Game-based Model-Checking of HyperLTL

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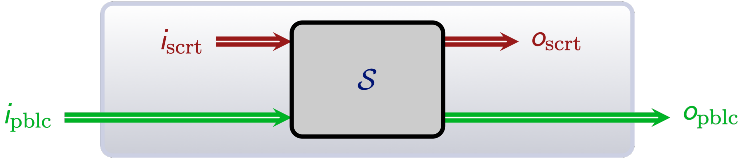
Aalborg University

Panhellenic Logic Symposium 2026

Joint work with Sarah Winter
(IRIF & Université Paris Cité)

Reactive Systems



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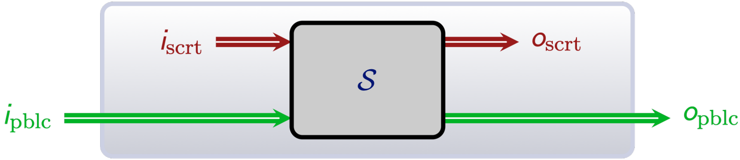

Trace-based view on $\mathcal{S}$: observe execution traces, i.e., infinite sequences over 2^{AP} for some set AP of atomic propositions.

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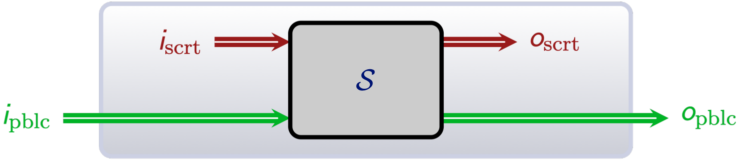

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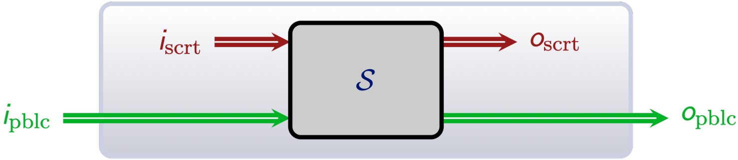

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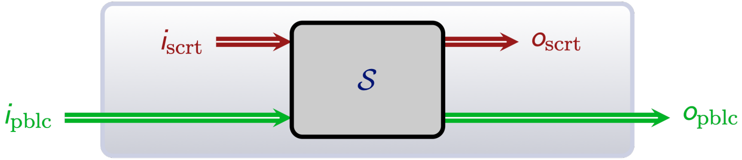

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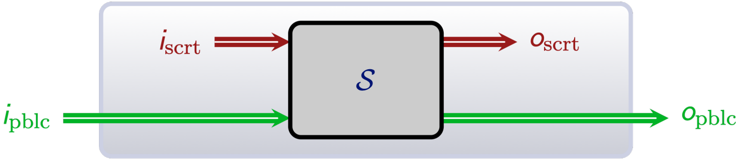

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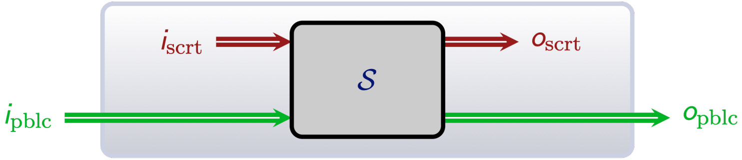

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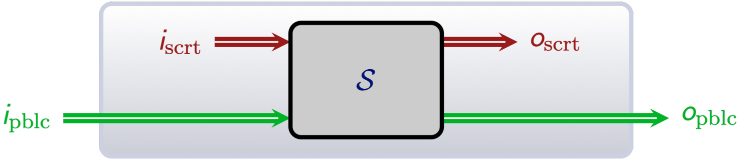

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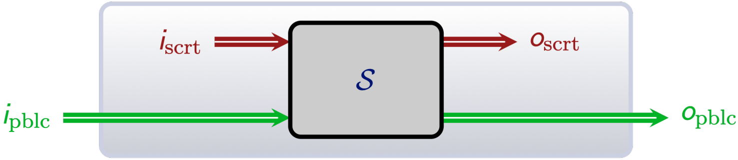

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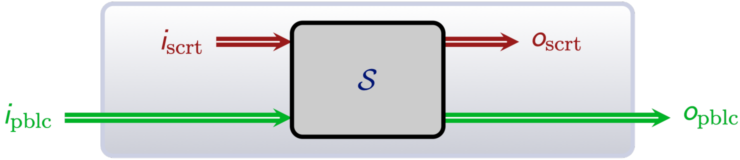
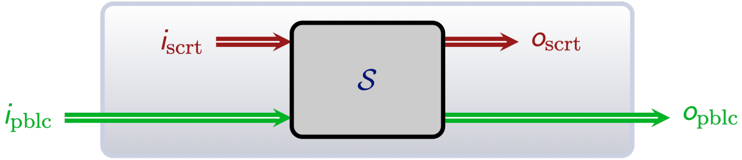


Typical specifications:

Reactive Systems



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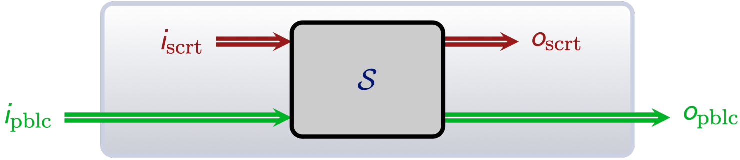

Typical specifications:

- $\mathcal{S}$ terminates
- $\mathcal{S}$ terminates within a uniform time bound

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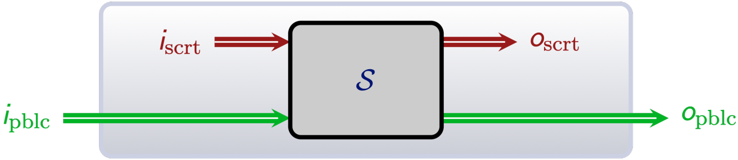

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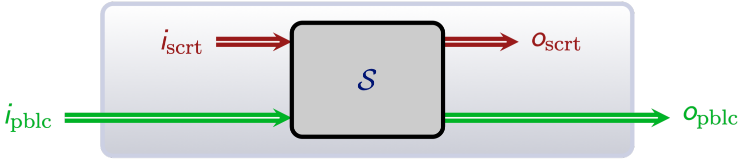

Typical specifications:

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- Noninterference for nondeterministic systems: for all traces t, t' of $\mathcal{S}$ there exists a trace t'' of $\mathcal{S}$ such that t'' and t coincide on their projection to public inputs and outputs and t'' and t' coincide on their projection to secret inputs.

Trace Properties vs. Hyperproperties

Definition

A trace property $T \subseteq (2^{\mathrm{AP}})^\omega$ is a set of traces. A system $\mathcal{S}$ satisfies T, if $\mathrm{Traces}(\mathcal{S}) \subseteq T$.

Example: The set of traces where `term` holds at least once.

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Definition

A hyperproperty $H \subseteq 2^{(2^{\mathrm{AP}})^\omega}$ is a set of sets of traces. A system $\mathcal{S}$ satisfies H if $\mathrm{Traces}(\mathcal{S}) \in H$.

Example: The set $\{T \subseteq T_n \mid n \in \mathbb{N}\}$ where T_n is the trace property containing the traces where `term` holds at least once within the first n positions.

LInear Temporal Logic in One Slide

Syntax

$$\varphi ::= a \mid \neg\varphi \mid \varphi \vee \varphi \mid \mathbf{X}\,\varphi \mid \varphi\,\mathbf{U}\,\varphi \qquad \text{where } a \in \mathrm{AP}$$

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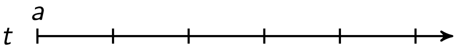

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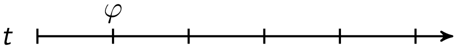

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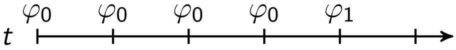

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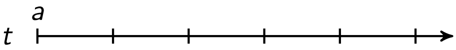

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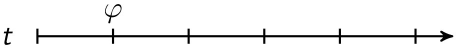

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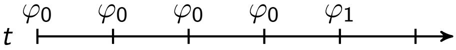

Syntactic Sugar

- $\mathbf{F}\,\psi = \mathbf{tt}\,\mathbf{U}\,\psi$
- $\mathbf{G}\,\psi = \neg\,\mathbf{F}\,\neg\psi$

HyperLTL

HyperLTL = LTL + trace quantification

$$\varphi ::= \exists \pi.\ \varphi \mid \forall \pi.\ \varphi \mid \psi$$
$$\psi ::= a_\pi \mid \neg\psi \mid \psi \vee \psi \mid \mathbf{X}\,\psi \mid \psi\,\mathbf{U}\,\psi$$

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- Prenex normal form, but
- closed under Boolean combinations.

Examples

- Noninterference:

$$\forall \pi.\ \forall \pi'.\ \mathbf{G}((i_{\mathsf{pblc}})_\pi \leftrightarrow (i_{\mathsf{pblc}})_{\pi'}) \rightarrow \mathbf{G}((o_{\mathsf{pblc}})_\pi \leftrightarrow (o_{\mathsf{pblc}})_{\pi'})$$

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- Noninterference for nondeterministic systems:

$$\begin{aligned}\forall\pi.\ \forall\pi'.\ \exists\pi''.\ & \mathbf{G}((i_{\mathsf{pblc}})_\pi \leftrightarrow (i_{\mathsf{pblc}})_{\pi''}) \wedge \\ & \mathbf{G}((o_{\mathsf{pblc}})_\pi \leftrightarrow (o_{\mathsf{pblc}})_{\pi''}) \wedge \\ & \mathbf{G}((i_{\mathsf{scrt}})_{\pi'} \leftrightarrow (i_{\mathsf{scrt}})_{\pi''})\end{aligned}$$

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- $\mathcal{S}$ terminates within a uniform time bound. Not expressible in HyperLTL.

Applications

- Uniform framework for information-flow control
  - Does a system leak information?
- Symmetries in distributed systems
  - Are clients treated symmetrically?
- Error resistant codes
  - Do codes for distinct inputs have at least Hamming distance d?
- "Software doping"
  - Think emission scandal in the automotive industry
- Network verification
  - Latency and congestion of computer networks

There are prototype tools for model checking, satisfiability checking, runtime verification, and synthesis.

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The HyperLTL model-checking problem:

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Theorem (Clarkson et al. '14, Rabe '16, Mascle & Z. '20)
The HyperLTL model-checking problem is TOWER-complete, with hardness holding for a fixed transition system with 5 states and formulas without nested temporal operators.

Skolem Functions

$\mathcal{S}$ satisfies a formula of the form $\forall\pi.\ \exists\pi'.\ \psi$ iff there is a (Skolem) function $f\colon \mathrm{Traces}(\mathcal{S}) \to \mathrm{Traces}(\mathcal{S})$ such that the assignment

$$[\pi \mapsto t, \pi' \mapsto f(t)]$$

satisfies ψ for all $t \in \mathrm{Tr}(\mathcal{S})$.

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- In general, if $\mathcal{S} \models \varphi$, then Skolem functions for the existentially quantified variables in φ explain why $\mathcal{S} \models \varphi$.
- Dually, if $\mathcal{S} \not\models \varphi$, then $\mathcal{S} \models \neg\varphi$ and Skolem functions for the existentially quantified variables in $\neg\psi$ are a "counterexample" for $\mathcal{S} \not\models \varphi$.

Computable Skolem Functions

To interpret and algorithmically handle Skolem functions, we represent them by finite automata with output (transducers).

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Example

Consider $\mathcal{S}$ with $\mathrm{Traces}(\mathcal{S}) = (2^{\{a\}})^\omega$, which satisfies

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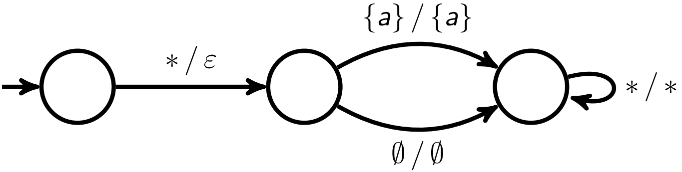
The following transducer represents a Skolem function for π':



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$$f(t) = \begin{cases} \{a\}\emptyset^\omega & \text{if } t \text{ contains an } a \text{ somewhere,} \\ \emptyset^\omega & \text{if } t \text{ does not contain an } a \text{ anywhere.} \end{cases}$$

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- However, this, and any other Skolem function, is not representable by a transducer.

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π_2

π_3

π_4

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Given $\mathcal{S}$ and φ such that $\mathcal{S} \models \varphi$, is $\mathcal{S} \models \varphi$ witnessed by Skolem functions representable by finite transducers?

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- The Skolem function for π_1 may only depend on the trace assigned to π_0, but not those assigned to π_2 and π_4.

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- The information is hierarchical $\Rightarrow$ solving games with ω-regular winning conditions is decidable.

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Results

Theorem (Winter & Z. '24)

There is a block size (effectively computable from $\mathcal{S}$ and φ) such that the following are equivalent:

1. *The coalition of players for the existentially quantified variables in φ has a collection of winning strategies.*
2. *$\mathcal{S} \models \varphi$ is witnessed by Skolem functions implemented by transducers.*

Furthermore, the game is effectively solvable and the transducers can be effectively computed.

Prophecies

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- In the example above, the prophecy is the language of words containing an a somewhere.

Results

What about arbitrary quantifier prefixes?

Theorem (Winter & Z. '25)

Given $\mathcal{S}$ and φ, there is an effectively computable and solvable imperfect information game such that the following are equivalent:

1. *The coalition of players for the existentially quantified variables in φ has a collection of winning strategies.*
2. $\mathcal{S} \models \varphi$.

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- Future work:
  - More expressive logics
  - Infinite-state systems
  - Complexity analysis