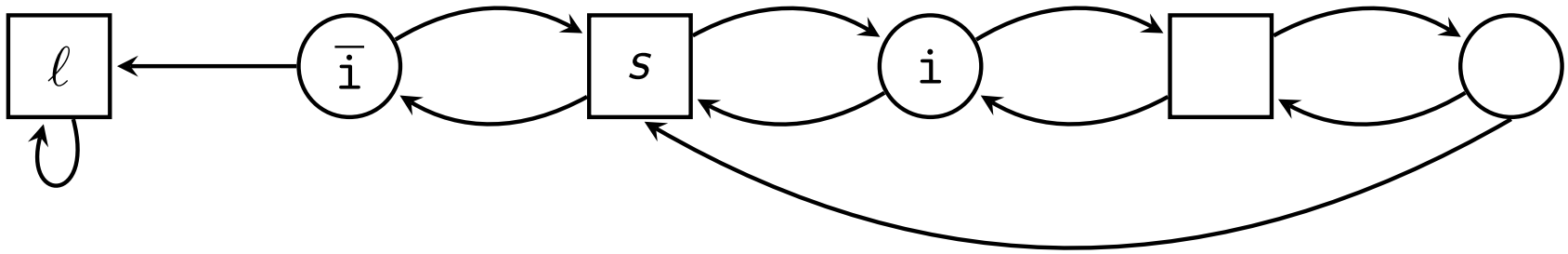

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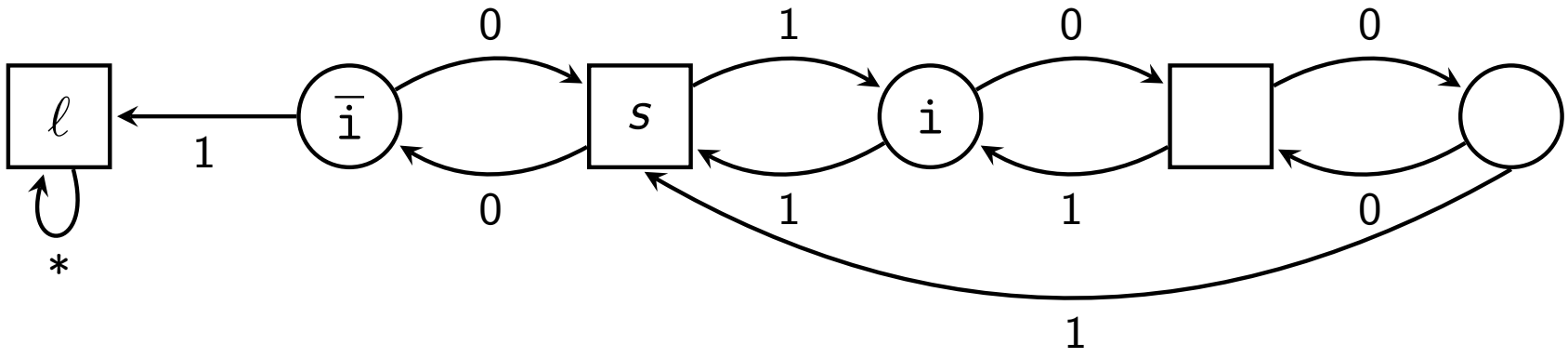

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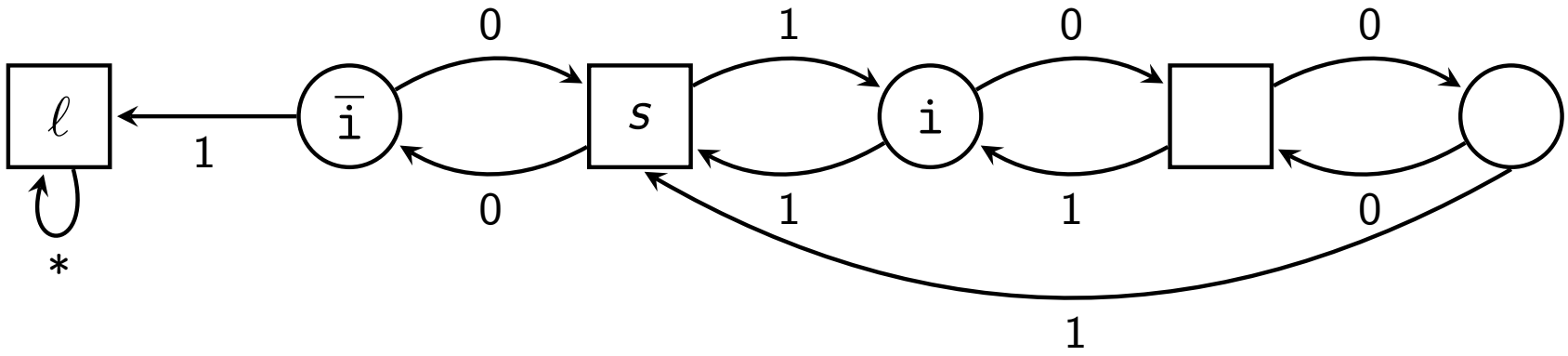

Now, the winning plays for the circuit player have to satisfy

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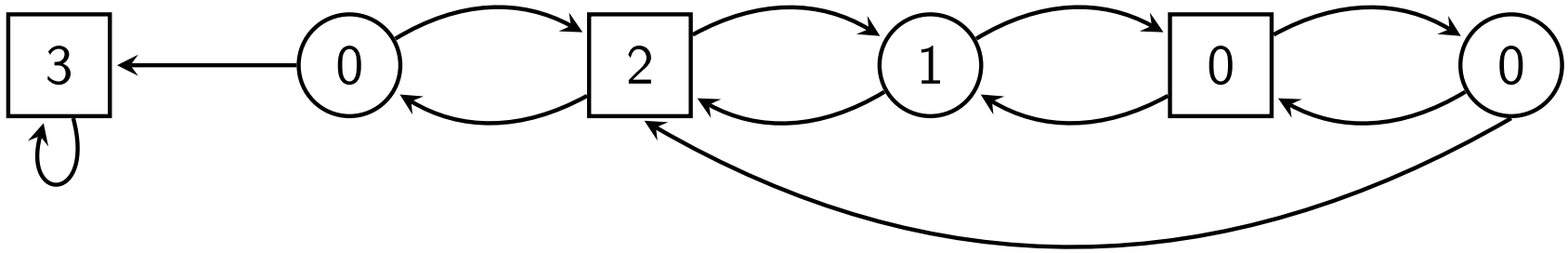


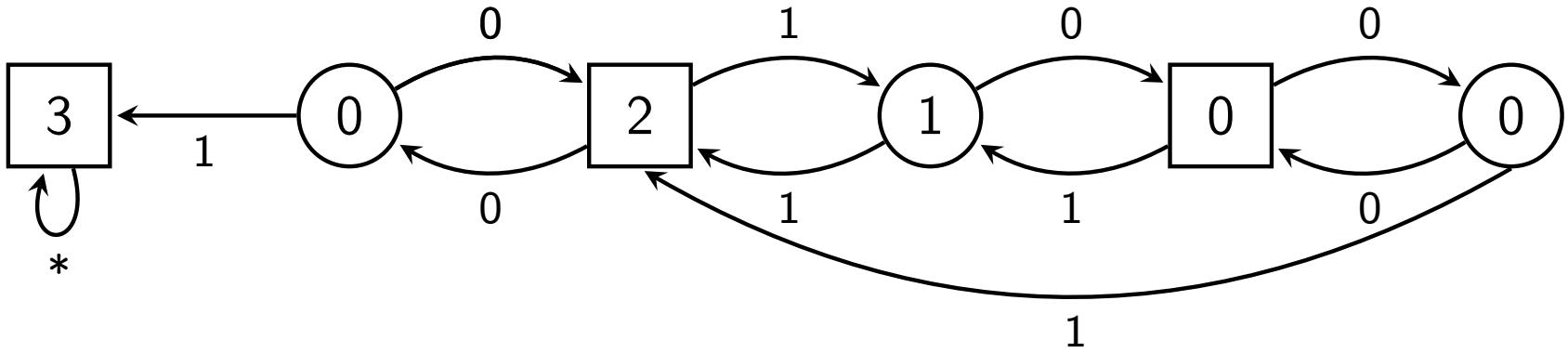
Now, the winning plays for the circuit player have to satisfy

1. if `i` is visited infinitely often, then s as well, and
2. ℓ is never visited.

Equivalently: color the vertices by natural numbers as above and require that almost all odd colors are followed by a larger even one.

This is the classical parity condition for ω-automata.

Büchi-Landweber in a Nutshell



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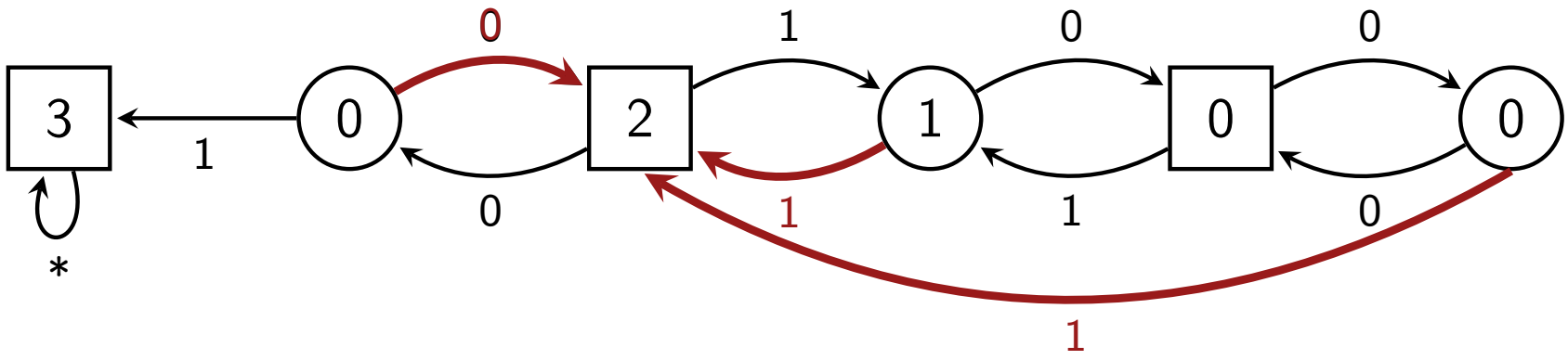

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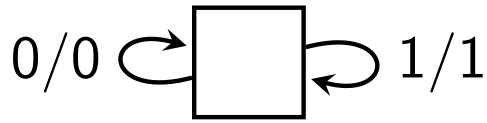

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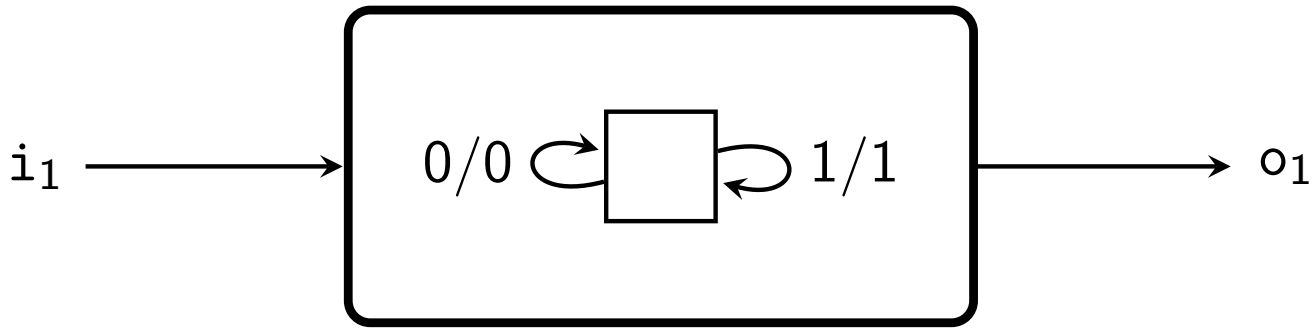

- Player 0 has a memoryless winning strategy,
- which can be turned into an automaton with output,
- which can be turned into a circuit satisfying the specification.

Not Everyone is Equal

Quality of winning strategies is measured in multiple dimensions:

1. Memory requirements
2. Degree of satisfaction of a (quantitative) winning condition
3. Computational complexity of computing such a strategy
4. Use of lookahead
5. Use of randomization
6. Informedness
7. Robustness
8. Non-determinism

In previous work, each dimension was studied in isolation.

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Goal of this thesis:
Understand the tradeoffs between (some of) these dimensions.

Overview

Typical questions we answered in this thesis:

- What is the price of optimality?
  - Is it harder to compute an optimal winning strategy in a quantitative game than an arbitrary winning strategy?
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  - Does optimality increase the memory requirements of winning strategies?
- Can the expressiveness of LTL be increased without increasing the complexity of solving games?
- How does the addition of lookahead change the characteristics of games?
  - Does lookahead increase the complexity of solving games?
  - Does lookahead allow to improve the quality of strategies in quantitative games?

Outline

1. **Playing Optimally in Variations of Parity Games**
2. **Playing (Approximatively) Optimally in LTL Games**
3. **More Tradeoffs**
   - Lookahead vs. Quality
   - Lookahead vs. Memory
   - Expressiveness vs. Complexity
   - Expressiveness vs. Memory
4. **Conclusion**

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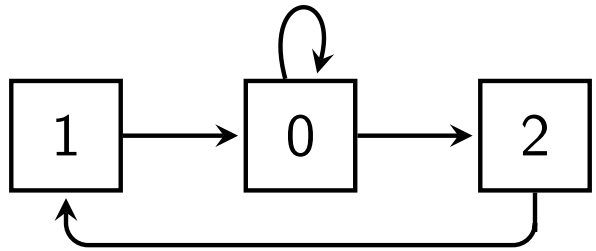

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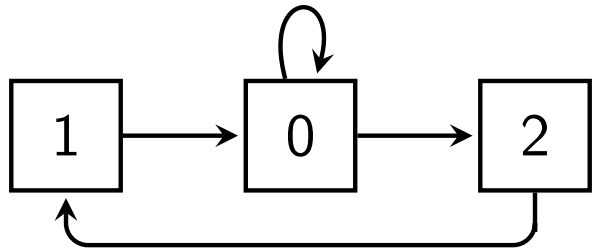

$1 \longrightarrow 0 \longrightarrow 2$

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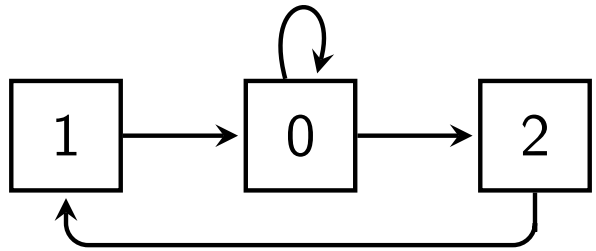
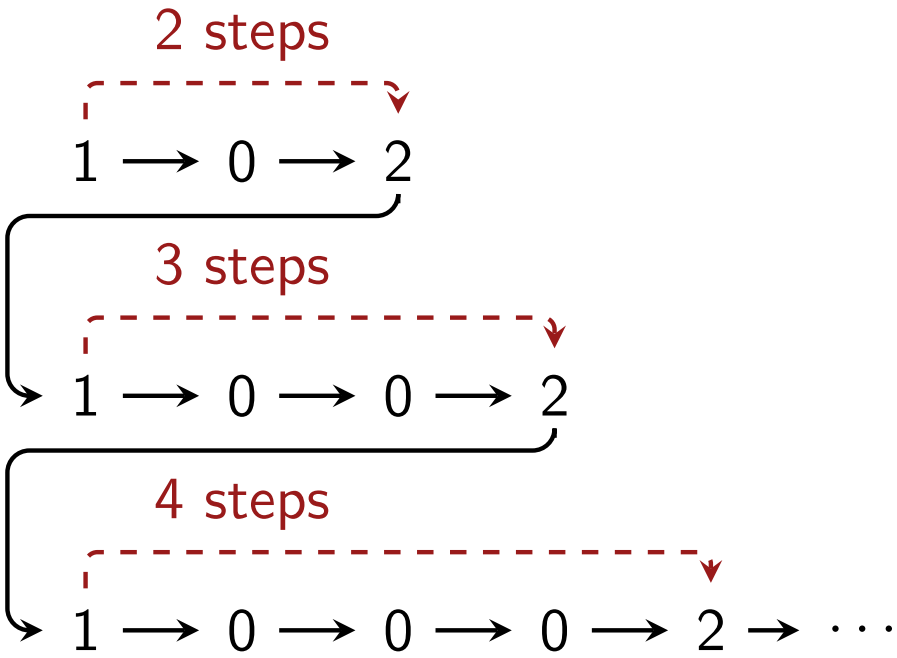
$1 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 2 \rightarrow \cdots$

A Parity Game





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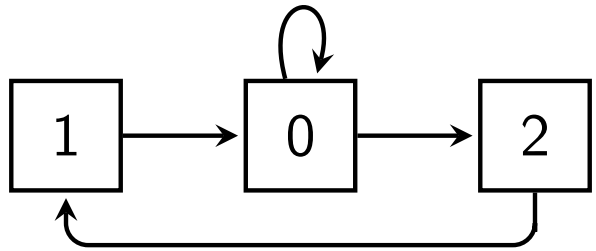

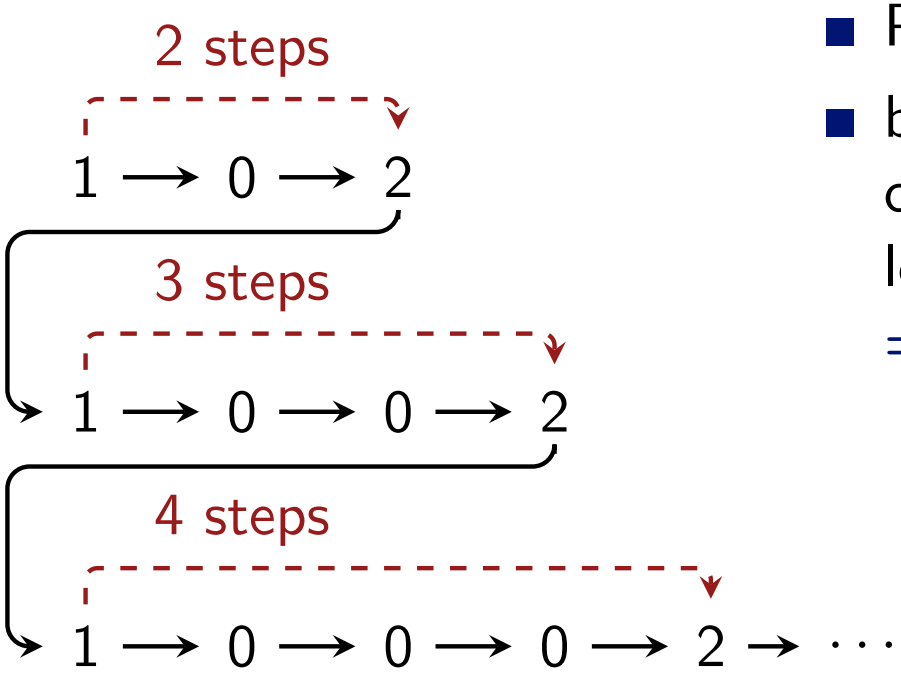

- Player 0 wins from every vertex,
- but Player 1 can delay between color 1 and color 2 longer and longer.
  $\Rightarrow$ undesired behavior.

Parity Games go Quantitative

During the last two decades, various quantitative variants of parity games have been introduced:

- Mean-payoff parity games **[CHJ05]**
- Finitary parity games **[CH06]**
- Energy parity games **[CD11]**

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- Window parity games **[BHR16]**

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Finitary Parity Games

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Condition	Complexity	Memory Pl. 0	Memory Pl. 1
Parity	quasi-poly	Memoryless	Memoryless
Finitary Parity	PTime	Memoryless	Infinite

Boundedness vs. Optimization

The bound b in the definition of finitary parity games is existentially quantified (and may depend on the play).

Corollary

If Player 0 *wins a finitary parity game* $\mathcal{G}$*, then a uniform bound* $b \leq |\mathcal{G}|$ *suffices.*

A trivial example shows that the upper bound $|\mathcal{G}|$ is tight.

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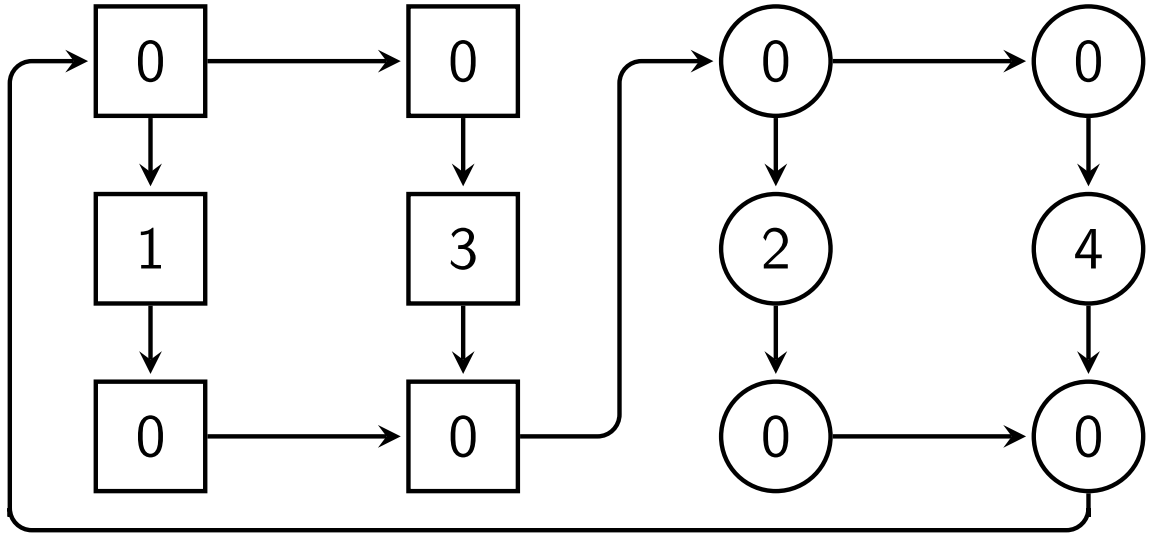
Questions

1. Does Player 0 need memory to achieve the optimal bound?
2. Is it harder to compute the optimal bound than checking whether a bound exists?

Chatterjee & Fijalkow



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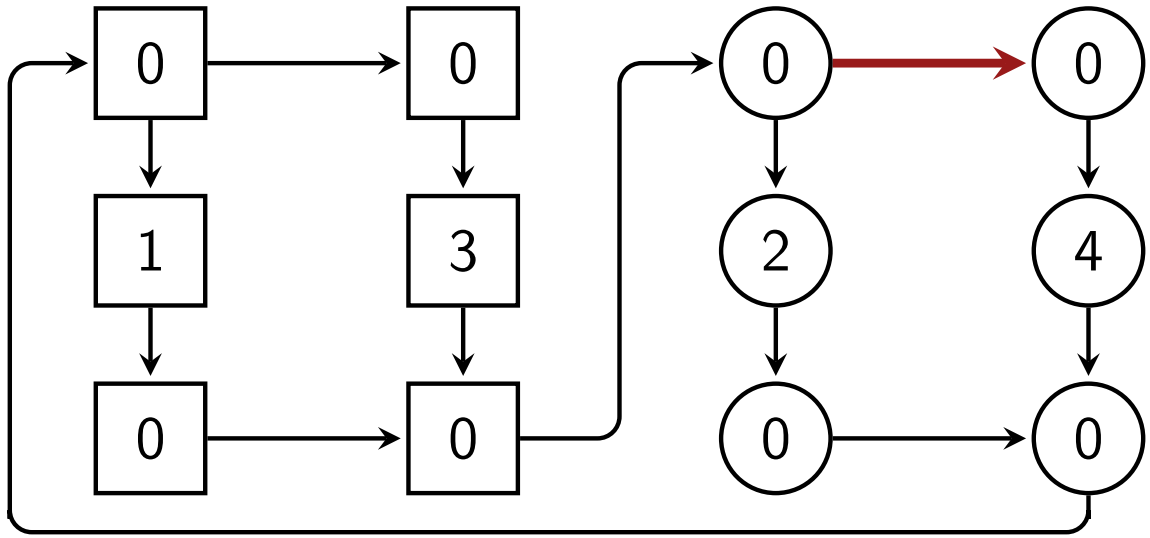

- Player 0 has a unique memoryless winning strategy, which achieves the bound five: from the 1 it takes five steps to the 4.

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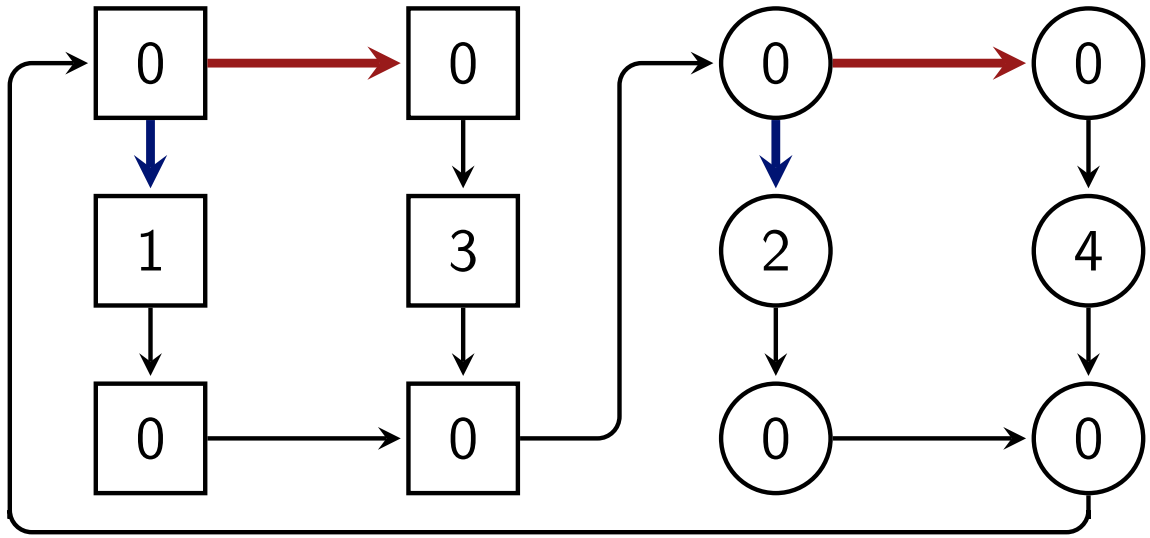

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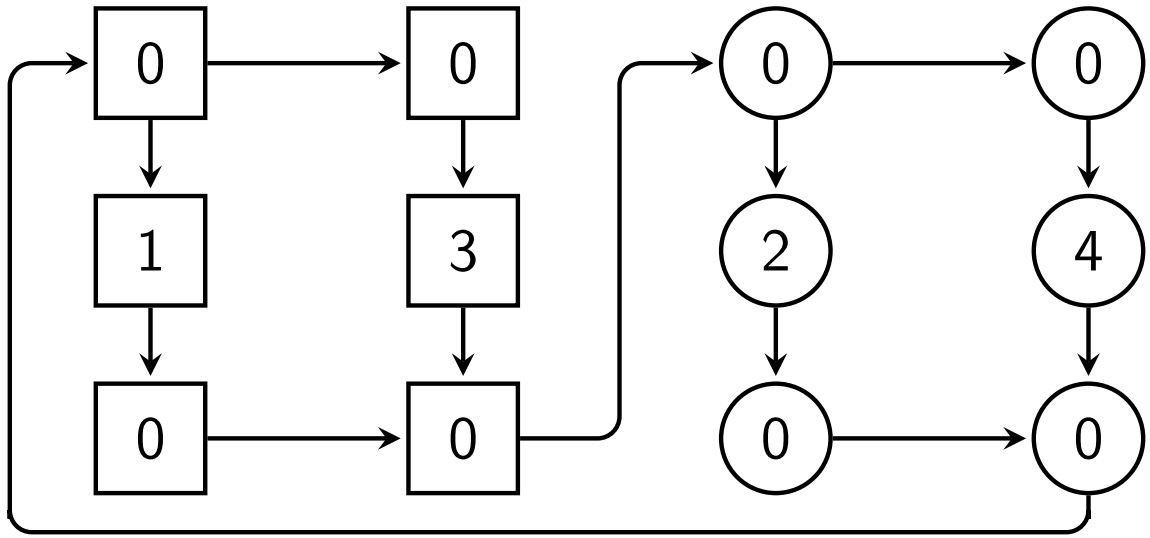
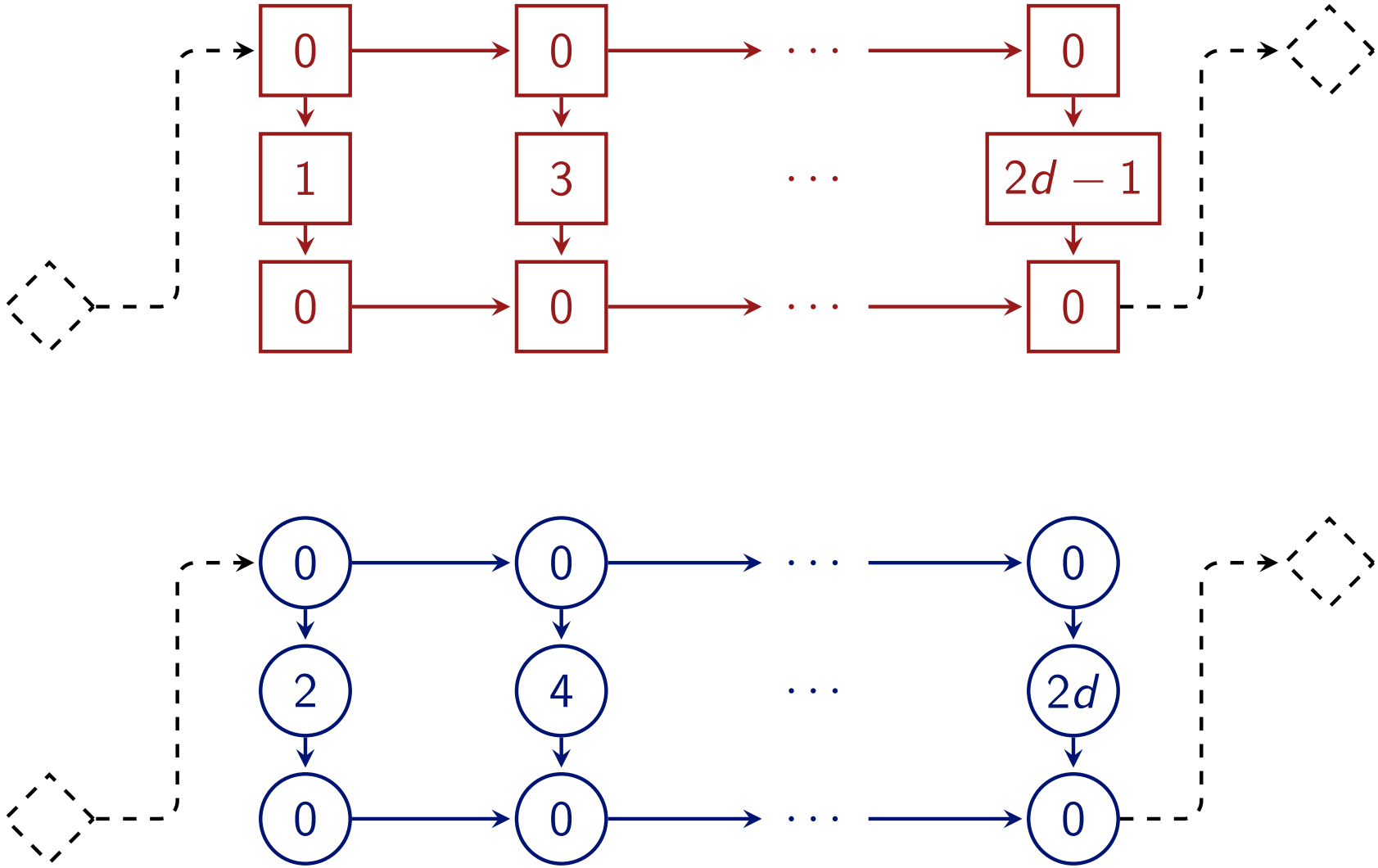


- Player 0 has a unique memoryless winning strategy, which achieves the bound five: from the 1 it takes five steps to the 4.
- With two memory states, she can achieve the bound four: "answer" a 1 by a 2 and a 3 by a 4.
- It is trivial to extend this example to d odd colors and d even colors requiring d memory states to play optimally.

$\Rightarrow$ In general, playing optimally requires memory; but how much?

Memory Requirements



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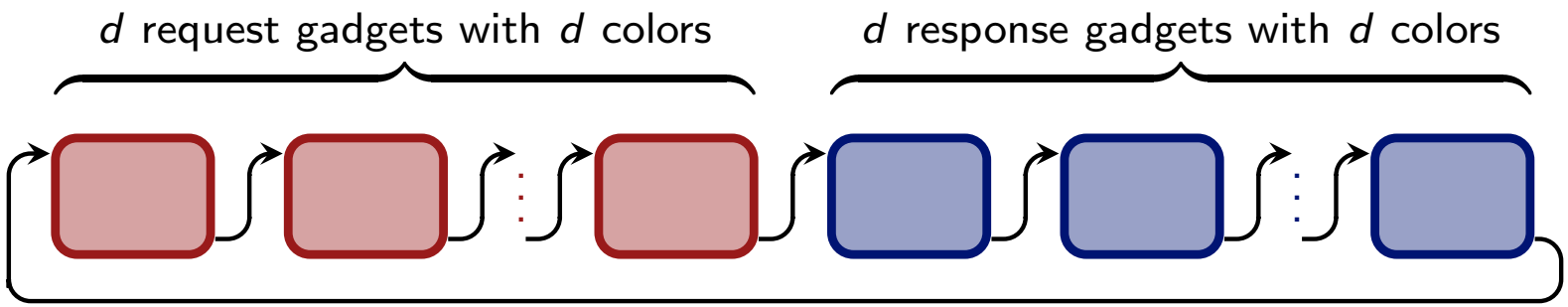

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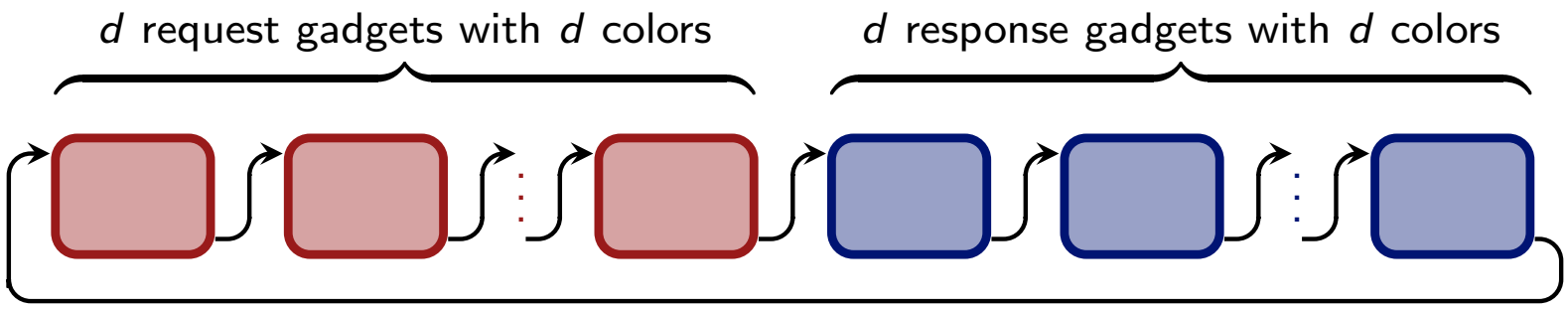

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  $\Rightarrow$ requires exponential memory (in d).

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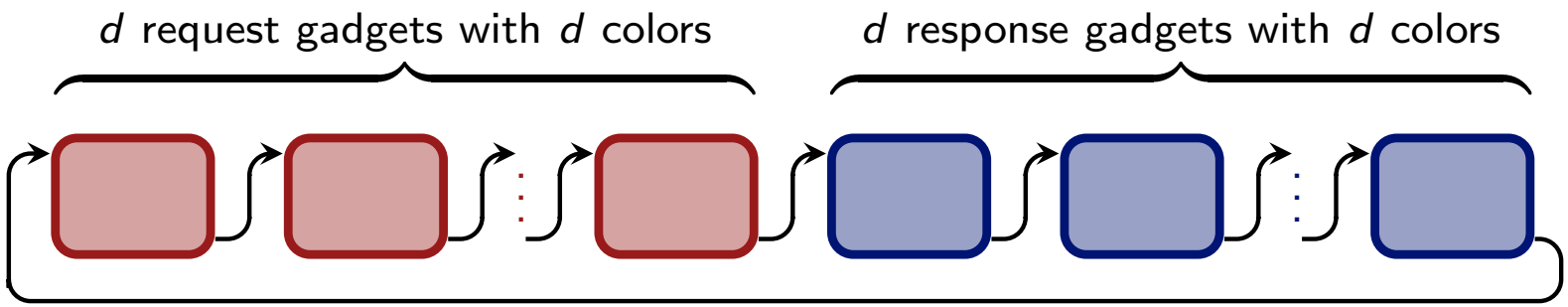

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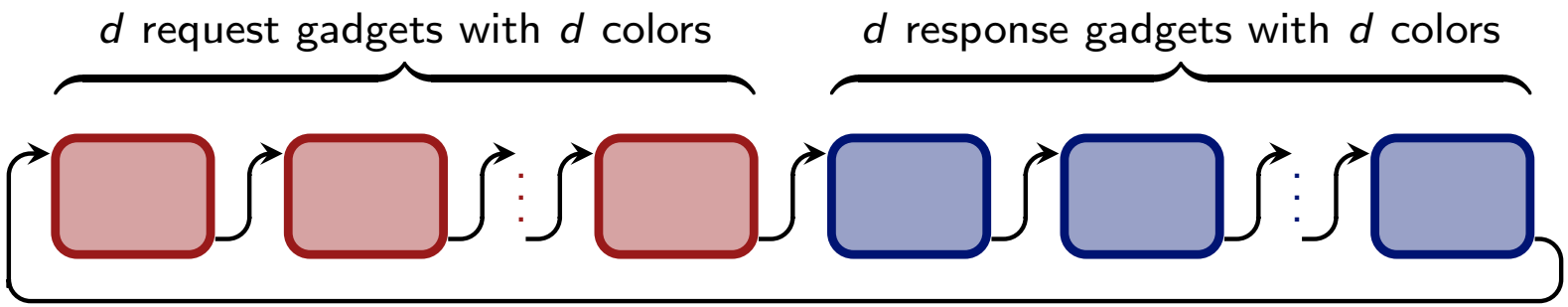

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Theorem (WZ16)

For every $d > 1$, there exists a finitary parity game $\mathcal{G}_d$ such that

- *$|\mathcal{G}_d| \in \mathcal{O}(d^2)$ and $\mathcal{G}_d$ has d odd colors, and*
- *every optimal strategy for Player 0 has at least size 2^{d-1}.*

PSPACE-Hardness

Lemma (WZ16)

The following problem is PSPACE*-hard: "Given a finitary parity game* $\mathcal{G}$ *and a bound* $b \in \mathbb{N}$*, does Player* 0 *have a strategy for* $\mathcal{G}$ *whose cost is at most* b*?"*

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- By a reduction from QBF (w.l.o.g. in CNF).
- Checking the truth of $\varphi = \forall x \exists y.\ (x \vee \neg y) \wedge (\neg x \vee y)$ as a two-player game (Player 0 wants to prove truth of φ):

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  1. Player 1 picks truth value for x.
  2. Player 0 picks truth value for y.
  3. Player 1 picks clause C.
  4. Player 0 picks literal ℓ from C.
  5. Player 0 wins $\Leftrightarrow$ ℓ is picked to be satisfied in step 1 or 2.

The Reduction

$$\varphi = \forall x \ \exists y \,.\ \overbrace{(\, x \ \vee \ \neg y \,) \wedge (\, \neg x \ \vee \ y \,)}^{\psi}$$

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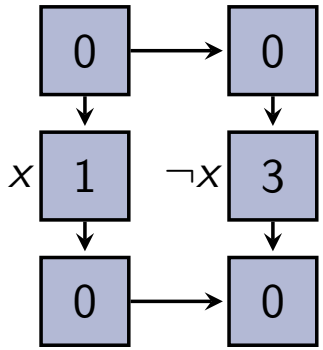

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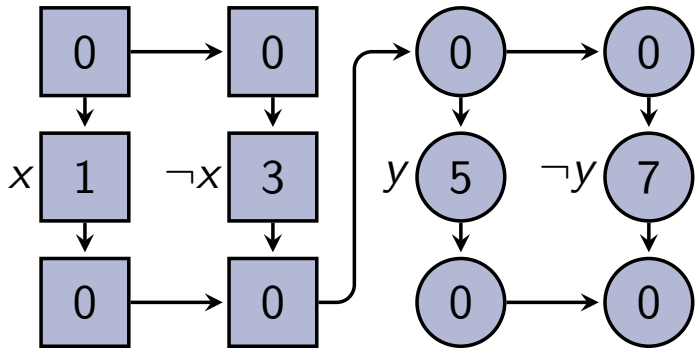

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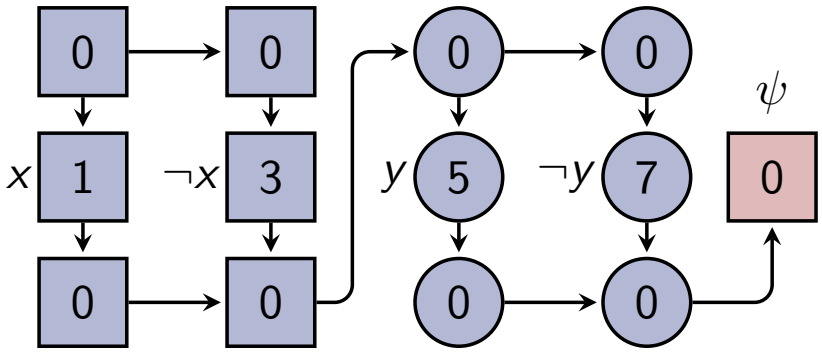

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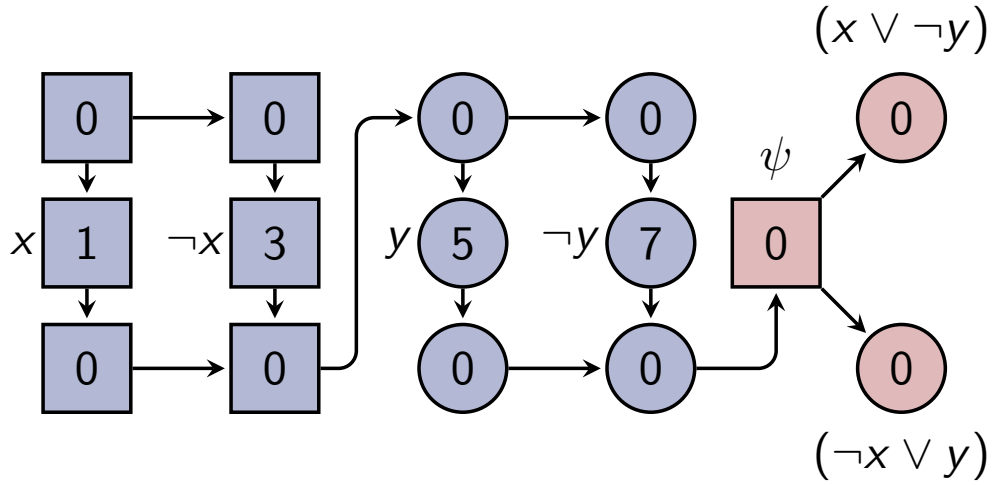

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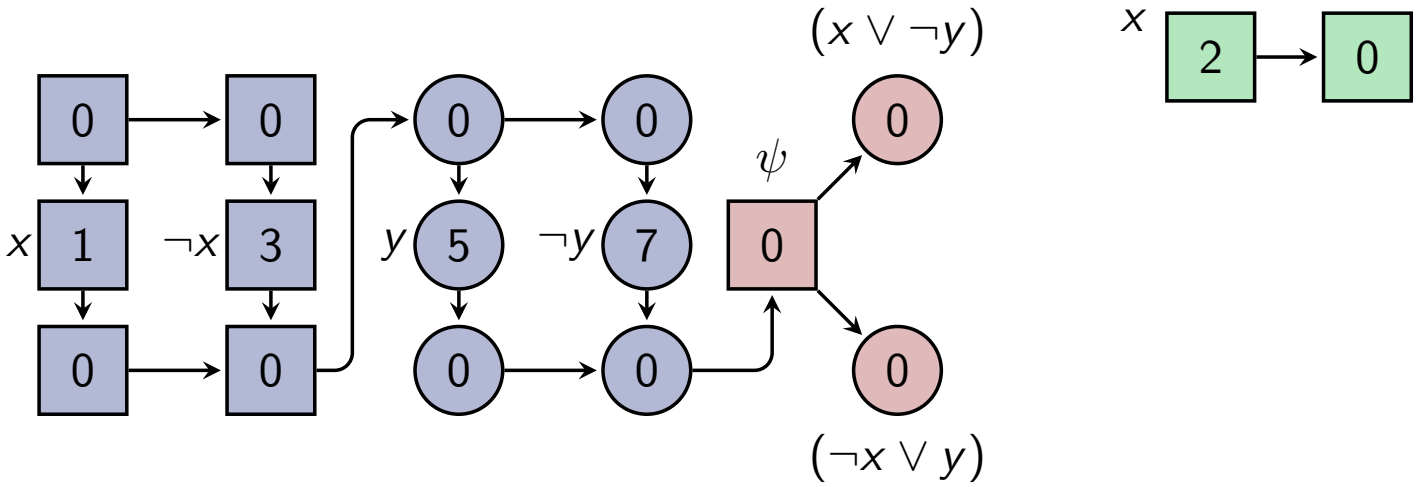

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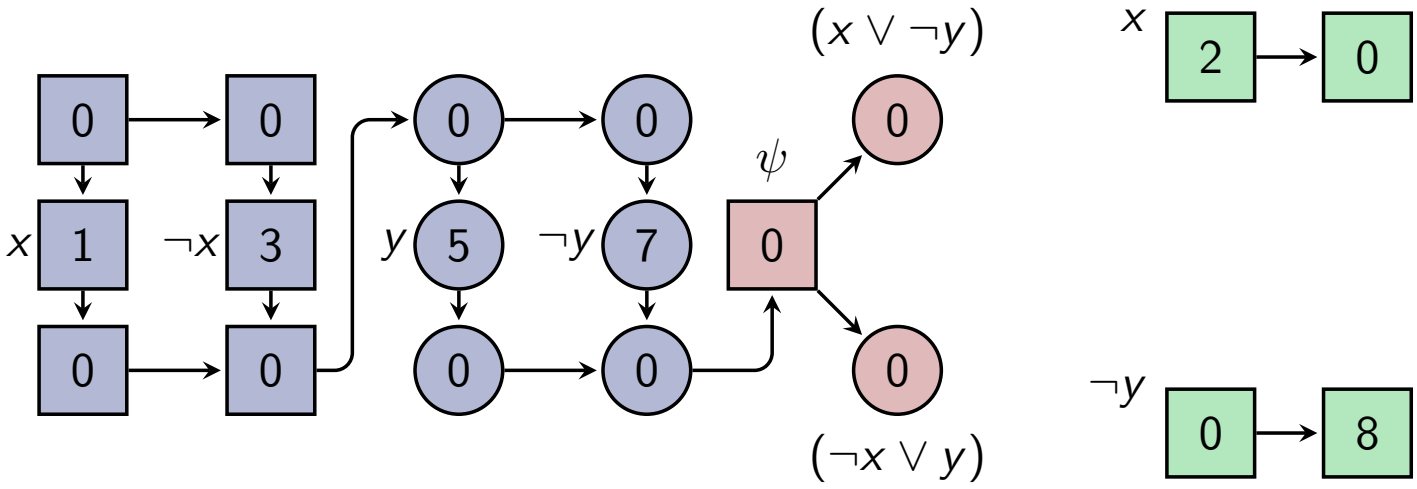

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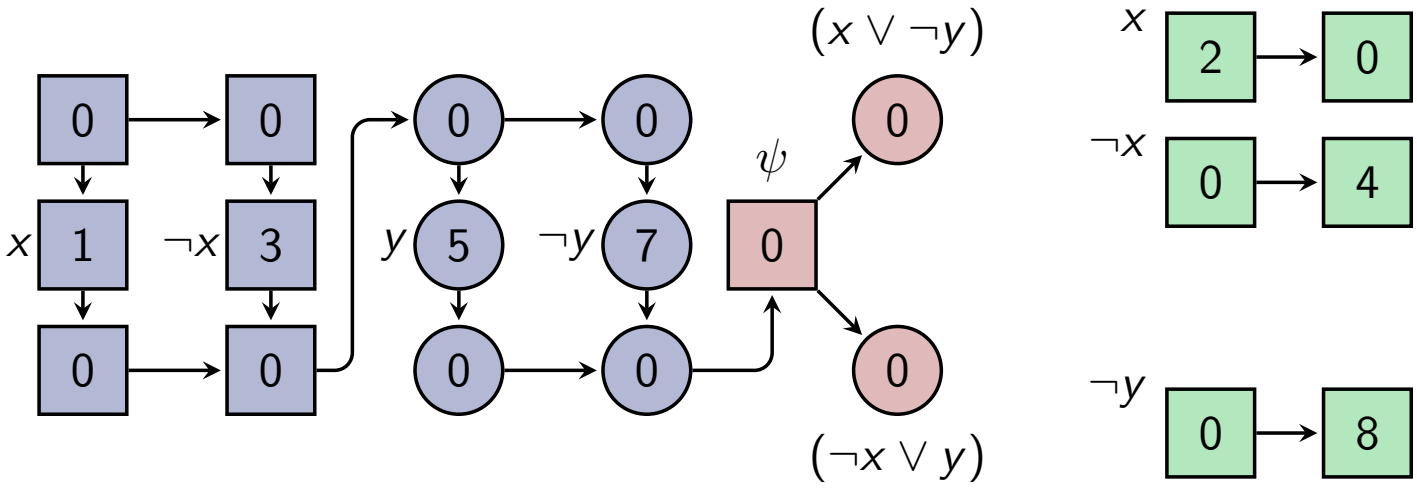

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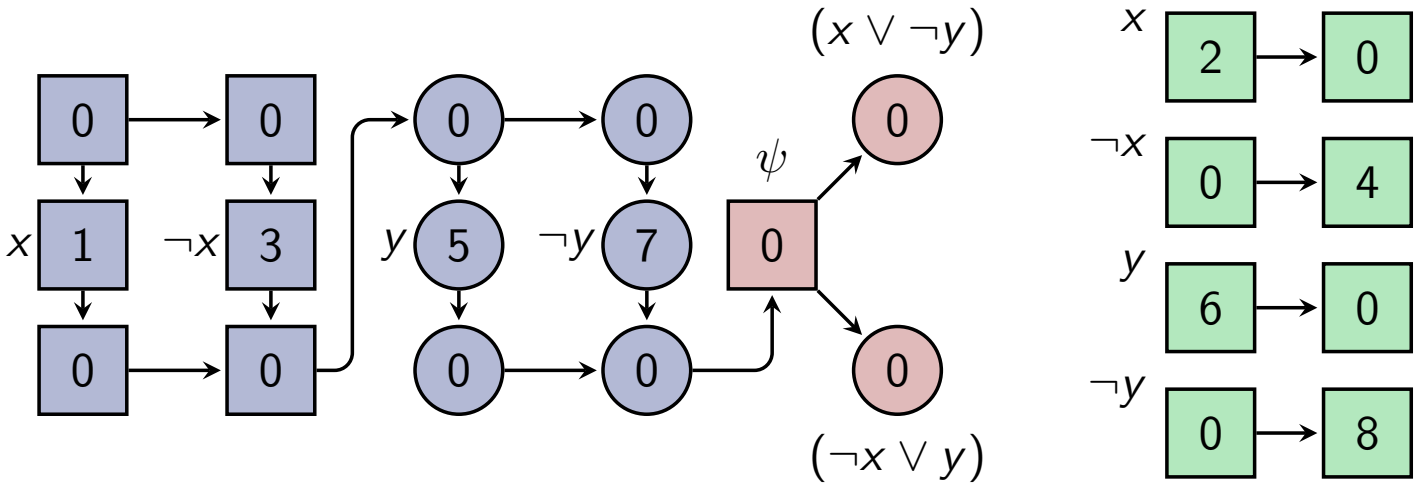

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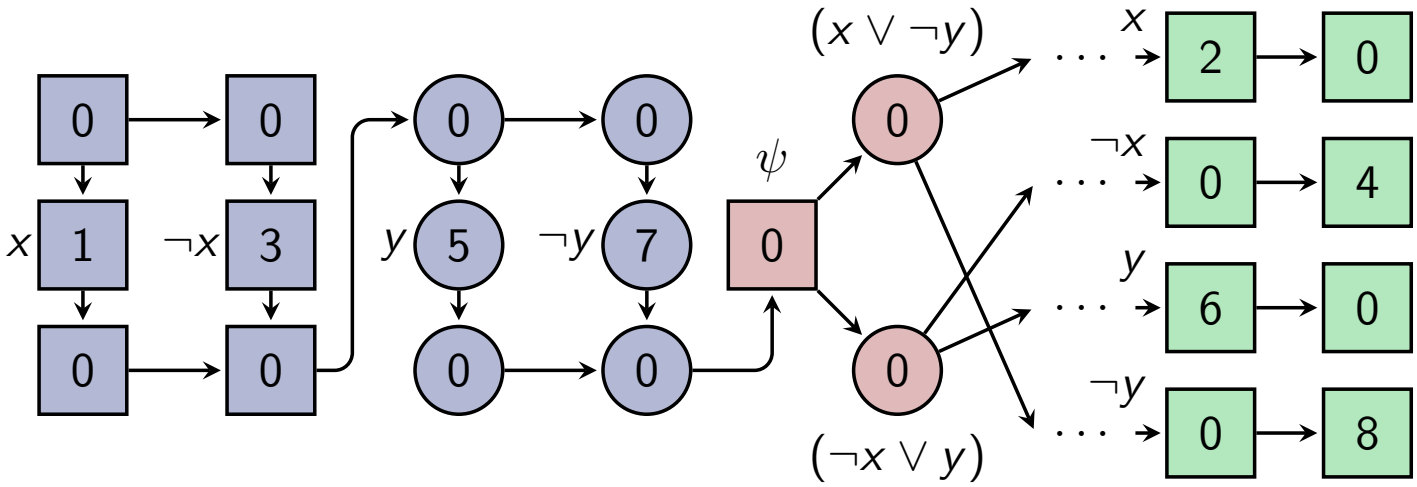

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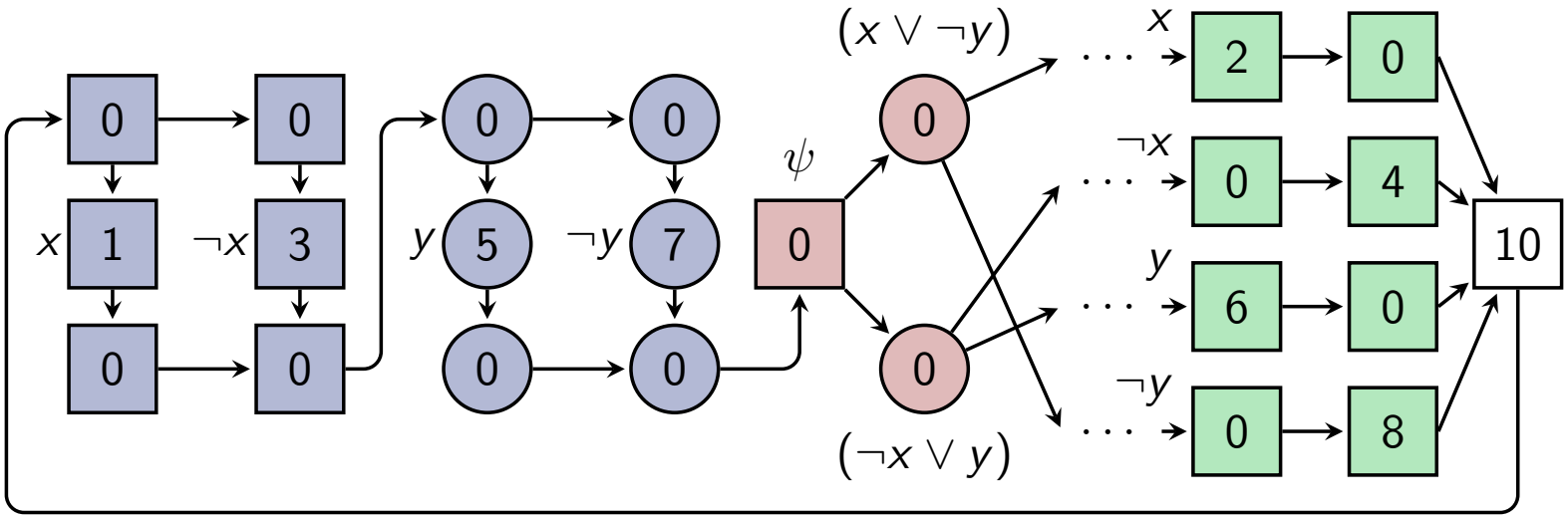

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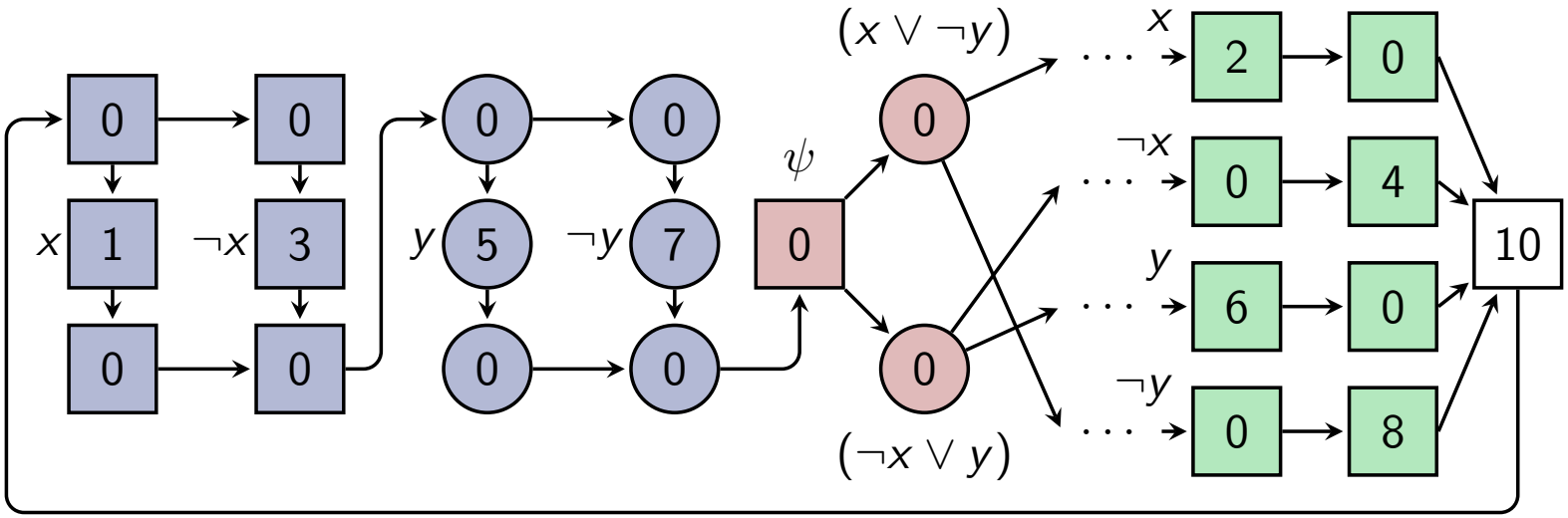

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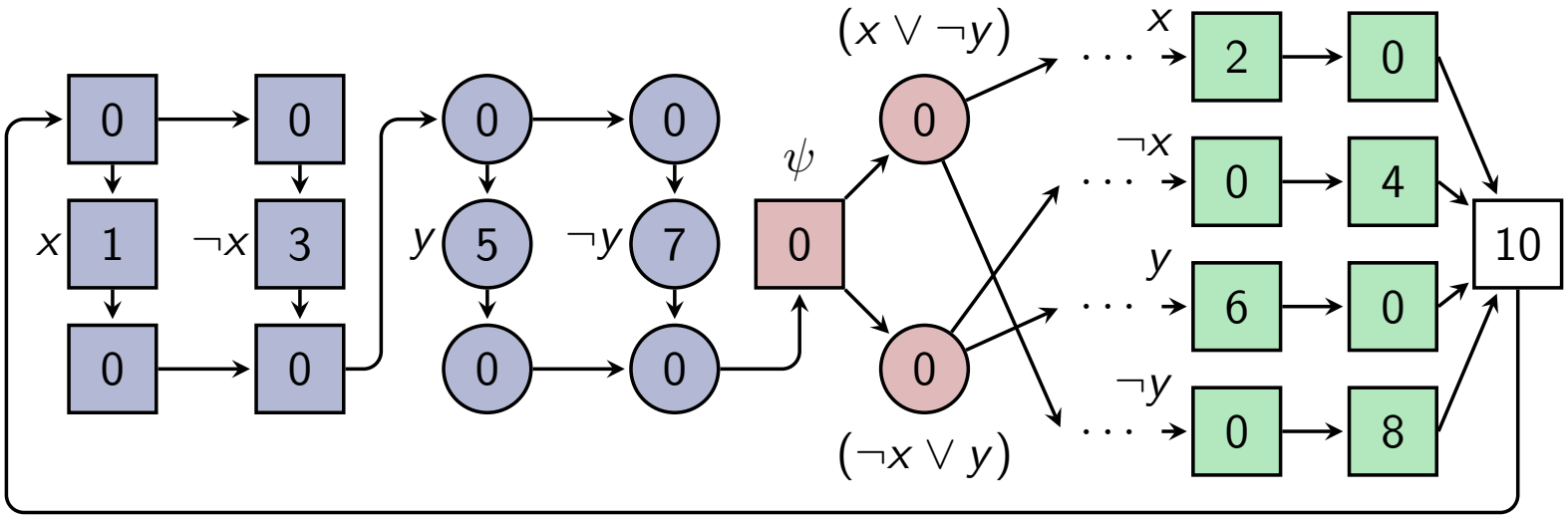

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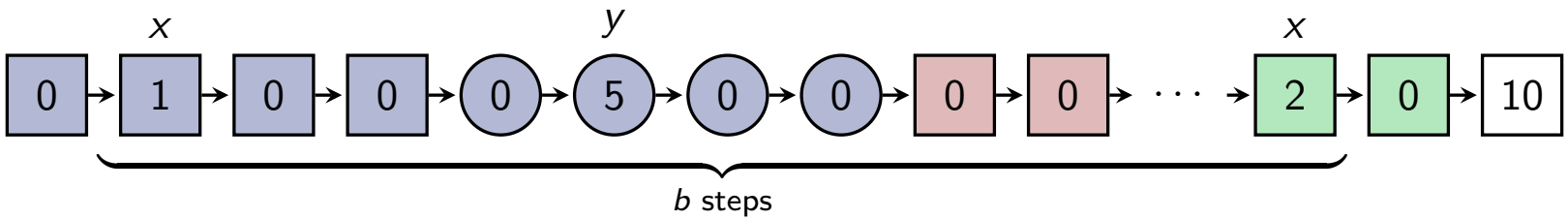

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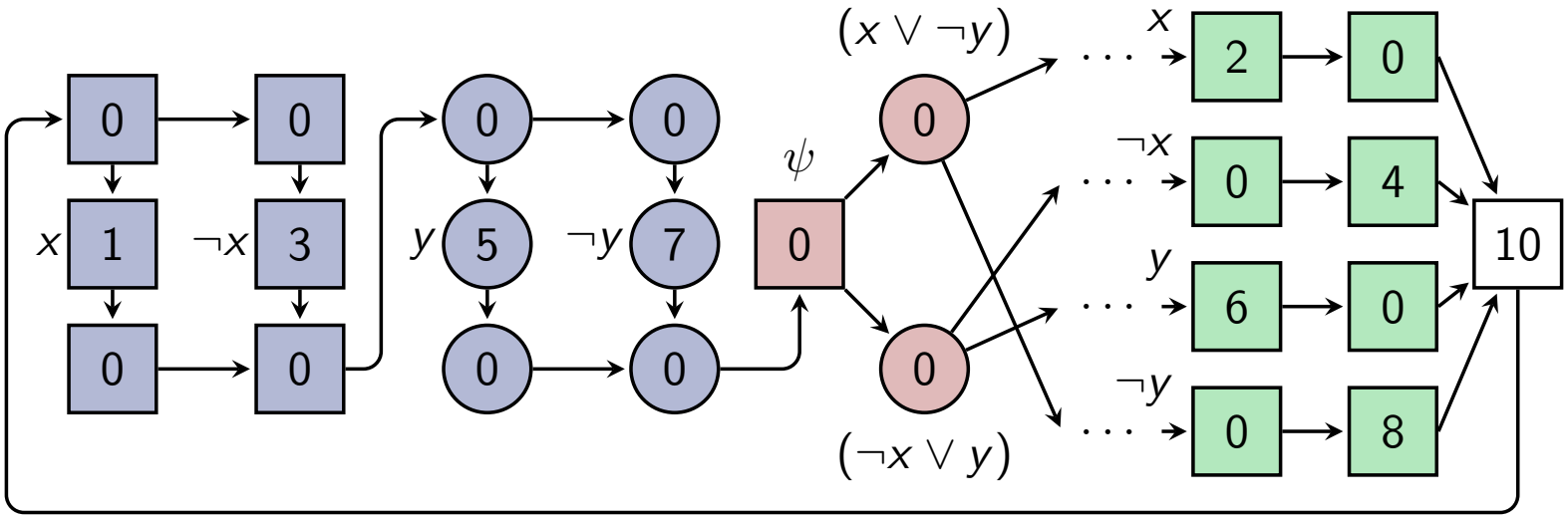

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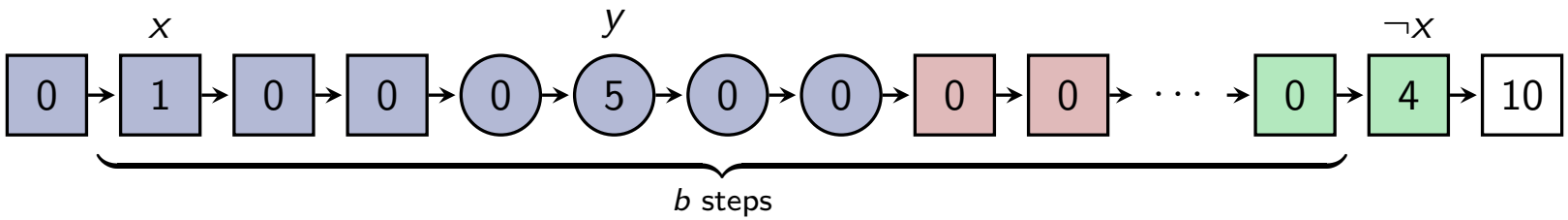

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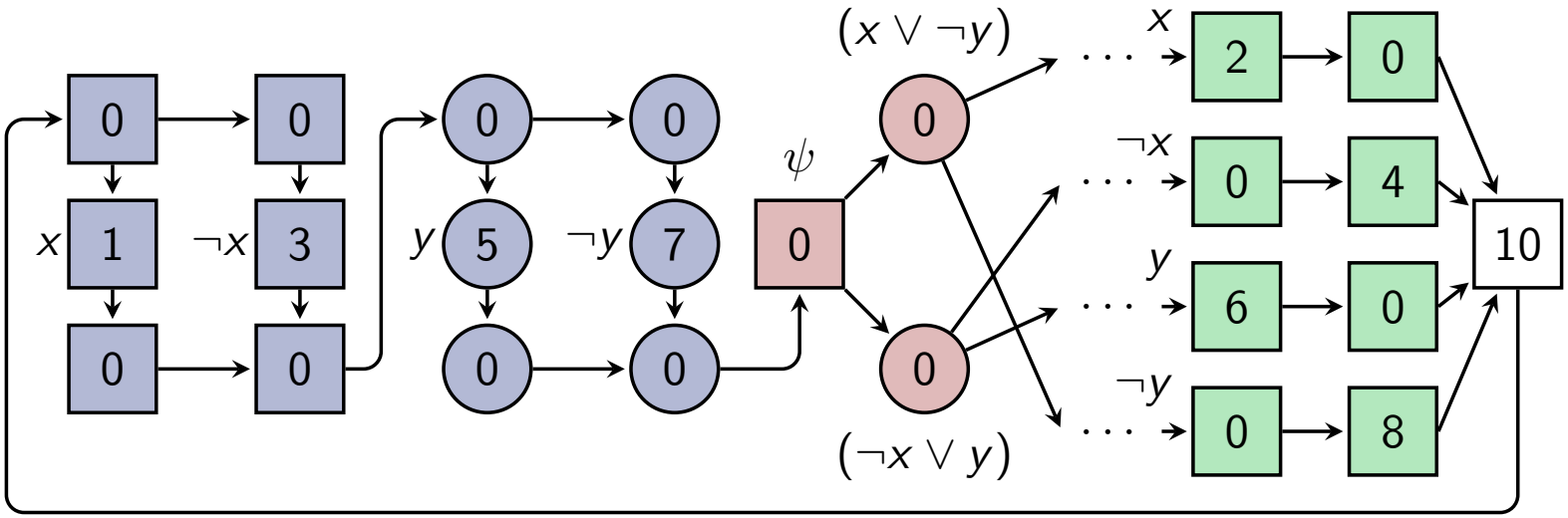

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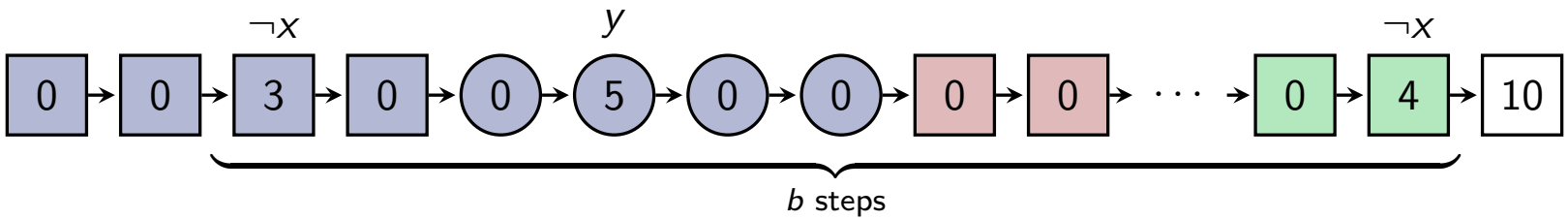

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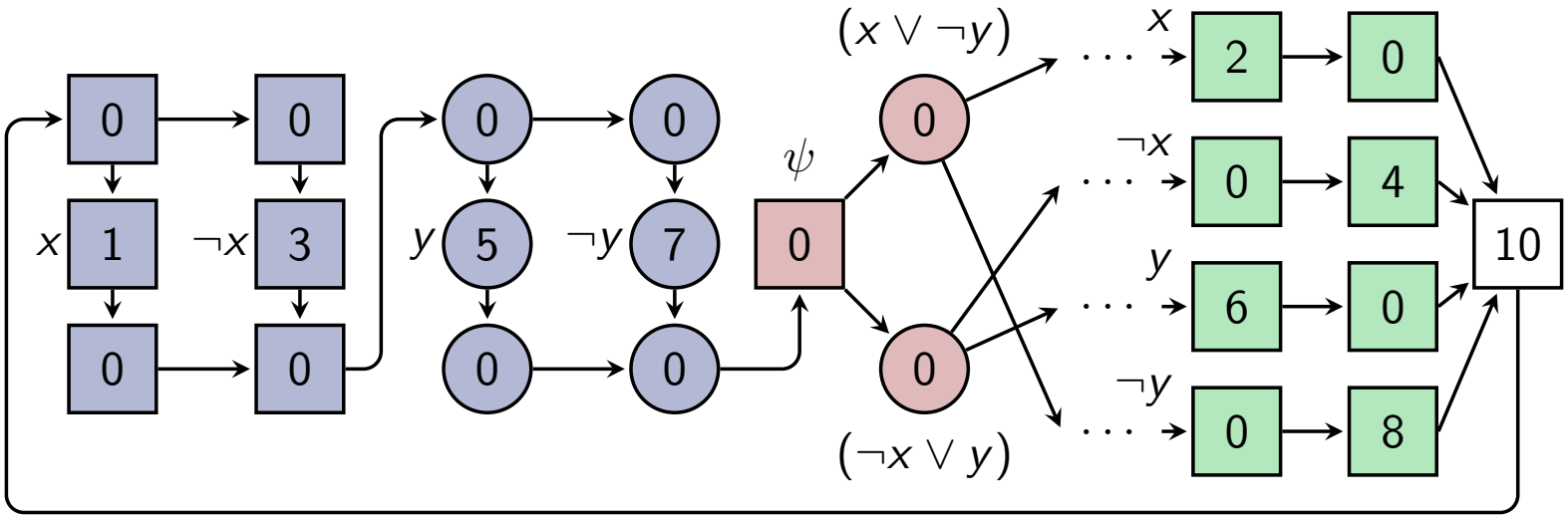

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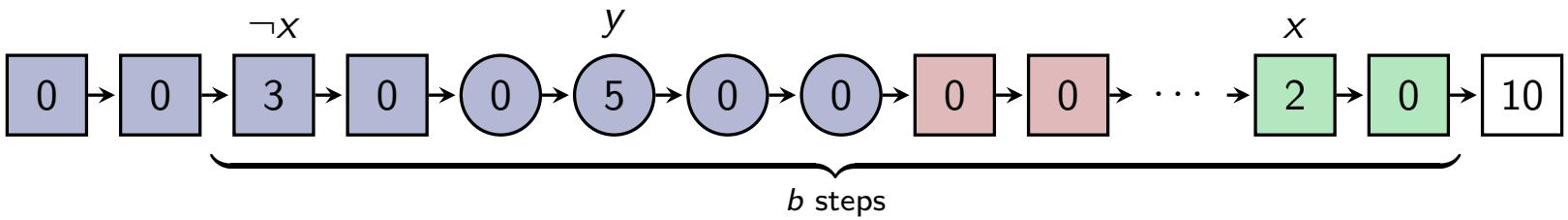

PSPACE-Membership

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Proof Sketch

Fix $\mathcal{G}$ and b (w.l.o.g. $b \leq |\mathcal{G}|$).

1. Construct equivalent parity game $\mathcal{G}'$ storing the costs of open requests (up to bound b) and the number of "overflows" (up to bound $|\mathcal{G}|$) $\Rightarrow |\mathcal{G}'| \in |\mathcal{G}|^{\mathcal{O}(d)}$.

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2. Define equivalent finite-duration variant $\mathcal{G}'_f$ of $\mathcal{G}'$ with polynomial play-length.
3. $\mathcal{G}'_f$ can be solved on alternating polynomial-time Turing machine.
4. APTIME = PSPACE concludes the proof.

Upper Bounds on Memory

Equivalence between finitary parity game $\mathcal{G}$ w.r.t. bound b and parity game $\mathcal{G}'$ yields upper bounds on memory requirements.

Corollary

Let $\mathcal{G}$ be a finitary parity game with costs with d odd colors. If Player 0 has a strategy for $\mathcal{G}$ with cost b, then she also has a strategy with cost b and size $(b+2)^d = 2^{d \log(b+2)}$.

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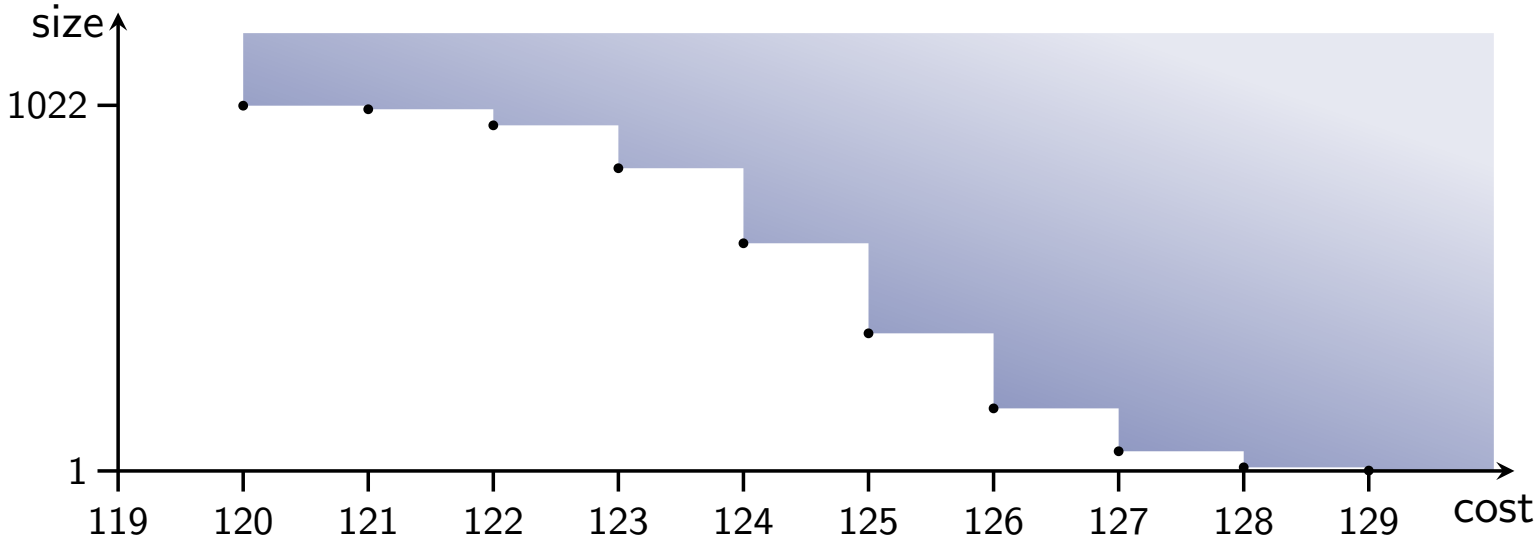
- Recall: lower bound 2^{d-1}.
- The same bounds hold for Player 1.

Tradeoffs

Theorem (WZ16)

Fix some finitary parity game $\mathcal{G}_d$ as before. For every i with $1 \leq i \leq d$ there exists a strategy σ_i for Player 0 in $\mathcal{G}_d$ such that σ_i has cost $d^2 + 3d - i$ and size $\sum_{j=1}^{i-1} \binom{d}{j}$.

Also, every strategy σ' for Player 0 in $\mathcal{G}_d$ whose cost is at most the cost of σ_i has at least the size of σ_i.



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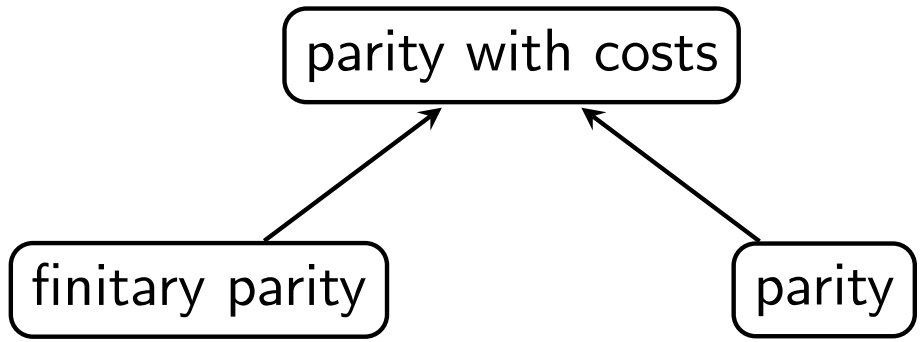

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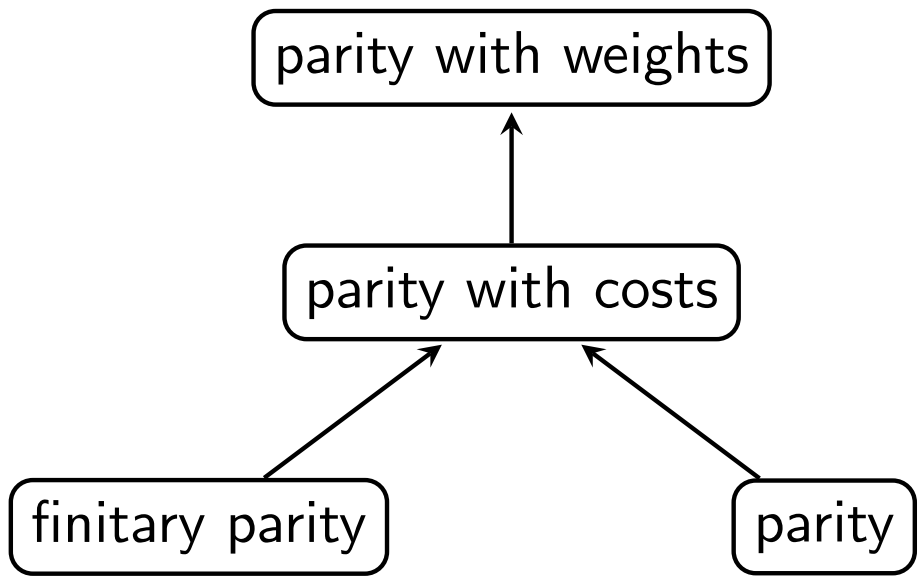

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Boundedness

Condition	Complexity	Memory Pl. 0	Memory Pl. 1
Parity	quasi-poly	Memoryless	Memoryless
Finitary Parity	PTime	Memoryless	Infinite
Parity w. Costs	quasi-poly	Memoryless	Infinite
Parity w. Weights	$\mathrm{NP} \cap \mathrm{co\text{-}NP}$	Exponential	Infinite

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Optimization

Condition	Complexity	Memory Pl. 0 & 1
Finitary Parity	PSPACE-complete	Exponential
Parity w. Costs	PSPACE-complete	Exponential
Parity w. Weights	PSPACE-hard	$\geq$ Exponential

- The results for parity games with costs hold for unary and binary encodings of the weights.

Outline

1. Playing Optimally in Variations of Parity Games
2. **Playing (Approximatively) Optimally in LTL Games**
3. More Tradeoffs
4. Conclusion

Introducing LTL by Examples

Atomic propositions $\mathtt{r}_i$ for requests and $\mathtt{g}_i$ for grants.

1. Answer every request: $\bigwedge_i \mathbf{G}\,(\mathtt{r}_i \to \mathbf{F}\,\mathtt{g}_i)$

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$$\equiv \bigwedge_i [\,(\mathtt{r}_i\,\mathbf{R}\,(\mathtt{r}_i \vee \neg \mathtt{g}_i))\,] \wedge [\,\mathbf{G}\,(\neg \mathtt{g}_i \vee \mathbf{X}\,(\mathtt{r}_i\,\mathbf{R}\,(\mathtt{r}_i \vee \neg \mathtt{g}_i)))\,]$$

A Problem with LTL

- Answer every request: $\bigwedge_i \mathbf{G}\,(\mathtt{r}_i \to \mathbf{F}\,\mathtt{g}_i)$

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Problem:

LTL is too weak to express timing-constraints: no guarantee when request is granted, only that it is granted eventually

LTL goes Quantitative

During the last two decades, various quantitative variants of LTL have been introduced:

- Parametric LTL **[AETP99]**
- PROMPT–LTL **[KPV07]**
- Parametric MTL **[GTN10]**

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PROMPT–LTL is distinguished, as all problems for the more general Parametric LTL are reducible to those for PROMPT–LTL.

Prompt-LTL

Syntax: Add prompt-eventually operator $\mathbf{F_P}$.

$$\varphi ::= p \mid \neg p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \mathbf{X}\,\varphi \mid \varphi\,\mathbf{U}\,\varphi \mid \varphi\,\mathbf{R}\,\varphi \mid \mathbf{F_P}\,\varphi$$

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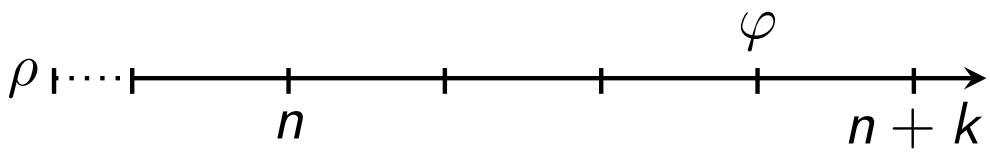
Semantics: Defined with respect to a fixed bound $k \in \mathbb{N}$.

- $(\rho, n, k) \models \mathbf{F_P}\,\varphi$: 

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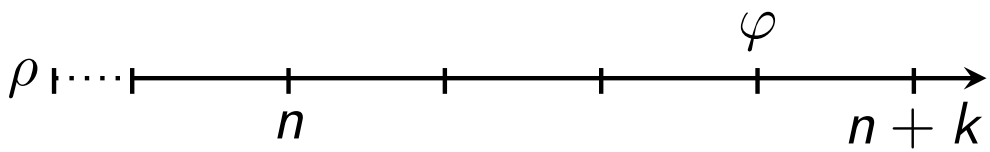
- $(\rho, n, k) \models \mathbf{F_P}\,\varphi$: 

Now: $\bigwedge_i \mathbf{G}\,(\mathtt{r}_i \to \mathbf{F_P}\,\mathtt{g}_i)$

Prompt-LTL Games

- Label the arena by atomic propositions.
- Winning condition: $\mathrm{PROMPT\text{–}LTL}$ formula φ.
- Player 0 wins if there is a uniform bound k and a strategy σ such that every play that is consistent with σ satisfies the winning condition φ w.r.t. k.

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$\mathrm{PROMPT\text{–}LTL}$ games are not harder than LTL games...

Theorem (KPV07)

1. *Determining the winner of* $\mathrm{PROMPT\text{–}LTL}$ *games is* 2ExpTime*-complete.*
2. *If Player* 0 *wins, then also with a finite-state strategy of size* $2^{2^{|\varphi|}}$ *and w.r.t. the bound* $k_\varphi = 2^{2^{|\varphi|}}$.

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...unless you optimize the bound.

Theorem (Z11)

1. *The* $\mathrm{PROMPT\text{–}LTL}$ *game optimization problem can be solved in triply-exponential time.*
2. *The bound* k_φ *is tight in general.*

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Questions

1. Is the optimization problem harder than the boundedness problem?
2. Can the optimum be approximated?

An Approximation Algorithm

Lemma (TWZ16)

Fix a PROMPT–LTL *game* $\mathcal{G}$ *with winning condition* φ *and* $k \leq k_\varphi$. *There is an* LTL *game* $\mathcal{G}_k$ *such that*

1. *if Player* 0 *wins* $\mathcal{G}$ *w.r.t.* k, *then she wins* $\mathcal{G}_k$,
2. *if Player* 0 *wins* $\mathcal{G}_k$, *then she wins* $\mathcal{G}$ *w.r.t.* $2k$, *and*
3. $\mathcal{G}_k$ *can be solved in doubly-exponential time in* $|\mathcal{G}|$.

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1. *if Player* 0 *wins* $\mathcal{G}$ *w.r.t.* k, *then she wins* $\mathcal{G}_k$,
2. *if Player* 0 *wins* $\mathcal{G}_k$, *then she wins* $\mathcal{G}$ *w.r.t.* $2k$, *and*
3. $\mathcal{G}_k$ *can be solved in doubly-exponential time in* $|\mathcal{G}|$.

The algorithm:

1: **for** $k = 0;\ k \leq k_\varphi;\ k \leftarrow k + 1$ **do**
2: **if** Player 0 wins $\mathcal{G}_k$ **then**
3: **return** $2k$

An Approximation Algorithm

Lemma (TWZ16)

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1: **for** $k = 0$; $k \leq k_\varphi$; $k \leftarrow k + 1$ **do**
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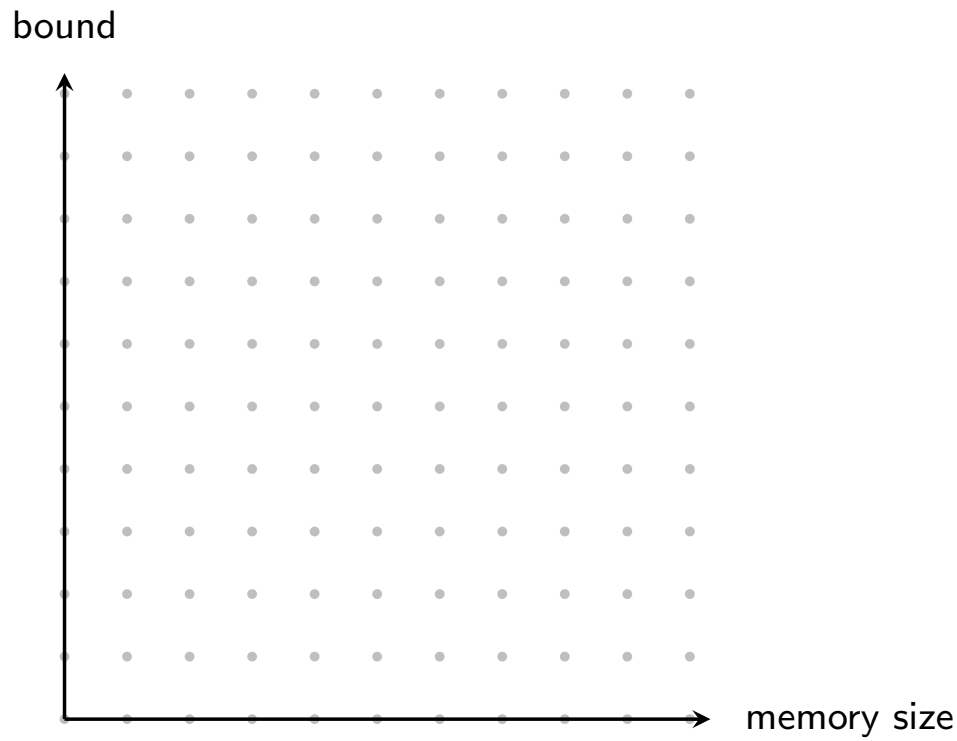
3: **return** $2k$

- Running time: doubly-exponential
- Approximation ratio: 2
- Yields winning strategy

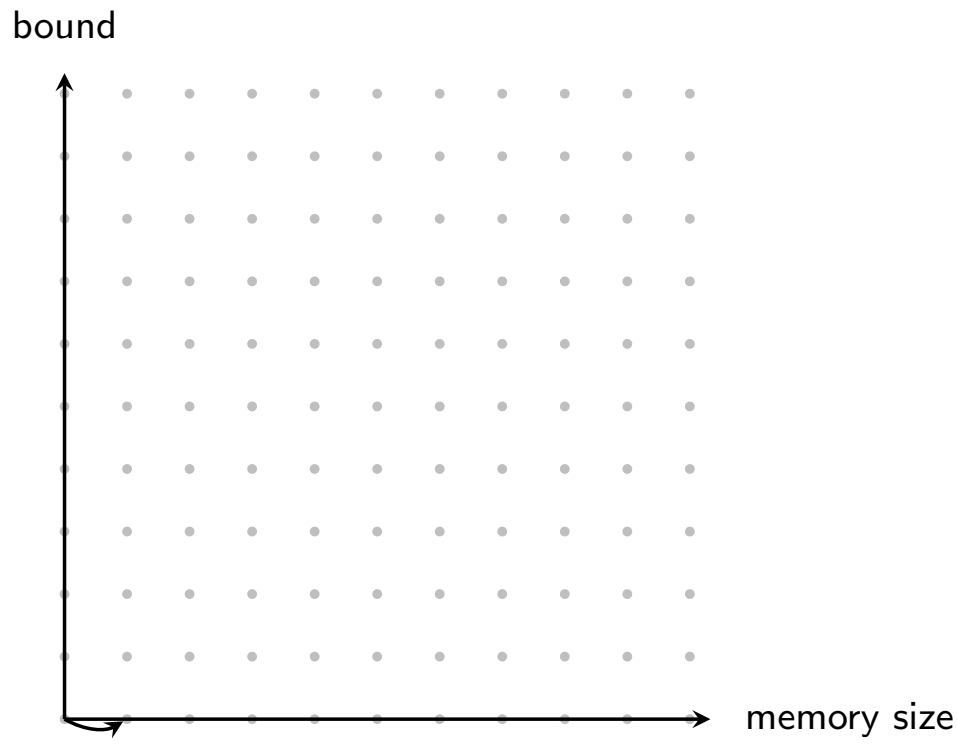
Tradeoffs



Implementation via bounded synthesis:

- Search for finite-state strategy of size n achieving bound k.
- Complete due to upper bounds on n and k.

Tradeoffs



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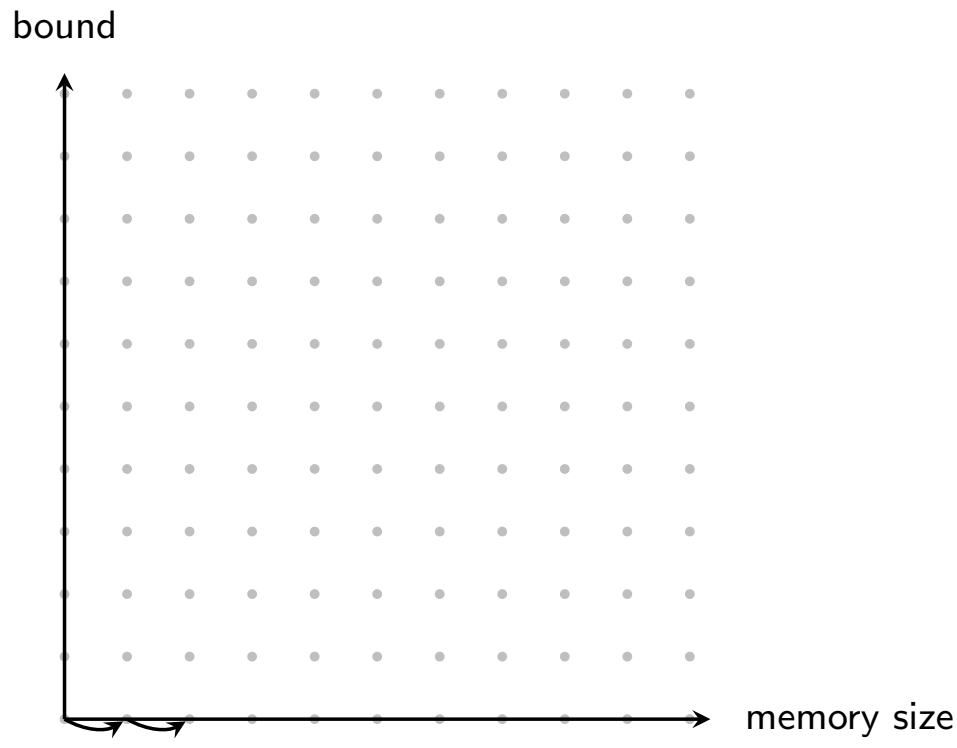

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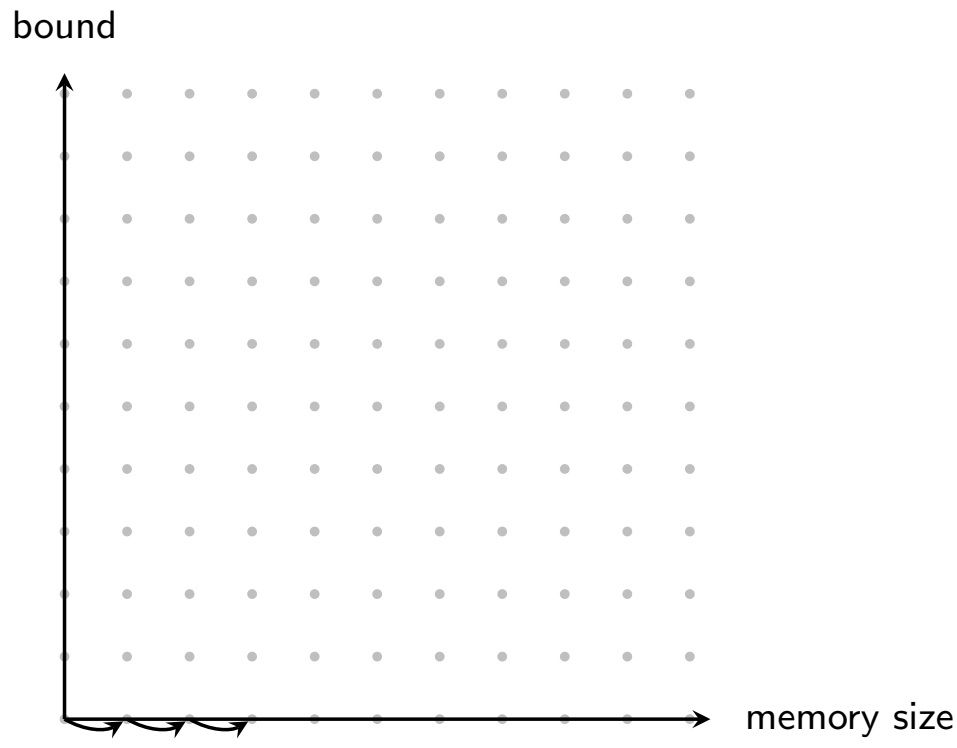

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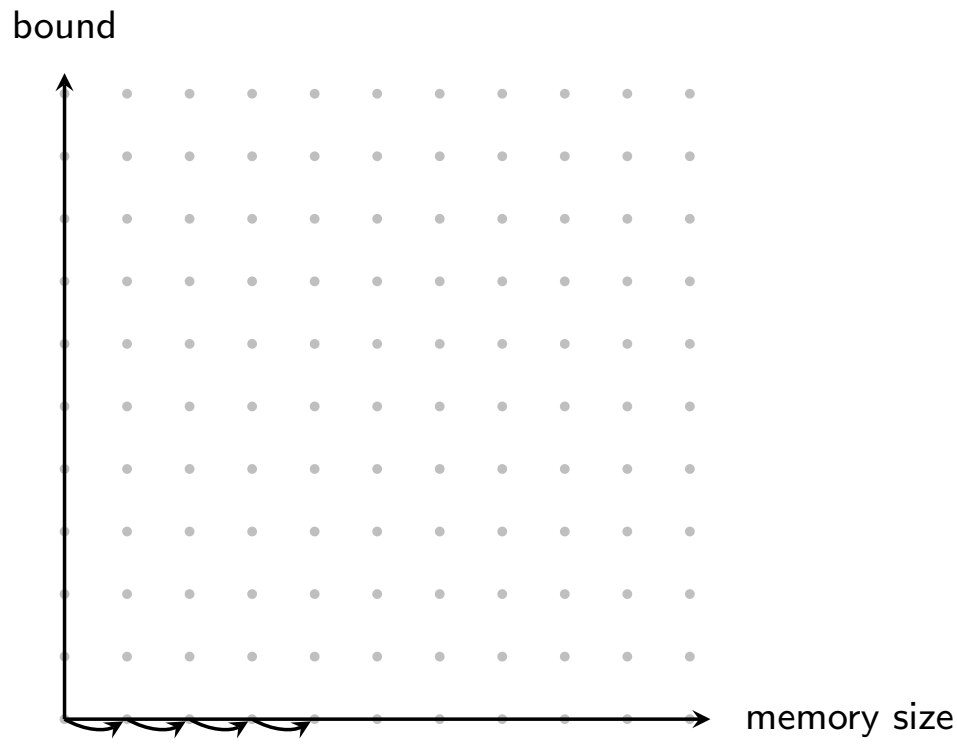

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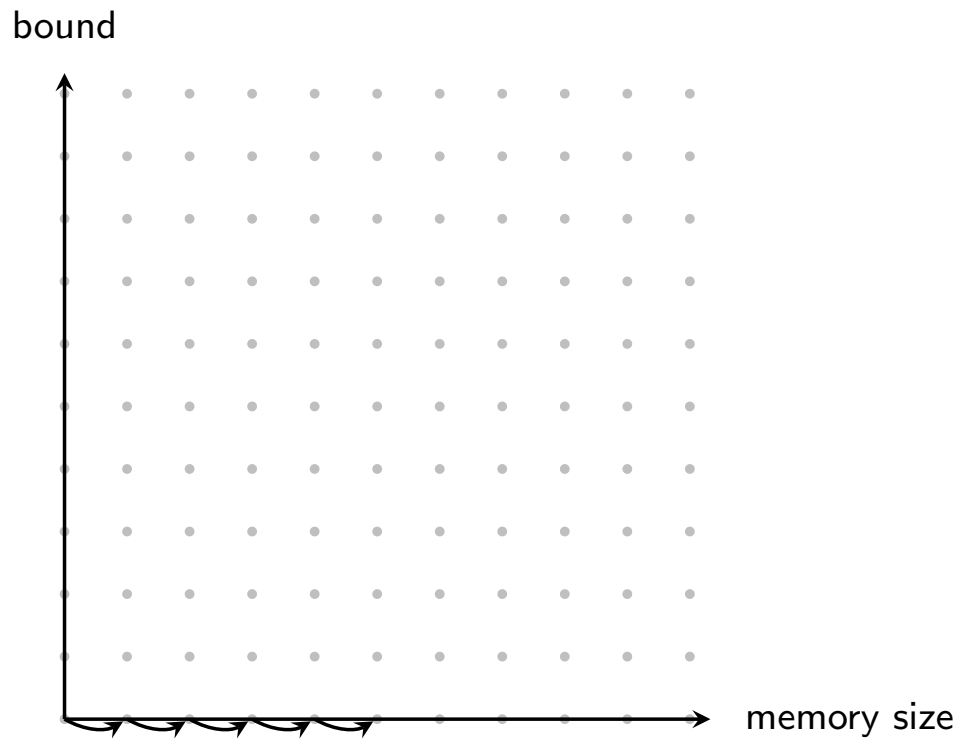

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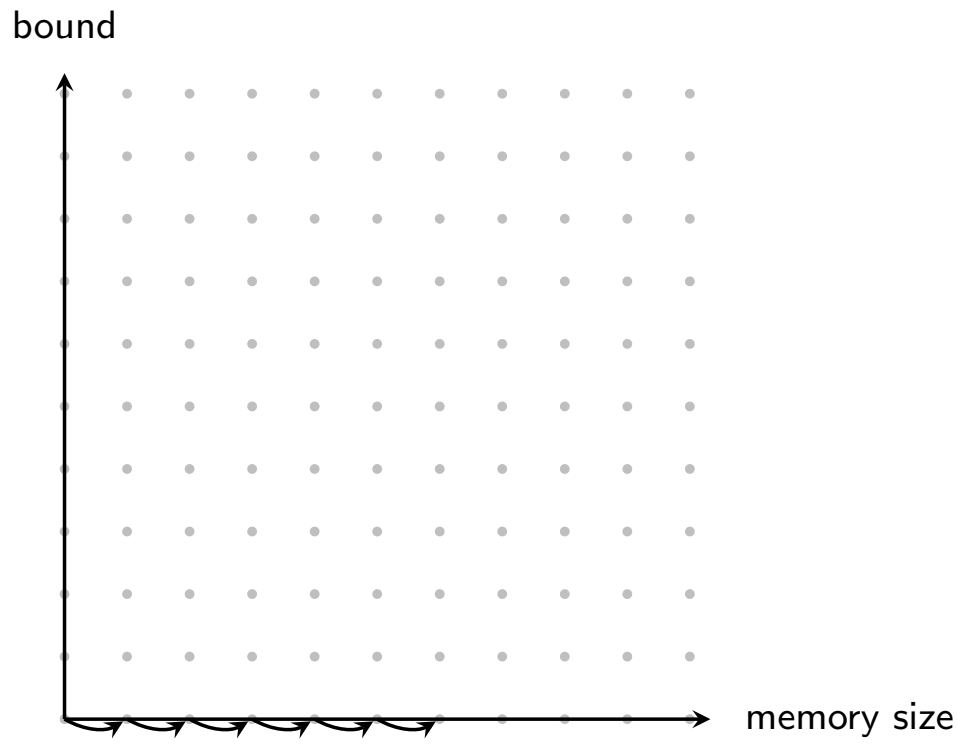

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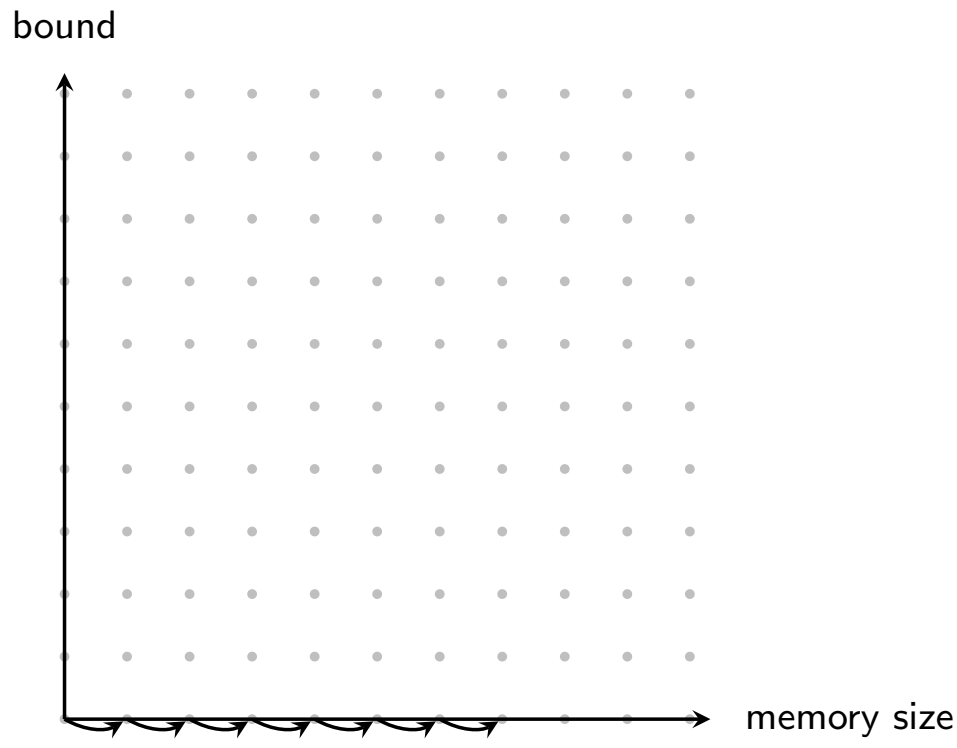

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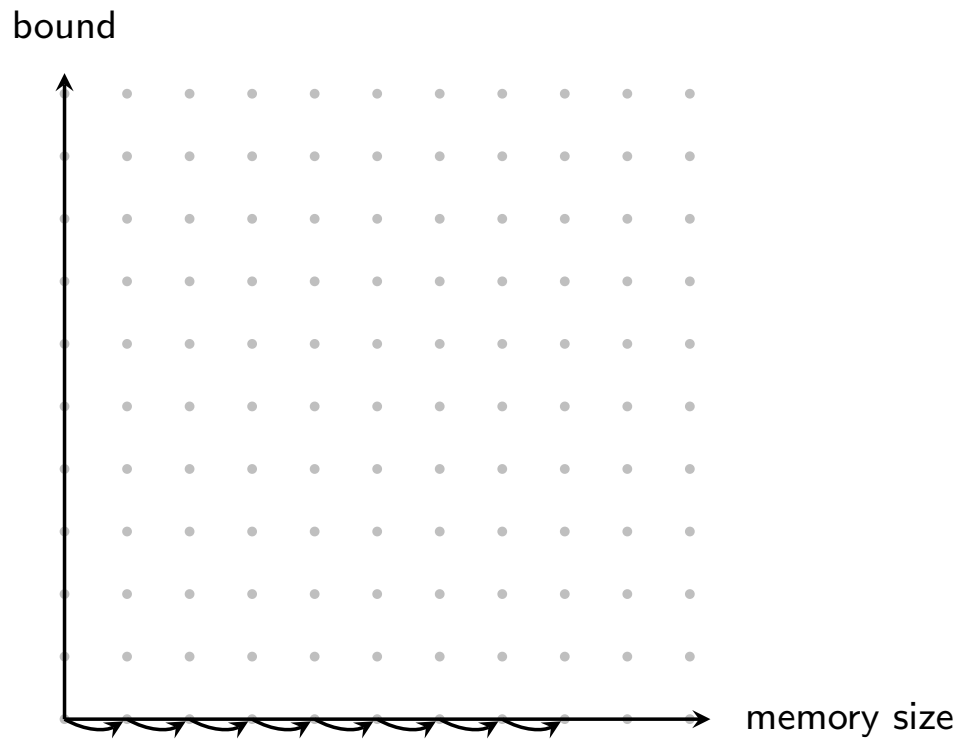

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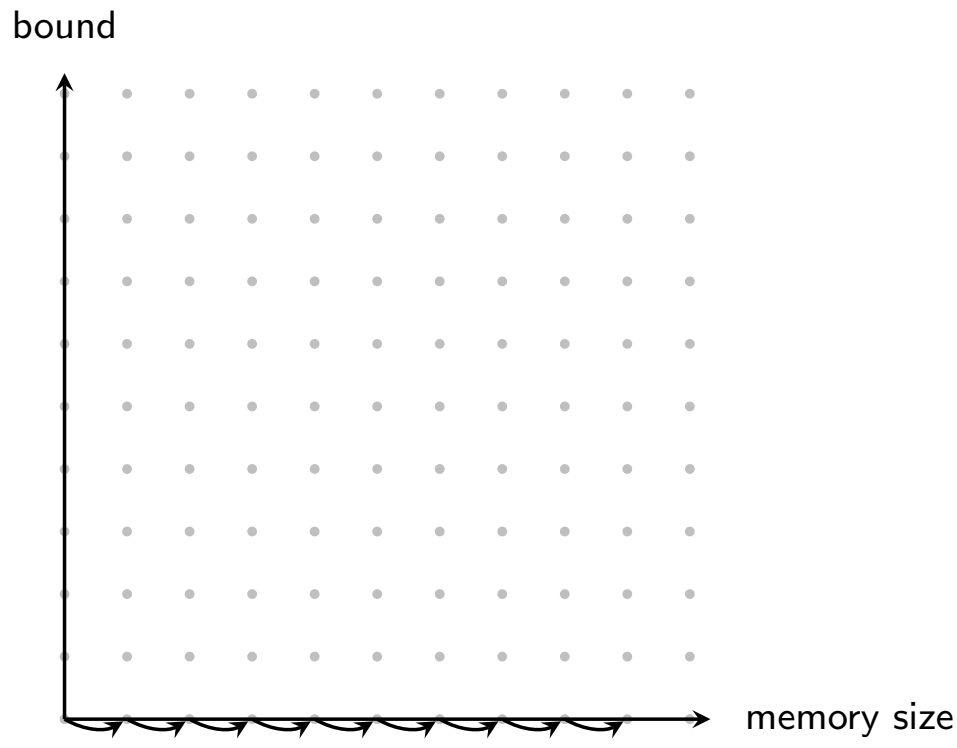

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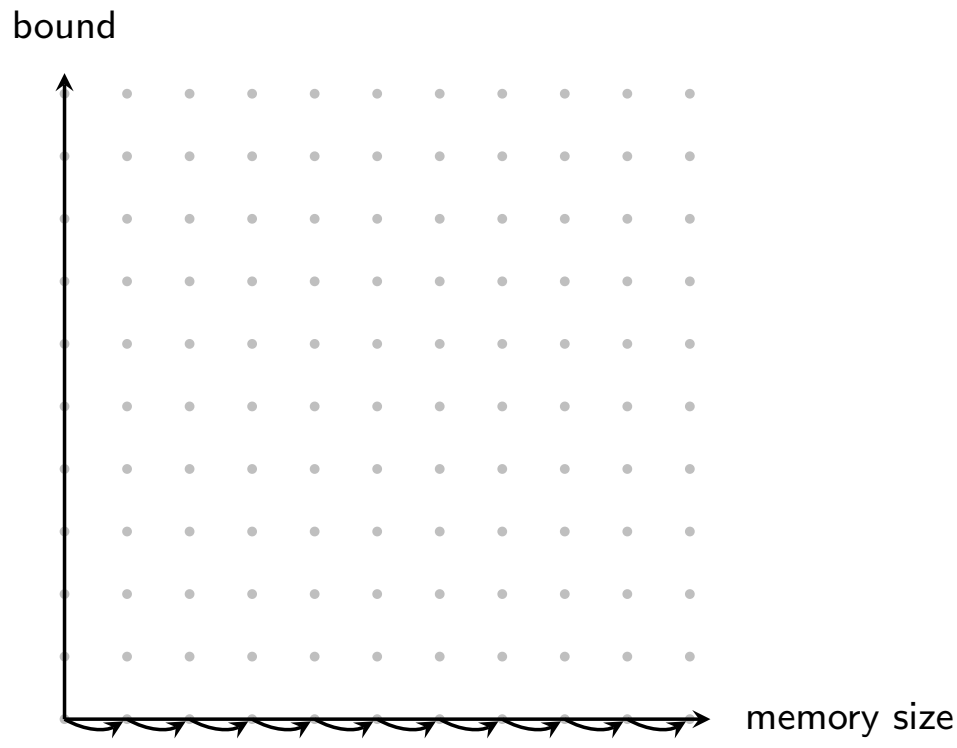

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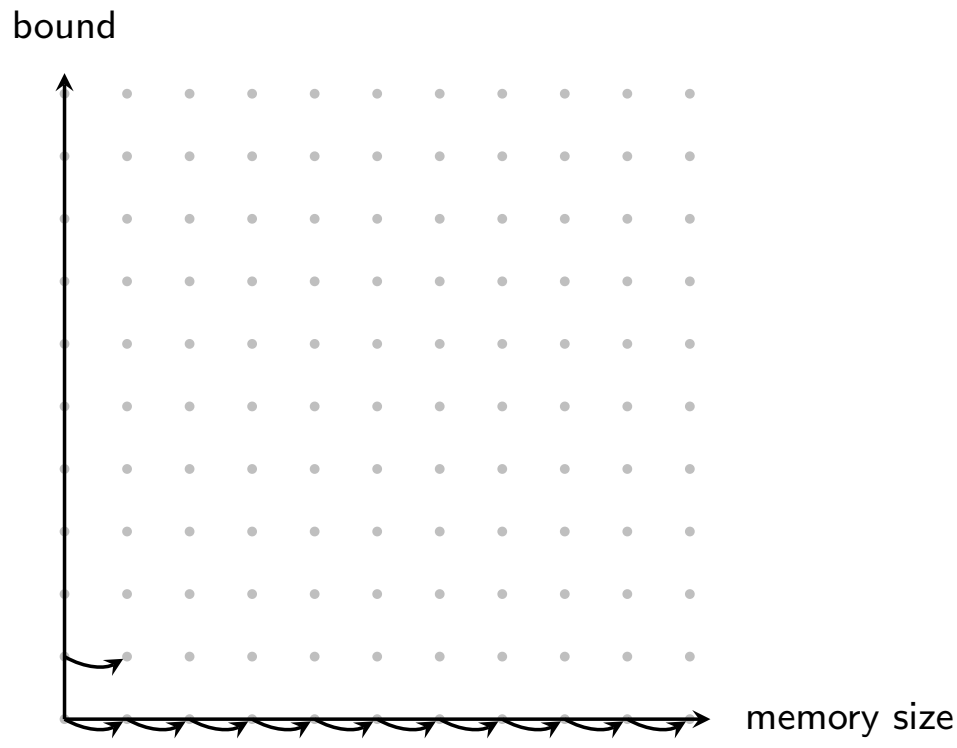

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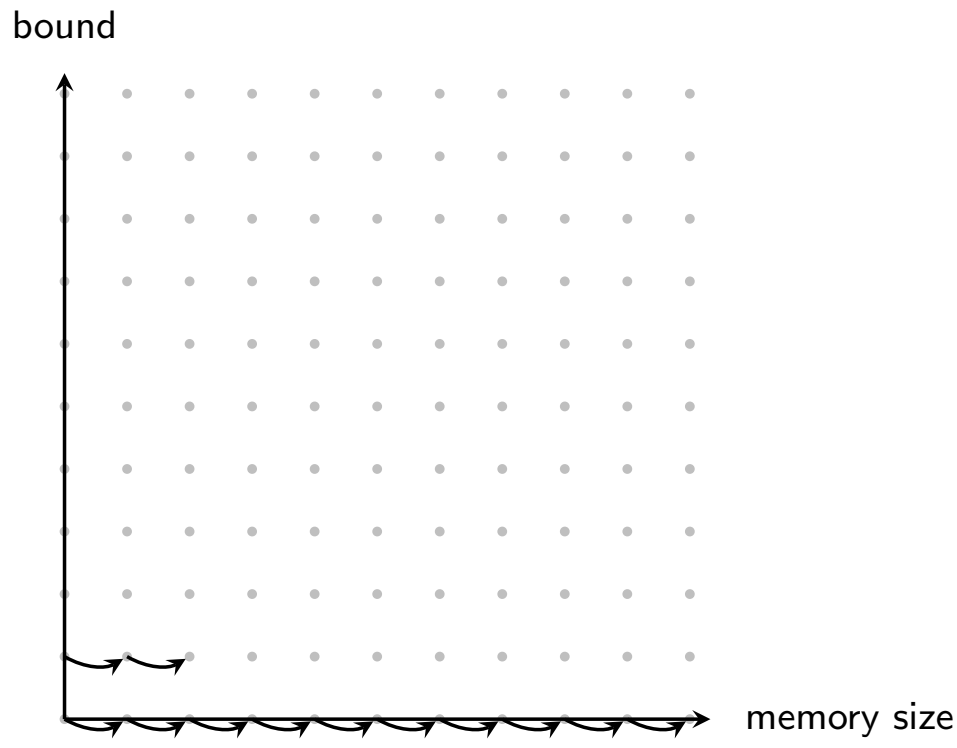

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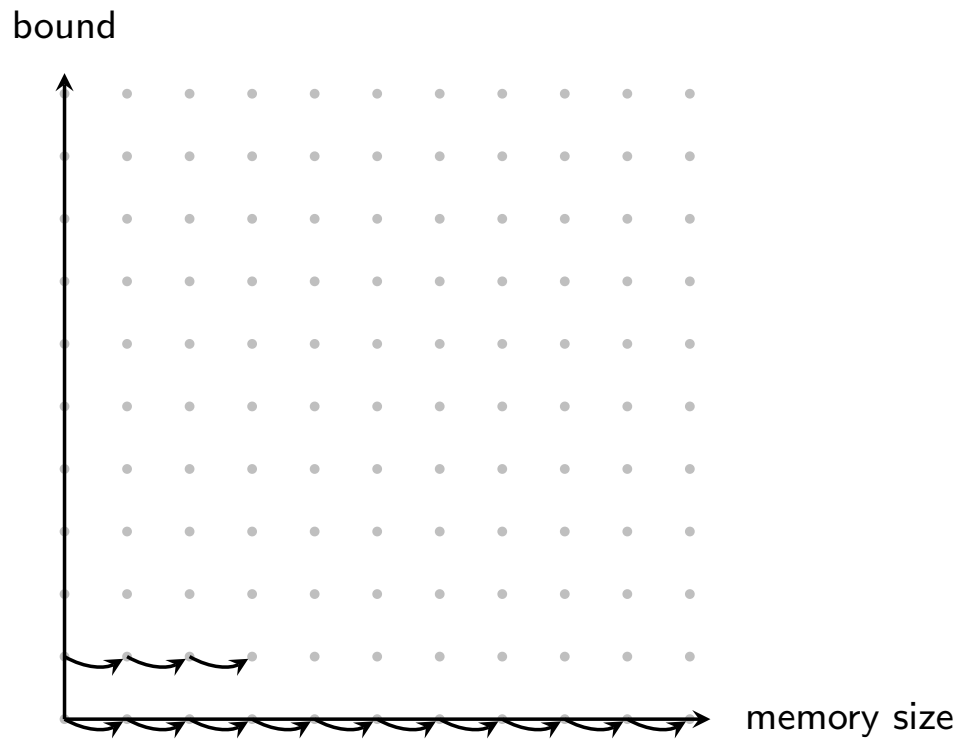

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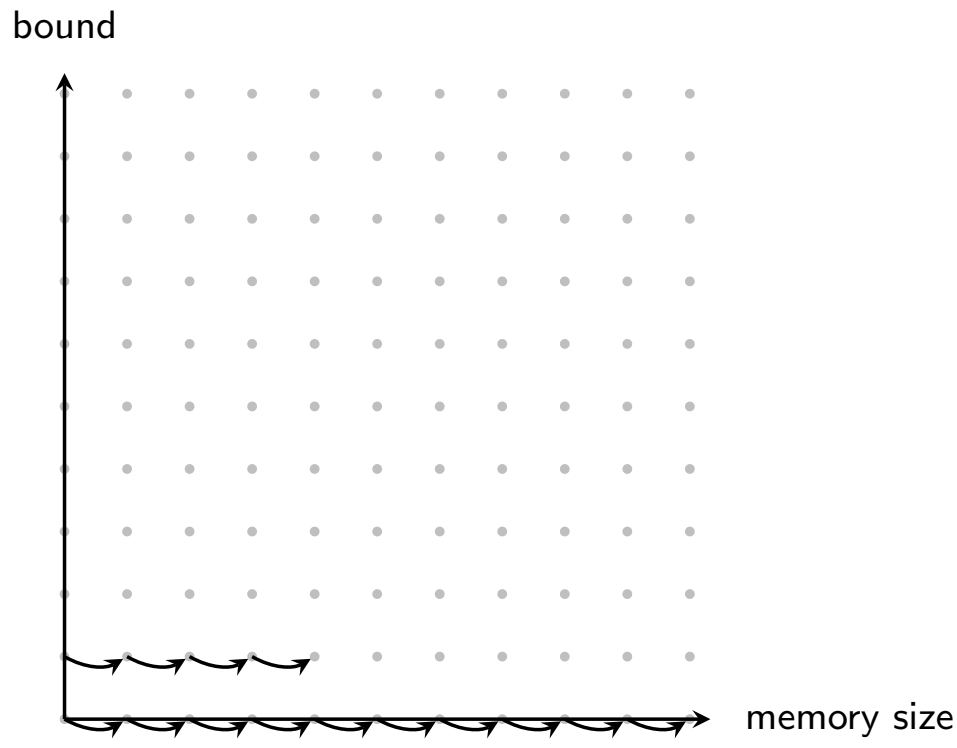

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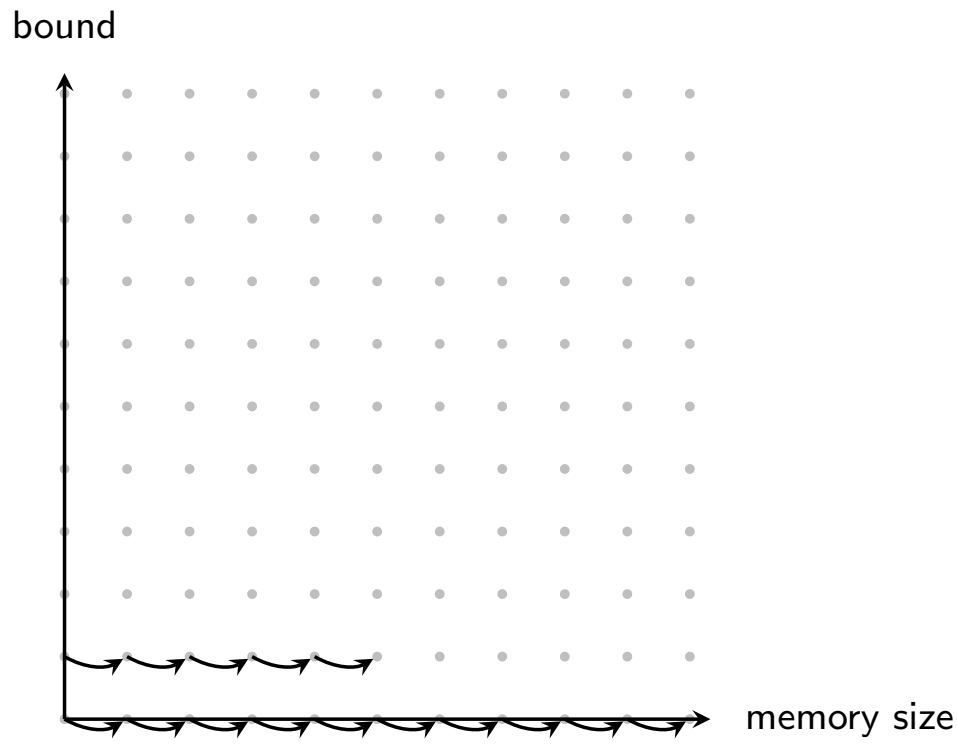

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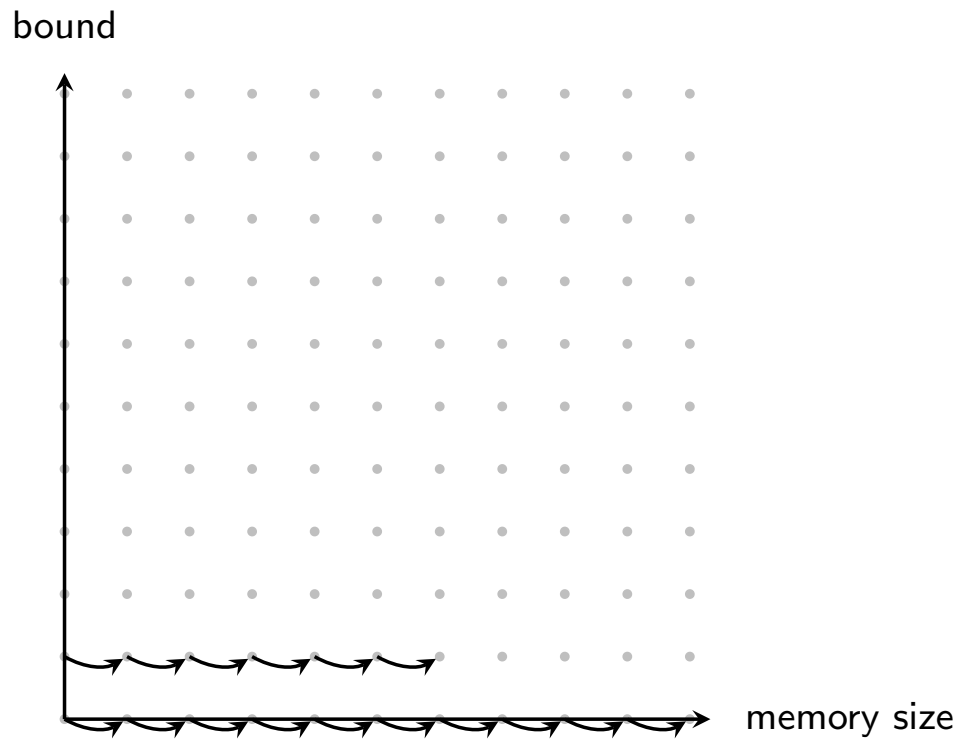

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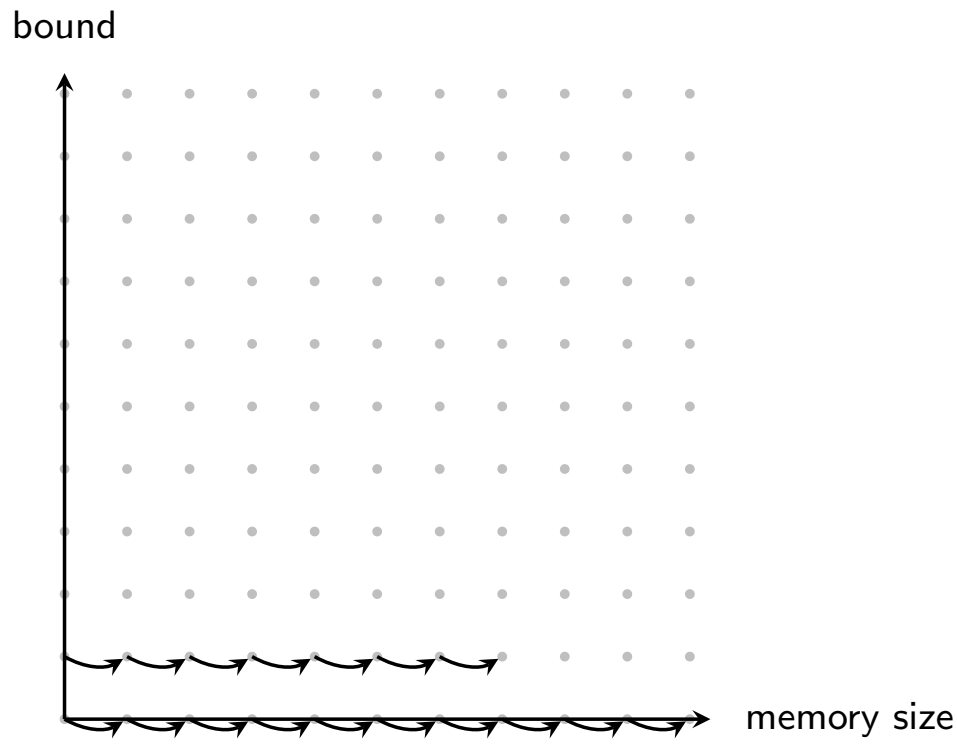

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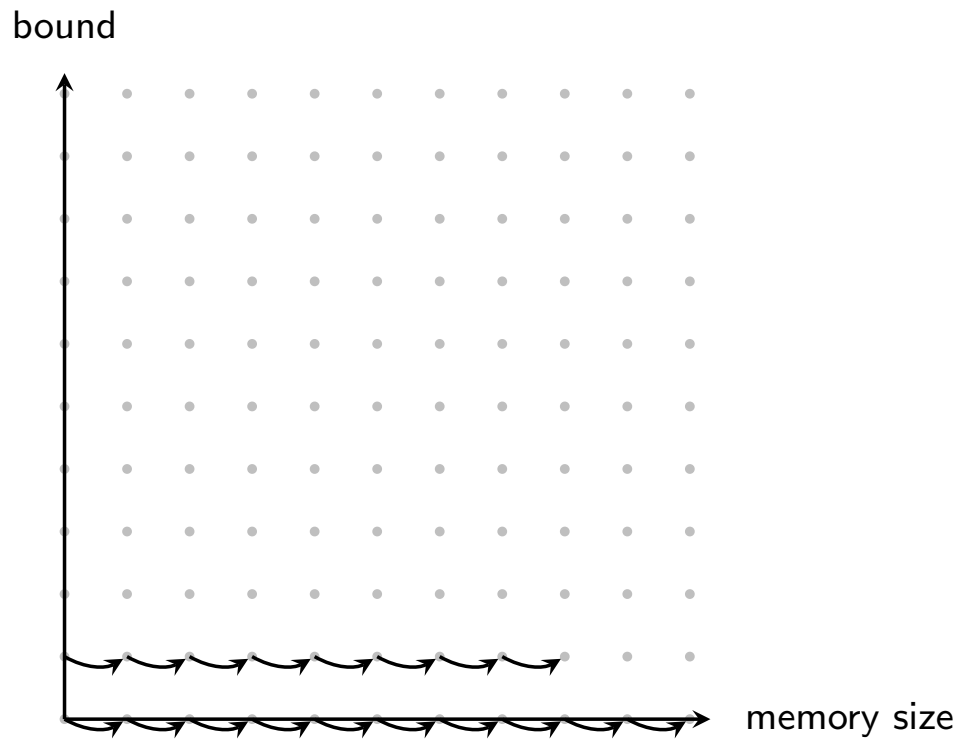

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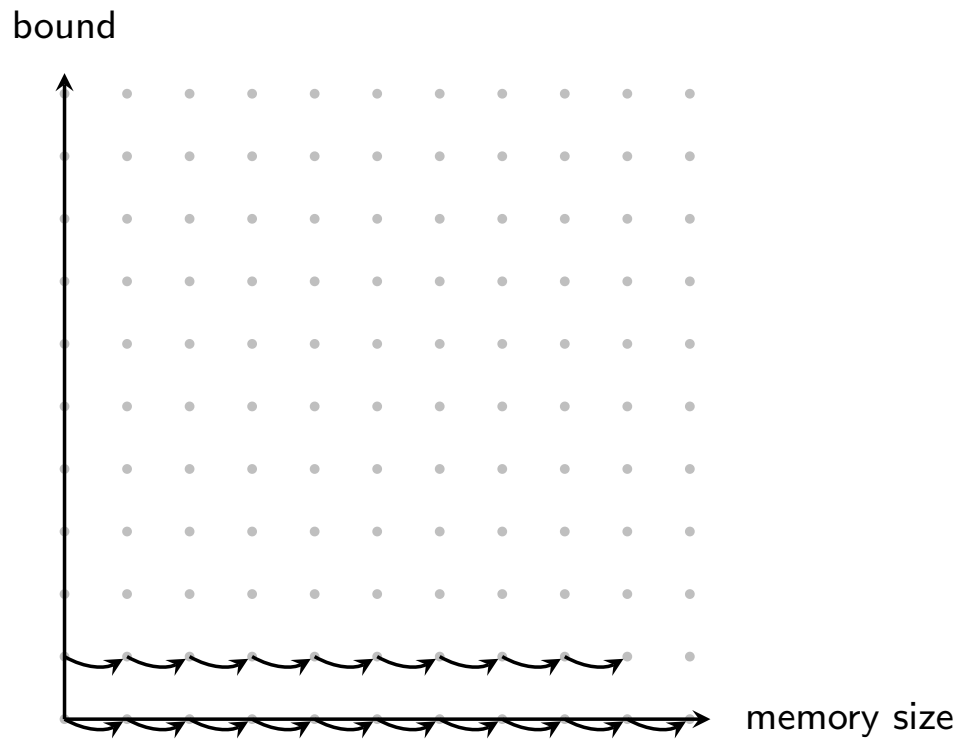

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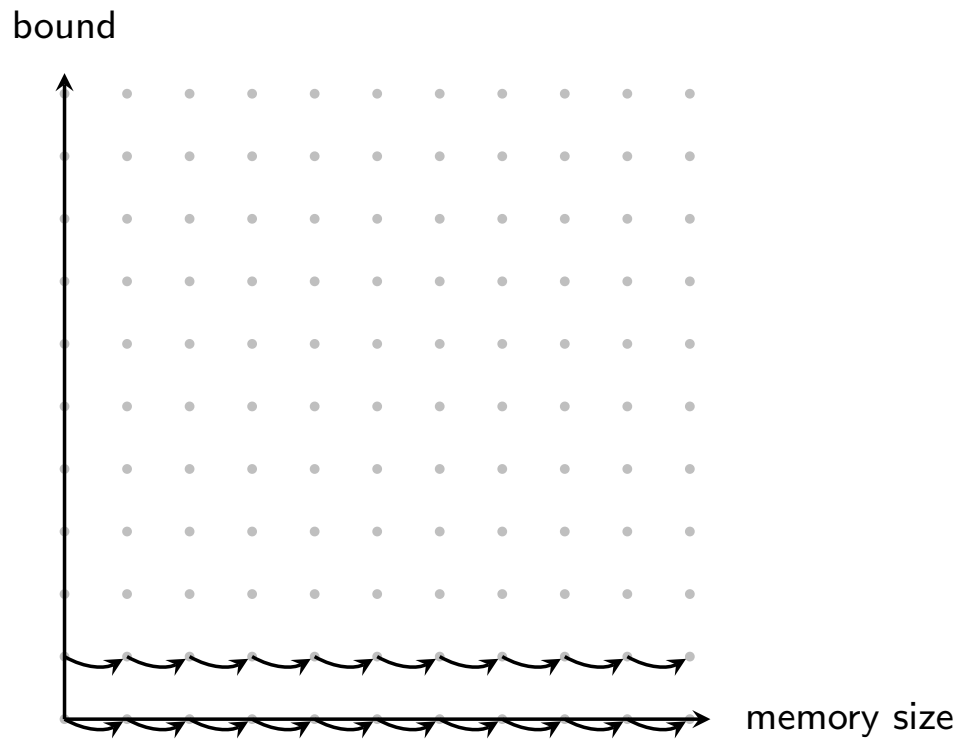

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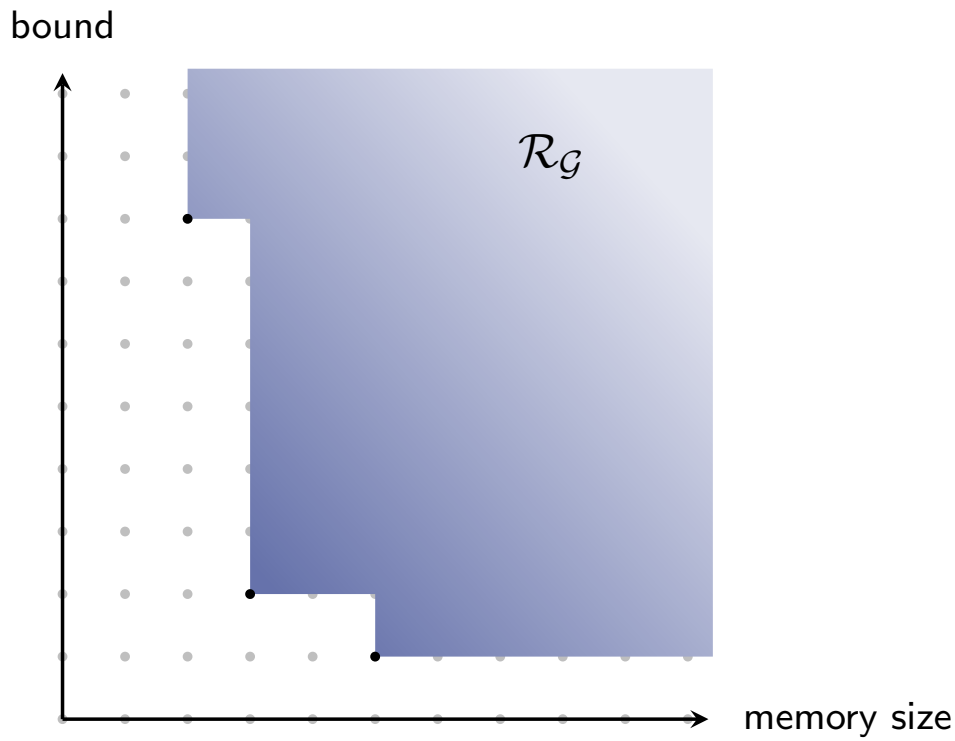

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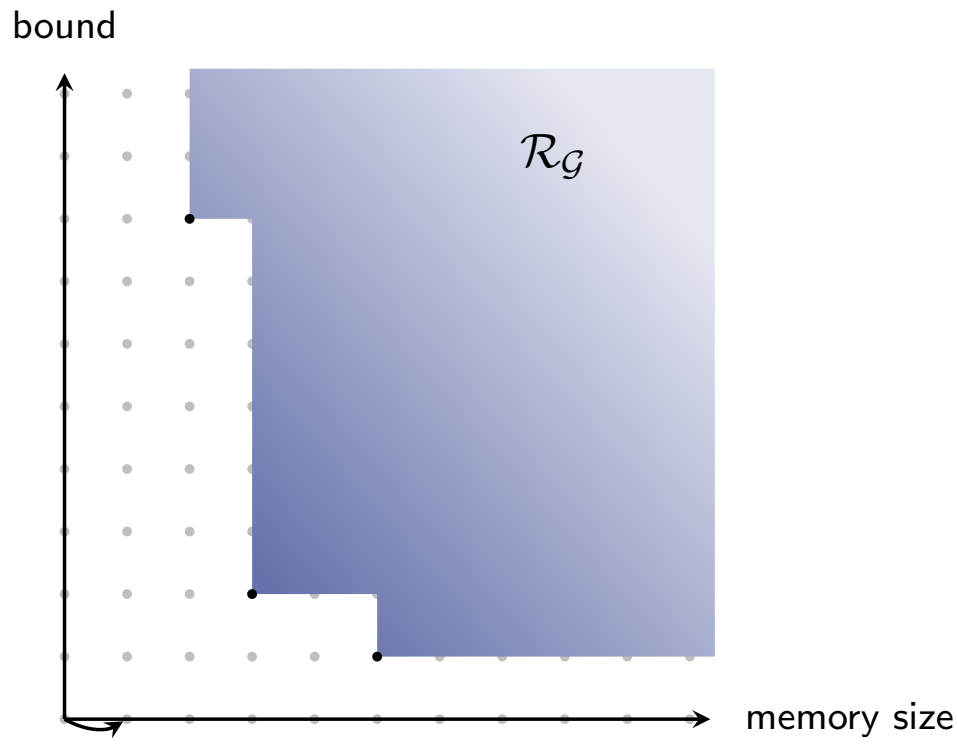
Tradeoffs



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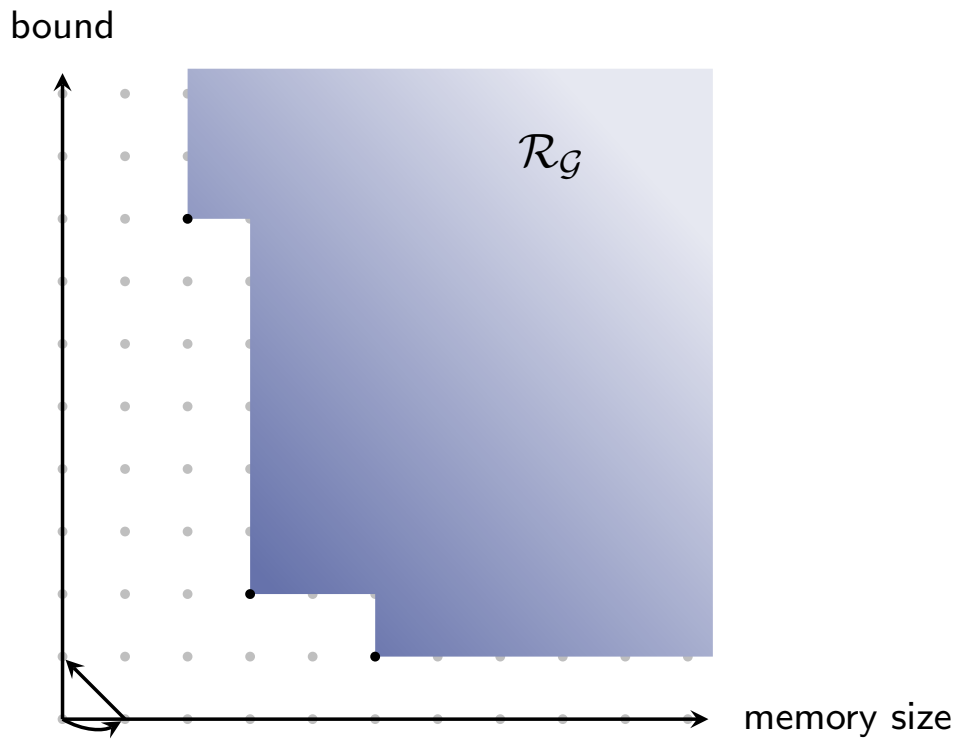

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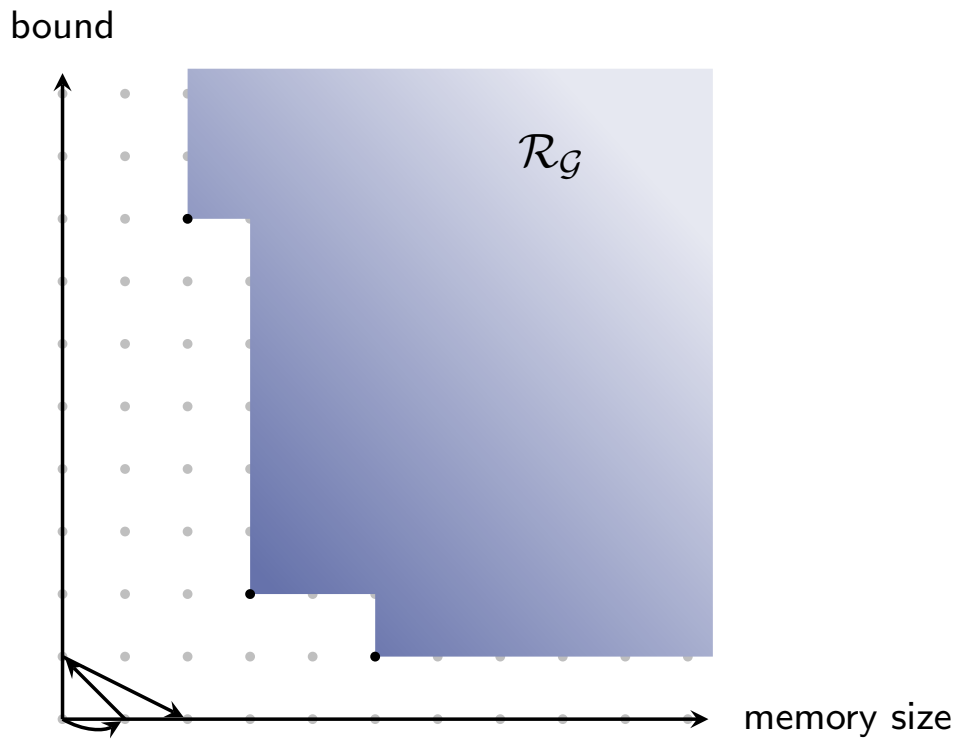

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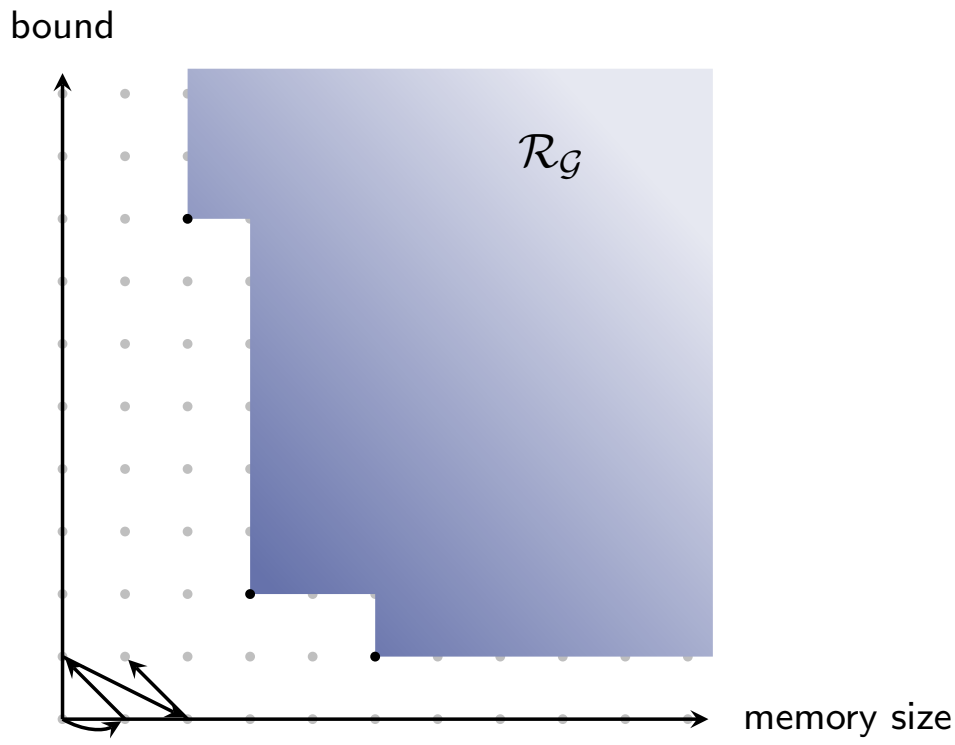

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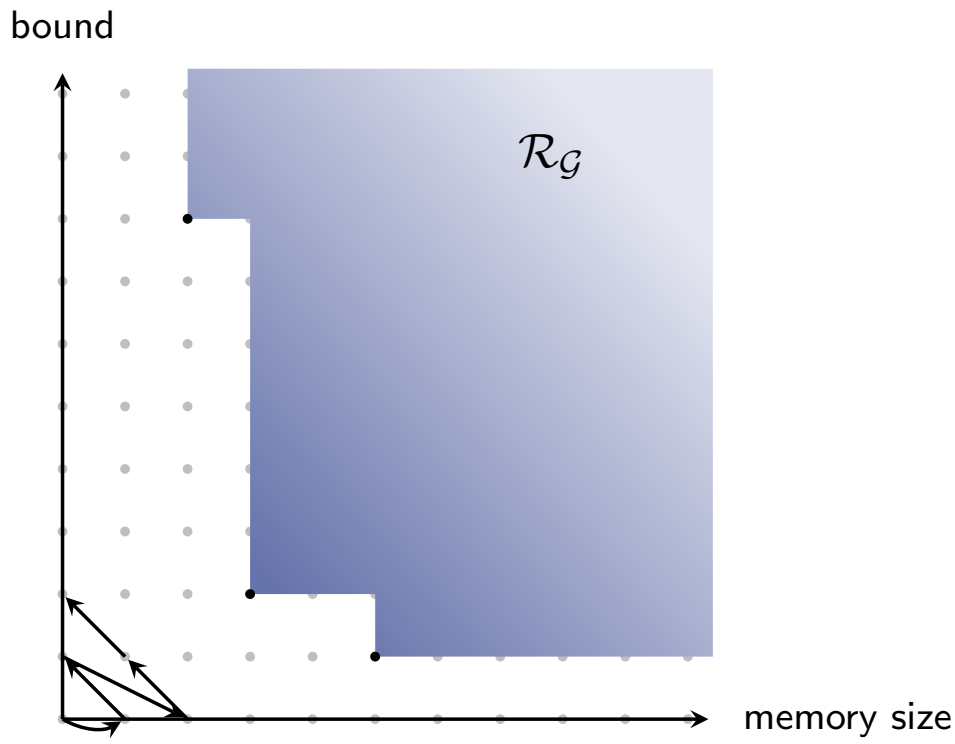

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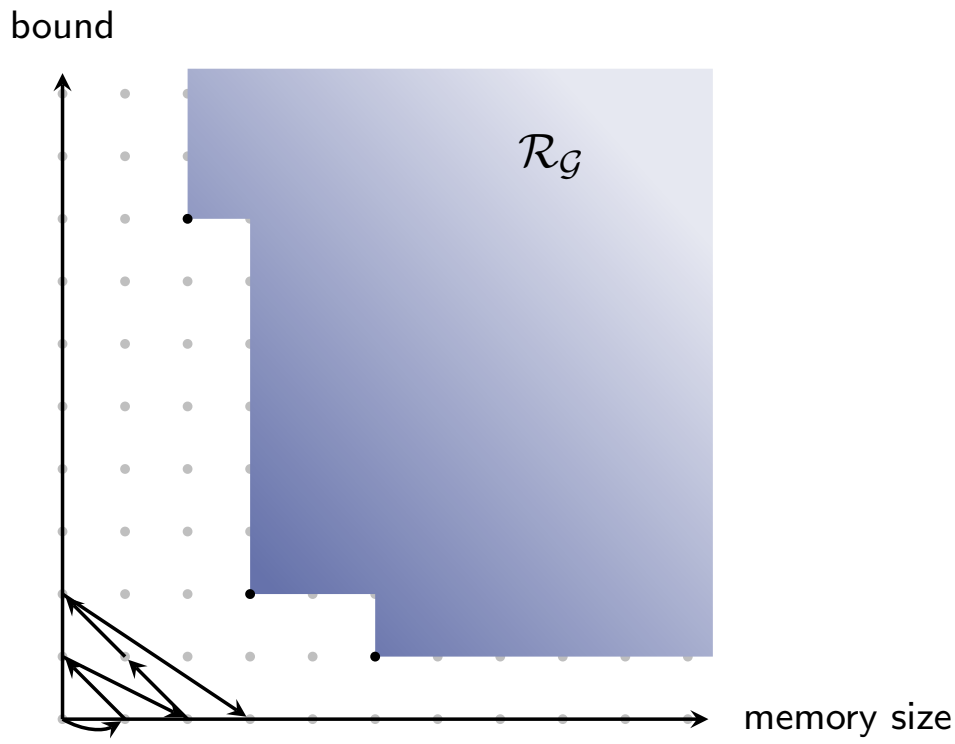

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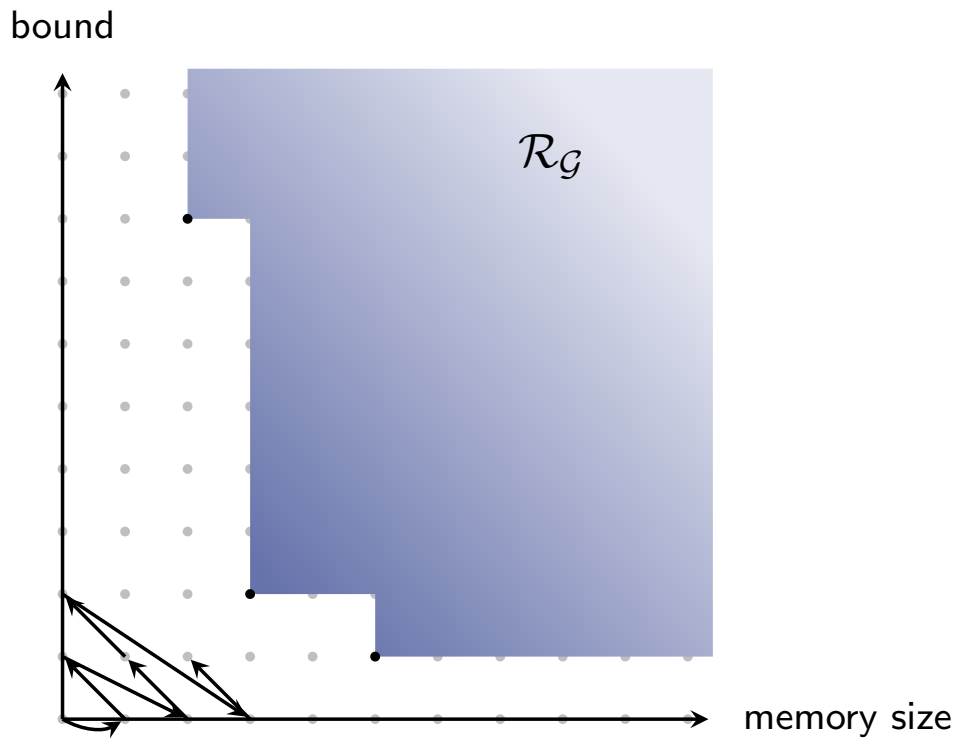

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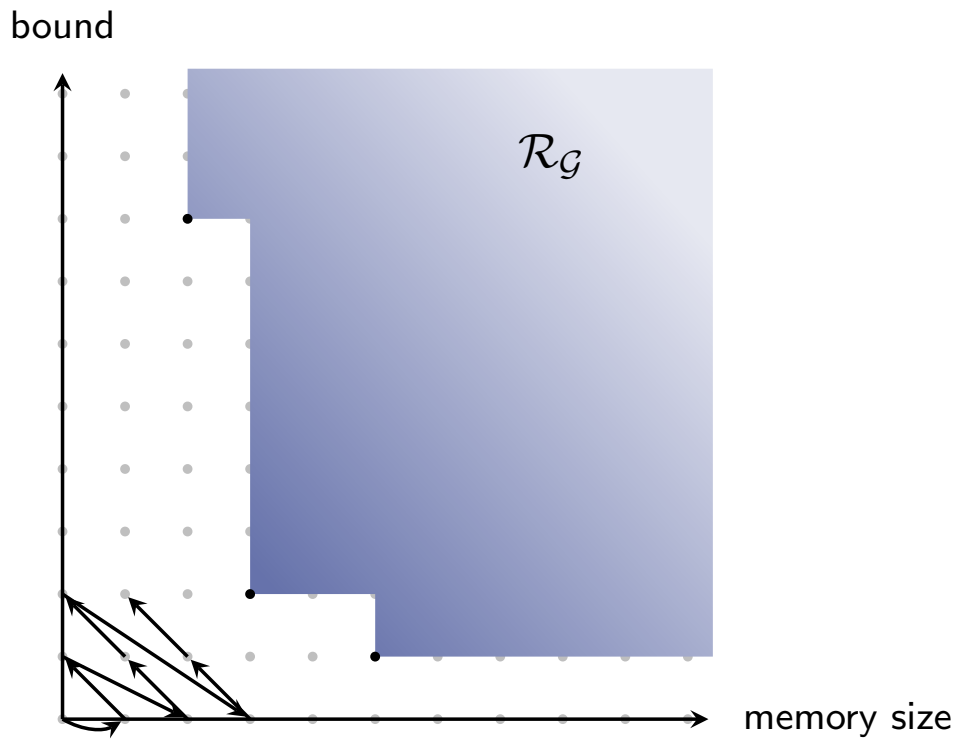

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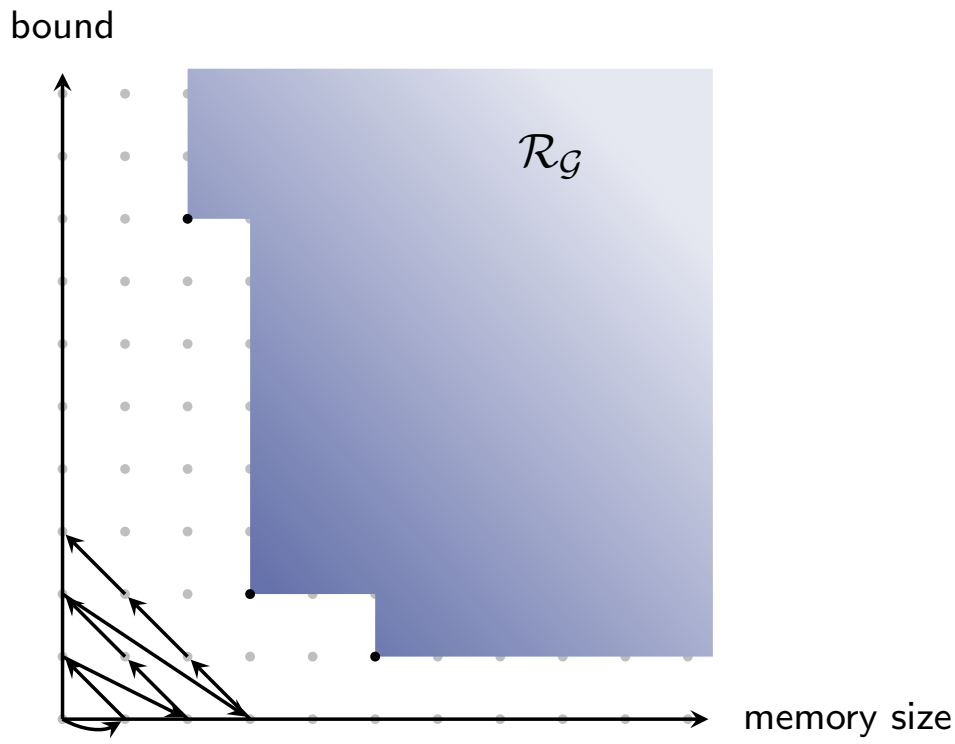

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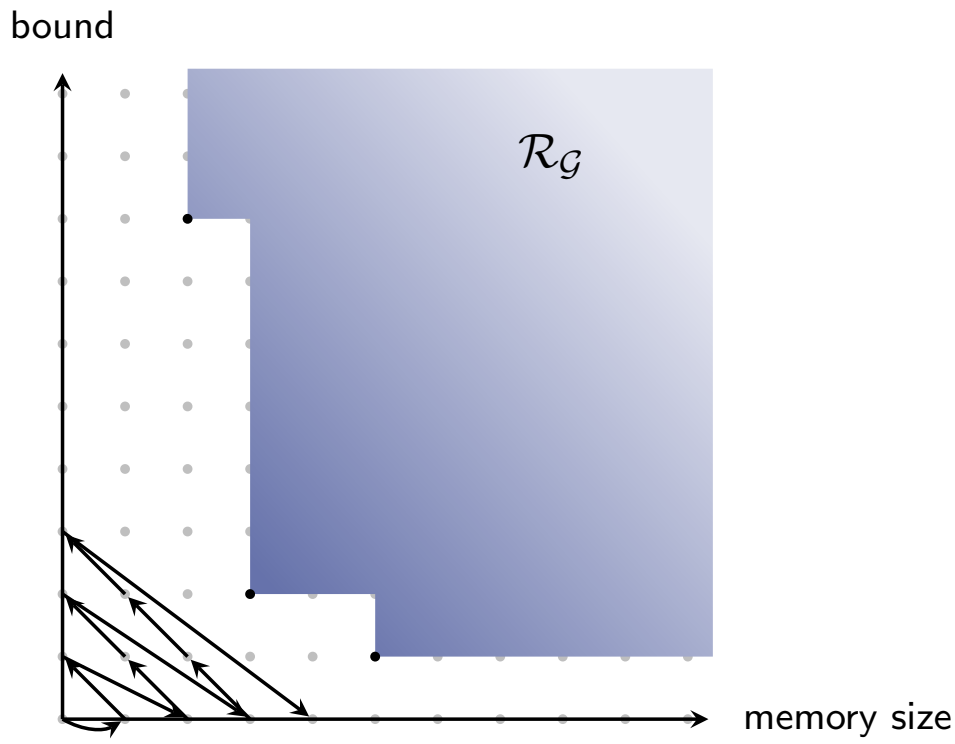

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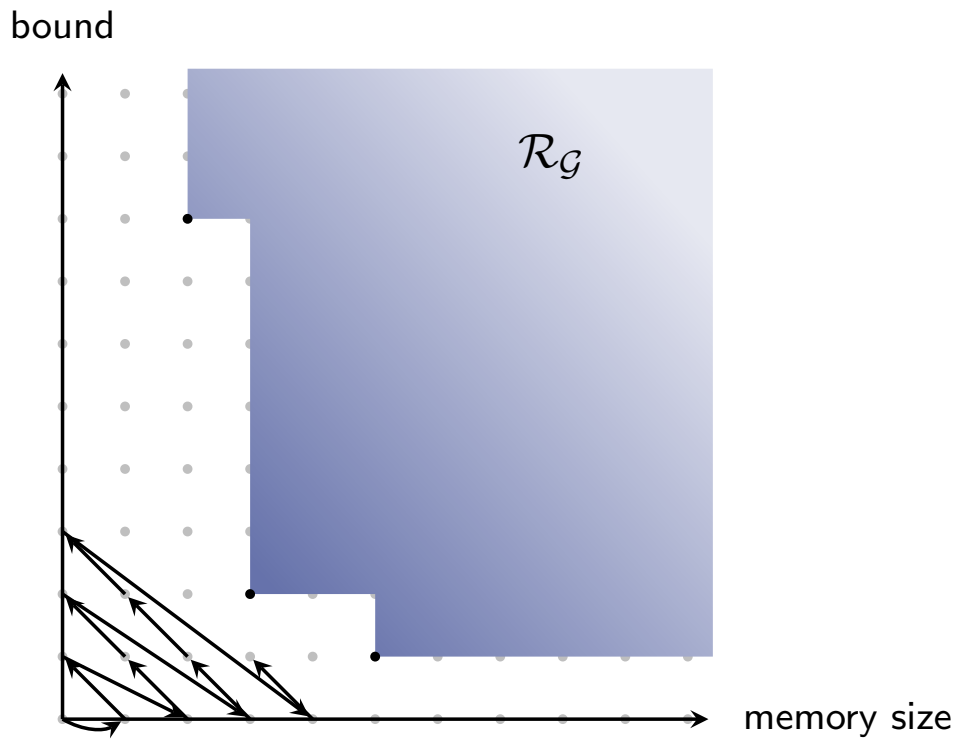

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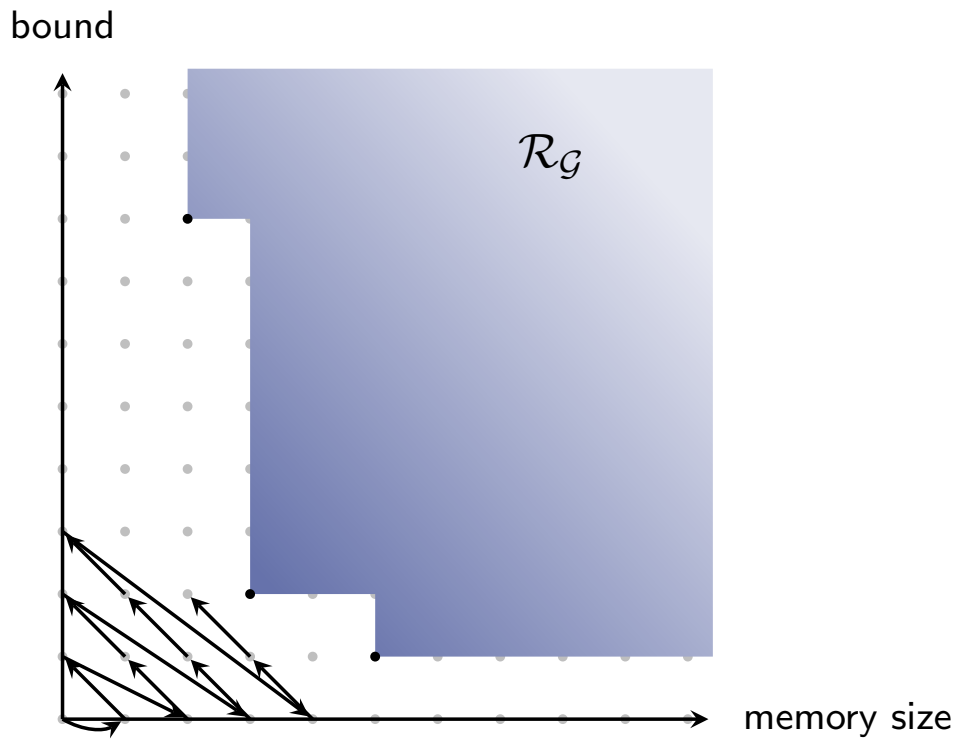

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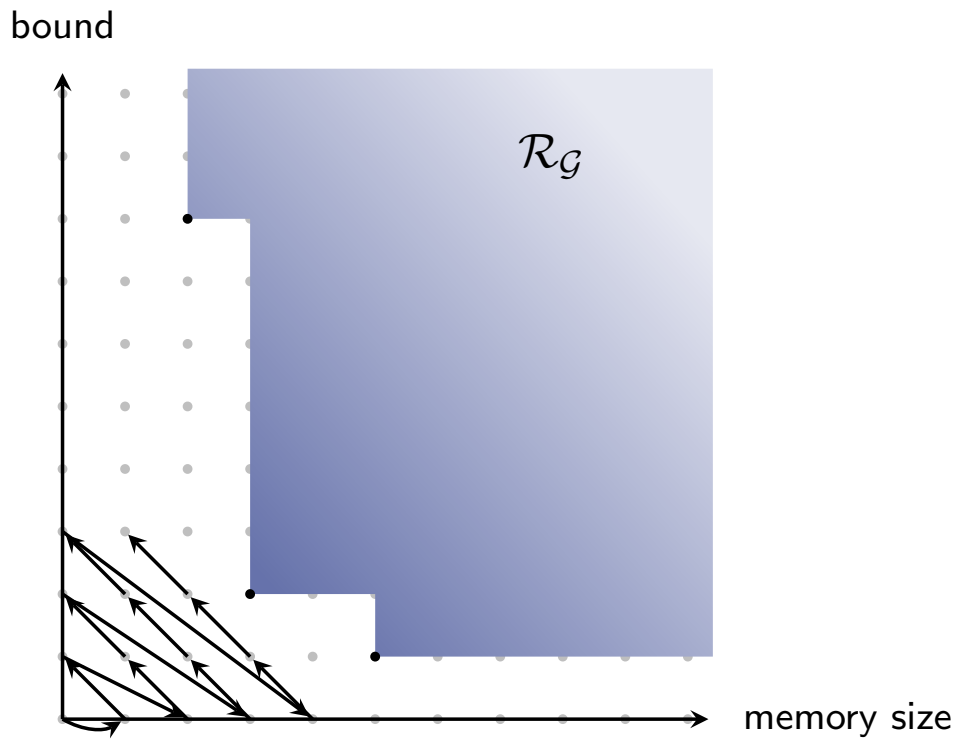

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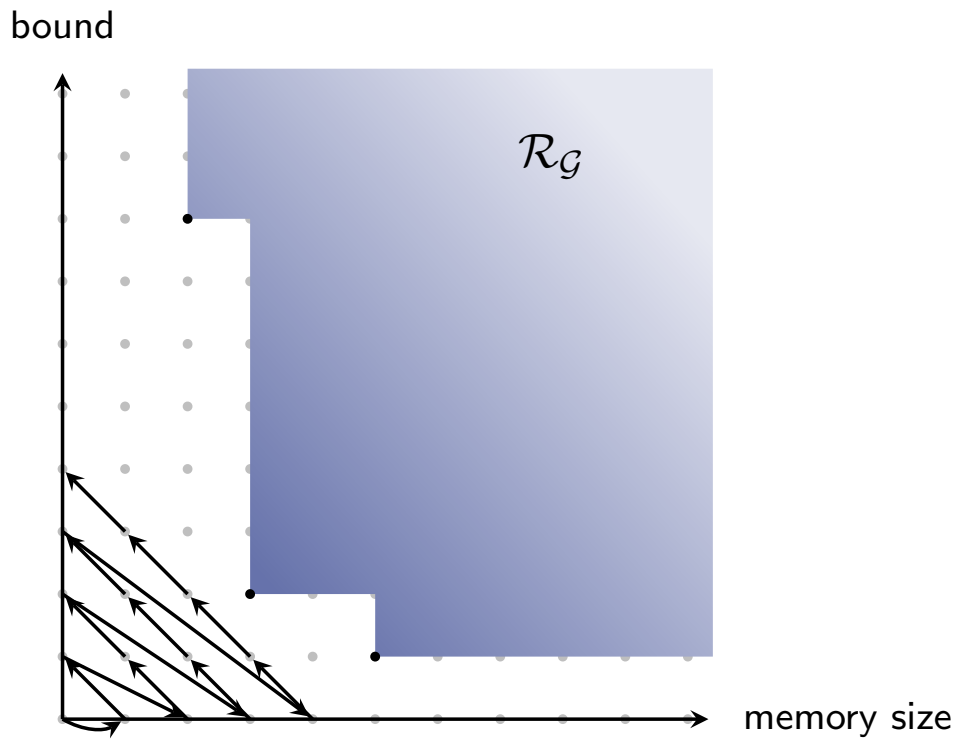

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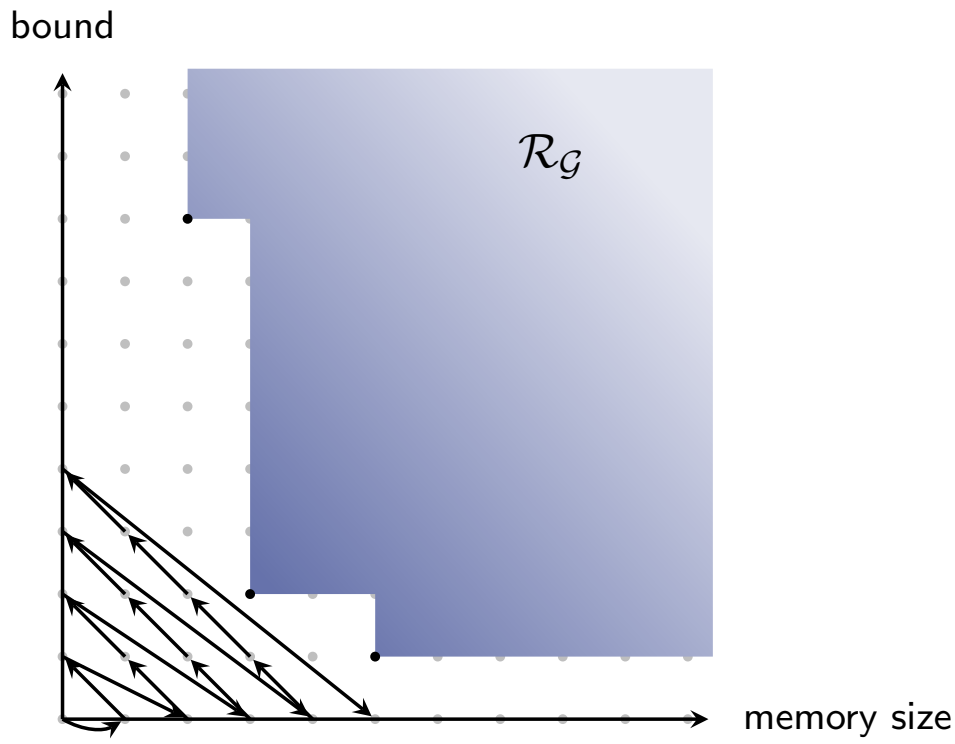

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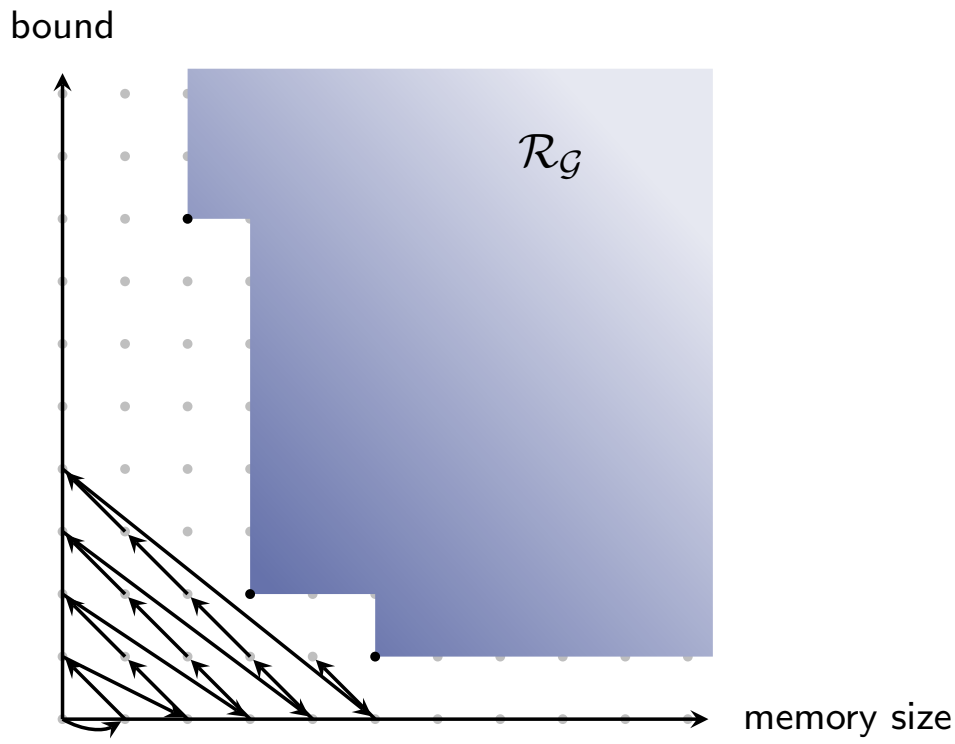

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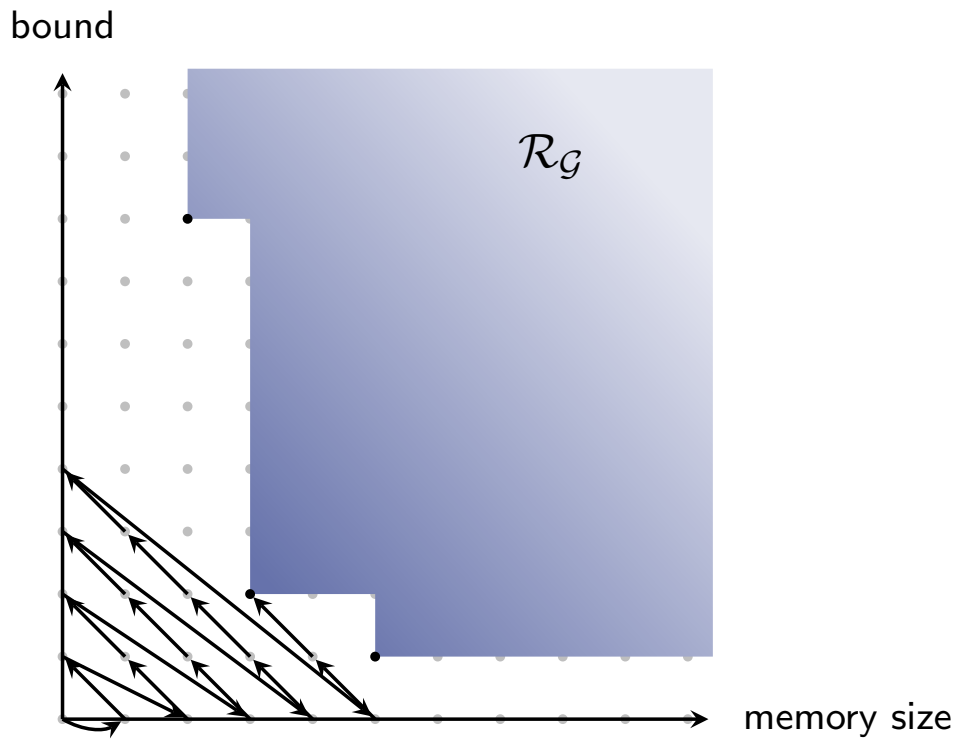

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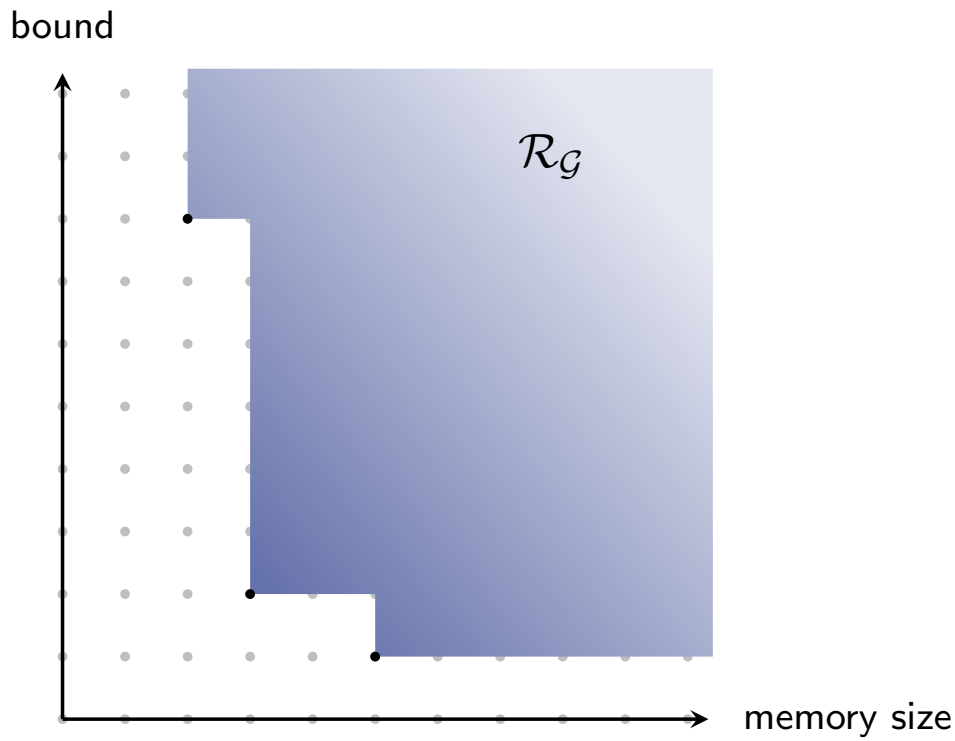
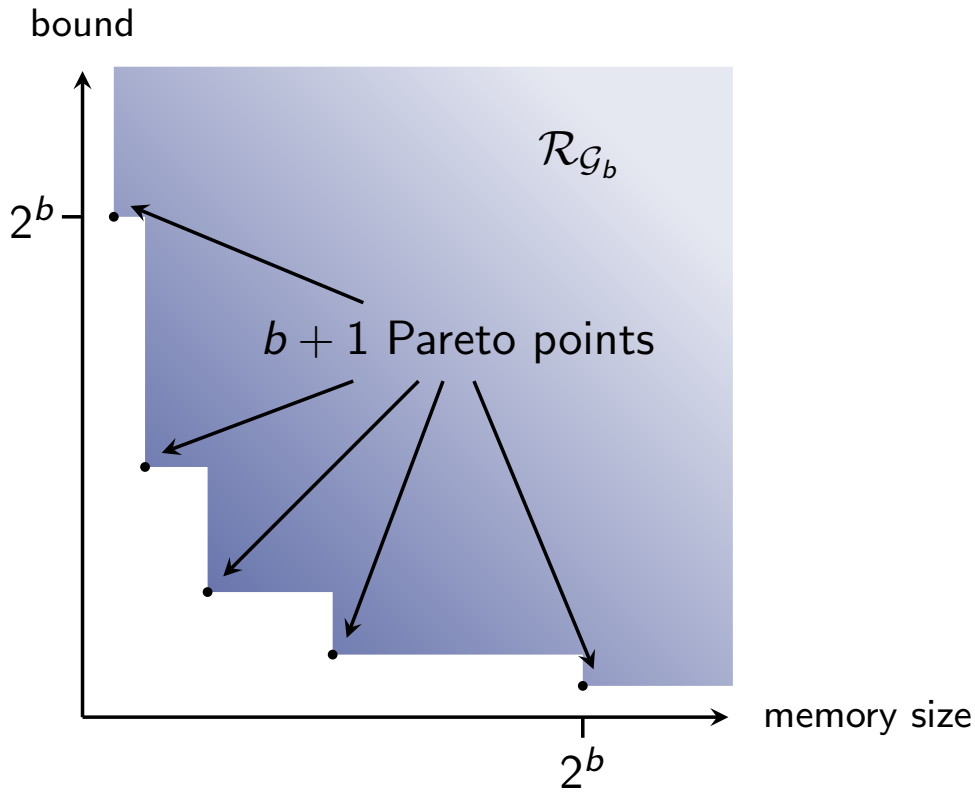


Lemma

1. *If* $(n, k) \in \mathcal{R}_\mathcal{G}$, *then* $(n, n2^{|\varphi|}) \in \mathcal{R}_\mathcal{G}$.
2. *If* $(n, k) \in \mathcal{R}_\mathcal{G}$, *then* $(2^{2^{k|\varphi|}}, k) \in \mathcal{R}_\mathcal{G}$.

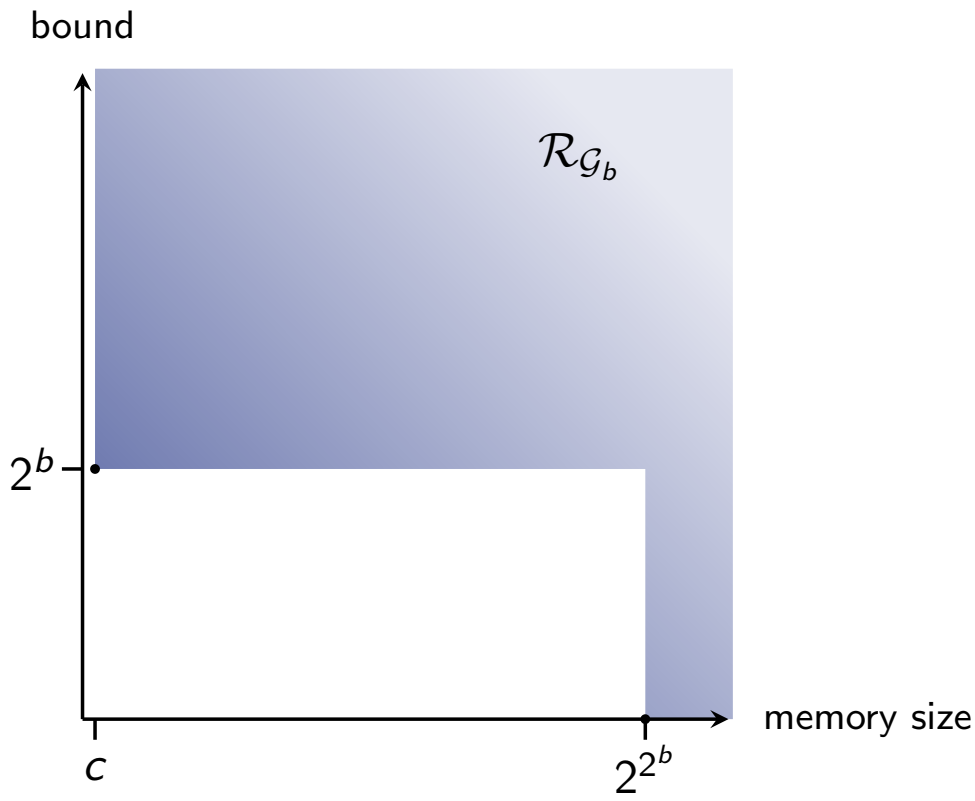
Tradeoffs



Theorem (TWZ16)

For every b, there is a game $\mathcal{G}_b$ of size $\mathcal{O}(b)$ such that $(2^j, 2^{b-j})$ is a Pareto point for every $j \leq b$.

Tradeoffs



Theorem (TWZ16)

For every b, there is a game $\mathcal{G}_b$ of size $\mathcal{O}(b)$ such that $(2^{2^b}, 0)$ and $(c, 2^b)$ are Pareto points for some constant c.

An Example

Five clients:

1. Answer every request of client 1 promptly: $\mathbf{G}\,(\mathtt{r}_1 \rightarrow \mathbf{F_P}\,\mathtt{g}_1)$
2. Answer every other request eventually: $\bigwedge_{i>1} \mathbf{G}\,(\mathtt{r}_i \rightarrow \mathbf{F}\,\mathtt{g}_i)$
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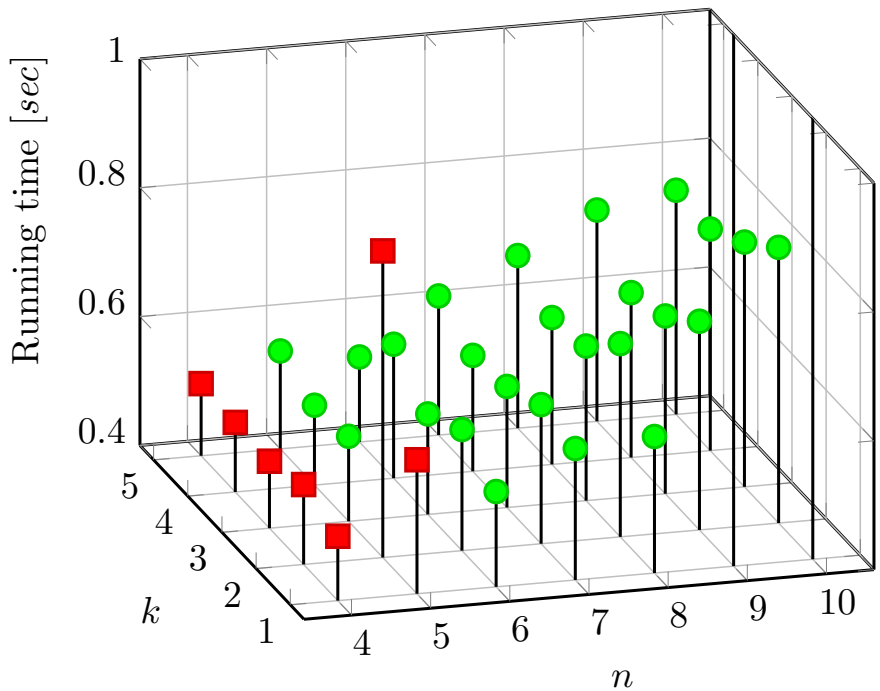

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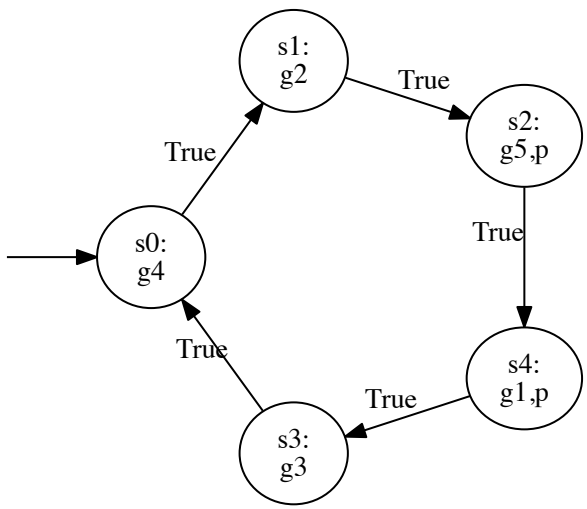
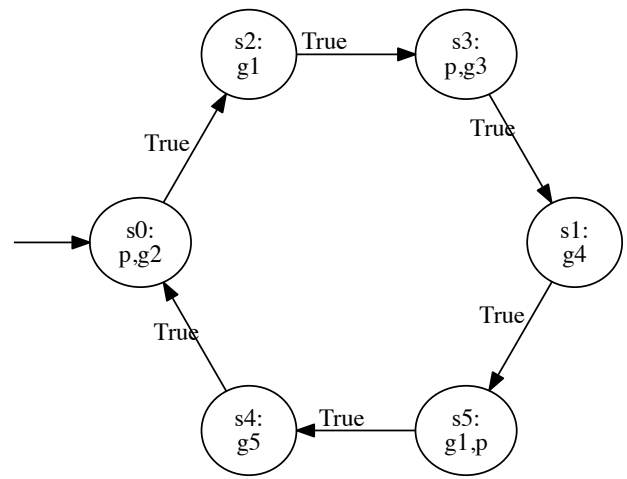
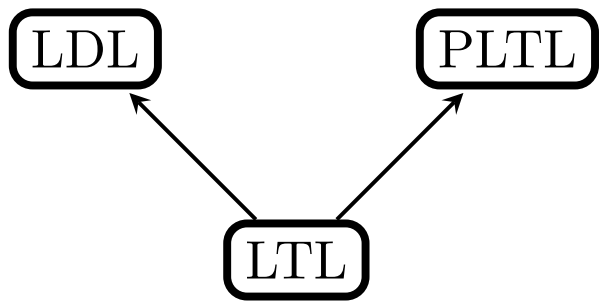


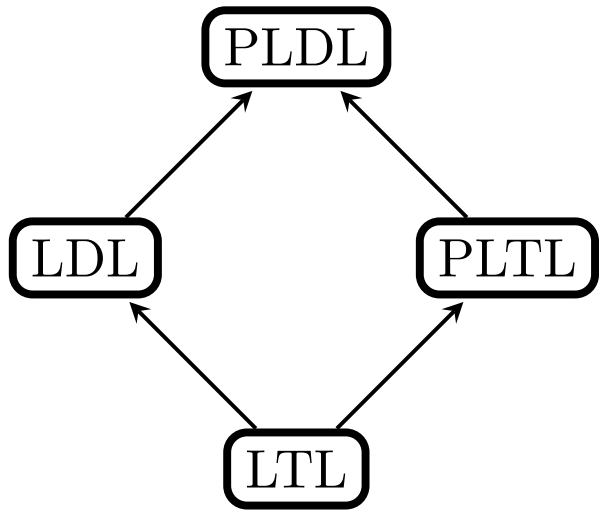


Generalizations



Generalizations

- Parametric LDL **[FZ14]**: full expressive power of the ω-regular languages and bounded operators.



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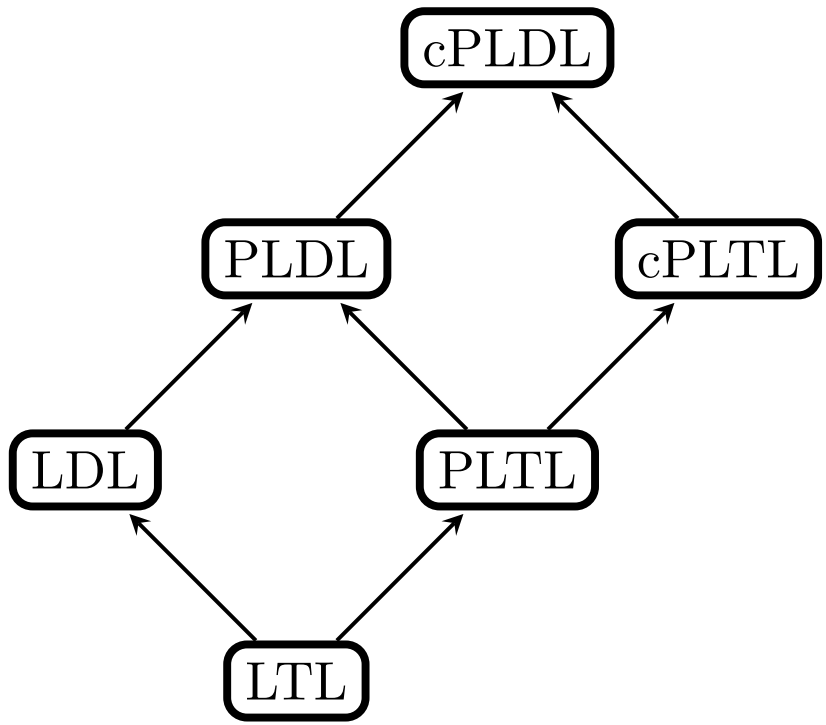

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Logic	Complexity	Memory Pl. 0 & 1
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- Visibly LDL **[WZ15]**: full expressive power of the ω-visibly pushdown languages.

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cPLDL	2ExpTime-complete	Doubly-exponential
VLDL	3ExpTime-complete	pushdown transducer

Outline

1. Playing Optimally in Variations of Parity Games
2. Playing (Approximatively) Optimally in LTL Games
3. **More Tradeoffs**
4. Conclusion

Delay Games

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Theorem (KZ15)

Solving ω-regular delay games is $\textsc{ExpTime}$*-complete and exponential lookahead is always sufficient and in general necessary.*

Best previous result: in $2\textsc{ExpTime}$ and doubly-exponential upper bound, no non-trivial lower bounds **[HKT10]**.

Delay Games

- **Landweber & Hosch**: Allow one player to delay her moves to obtain a lookahead on the opponent's moves.
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More results (incomplete)

- Solving LTL delay games is $3\mathrm{ExpTime}$-complete, triply-exponential lookahead sufficient and necessary **[KZ16]**.
- Lookahead can be traded for quality in delay games with finitary parity conditions **[Z17]**.
- Finite-state strategies for delay games **[WZ18]**: lookahead can be traded for memory.

More Results

In the thesis, but not covered in this talk:

- Solving an open problem on average-energy games **[BHMRZ17]**
- Optimal strategies for request-response games **[HTWZ15]**
- Distributed synthesis for PROMPT–LTL **[JTZ16]**
- A first-order logic for Hyperproperties **[FZ17]**
- Context-free delay games **[FLZ11]**
- Borel determinacy for delay games **[KZ15]**
- Delay games with WMSO+U winning conditions **[Z15]**

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- Lookahead allows to improve strategies and decrease memory requirements.

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- Lookahead allows to improve strategies and decrease memory requirements.

$\Rightarrow$ **Need to take tradeoffs into account when solving games.**