Martin Zimmermann
Aalborg University

History-deterministic Pushdown Automata

Based on joint work with Shibashis Guha, Ismaël Jecker and Karoliina Lehtinen

IRIF, Paris, August 2024

Let's Get Started

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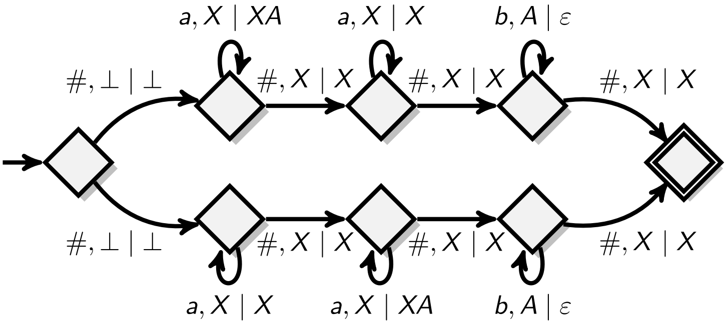

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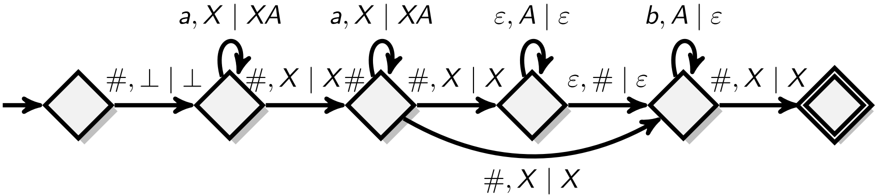

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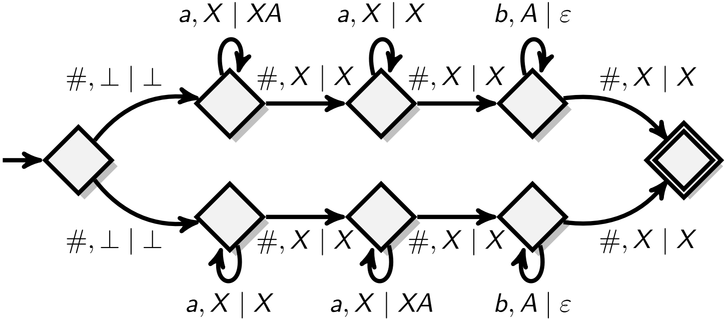
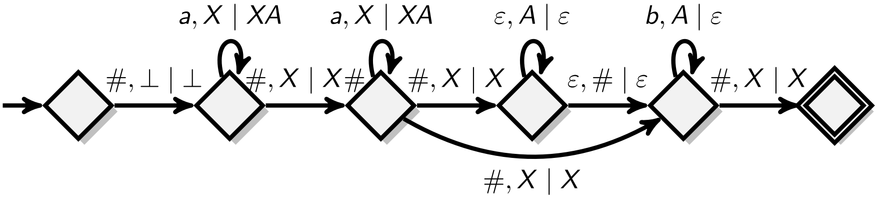
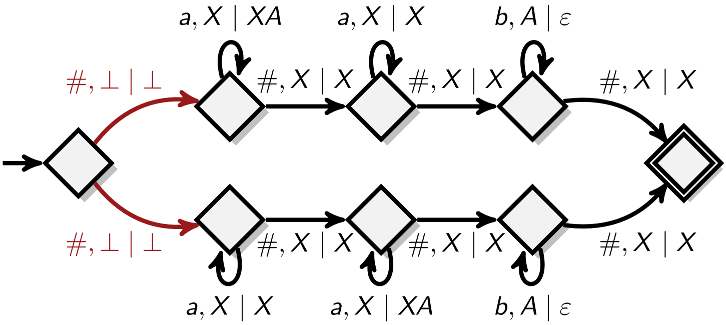




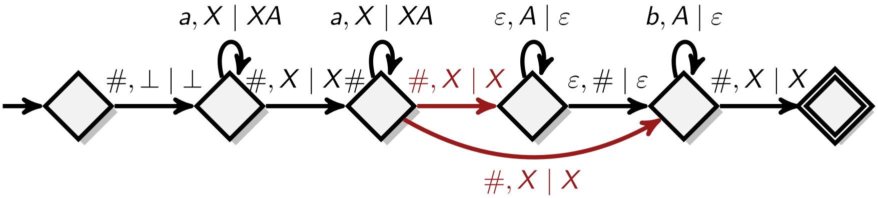
Not All Nondeterminism is Equal

- In the first automaton, one needs to know the whole word to make the nondeterministic choice.



Not All Nondeterminism is Equal

- In the first automaton, one needs to know the whole word to make the nondeterministic choice.
- In the second automaton, one only needs to know the prefix processed so far to make the nondeterministic choice.



Another Language

- Let $I = \{0, +, -\}$ and define the energy level $EL \colon I^* \to \mathbb{Z}$ of a finite word over I as

$$EL(w) = |w|_+ - |w|_-,$$

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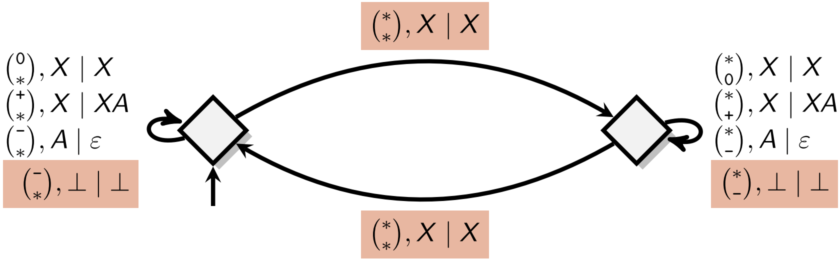
- A word $w \in I^\omega$ is eventually safe if it has a safe suffix.
- Let $\Sigma = I \times I$ and

$$L_{es} = \left\{ \binom{w_0}{w_1} \in \Sigma^\omega \;\middle|\; \text{some } w_i \text{ is eventually safe} \right\}.$$

An Automaton for L_{es}



Acceptance condition: Red transitions may only be taken finitely often.

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- *The following problem is* ExpTime*-complete: Does protagonist win a game with deterministic ω-contextfree winning condition?*

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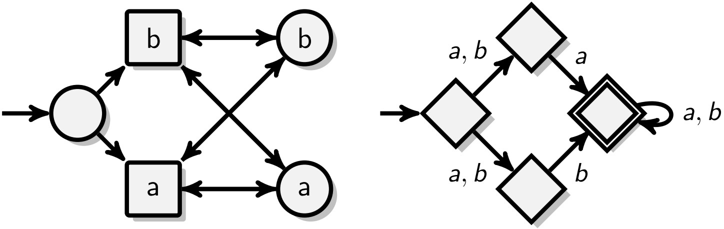

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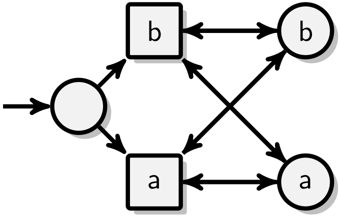

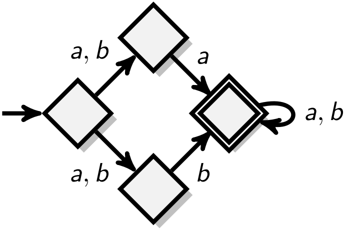

- Every play is winning for protagonist (round vertices), but..
- protagonist loses if she has to resolve nondeterminism on-the-fly.

History-Determinism

Let $\mathcal{P} = (Q, \Sigma, \Gamma, q_I, \Delta, \Omega)$ be a nondeterministic PDA.

- A resolver for $\mathcal{P}$ is a function $r\colon \Delta^* \times \Sigma \to \Delta$ such that for every $w \in L(\mathcal{P})$ the sequence $\tau_0\tau_1\tau_2 \cdots \in \Delta^\omega$ defined as

  $$\tau_n = r(\tau_0 \cdots \tau_{n-1}, w(|(\tau_0 \cdots \tau_{n-1})|_\Sigma))$$

  induces an accepting run of $\mathcal{P}$ on w.

  Here, $|(\tau_0 \cdots \tau_{n-1})|_\Sigma$ denotes the number of letters processed by the transitions $\tau_0 \cdots \tau_{n-1}$ ($\mathcal{P}$ may have ε-transitions), i.e., $w(|(\tau_0 \cdots \tau_{n-1})|_\Sigma)$ is the first letter of w not processed by $\tau_0 \cdots \tau_{n-1}$.

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- Definition slightly more tedious for finite words.

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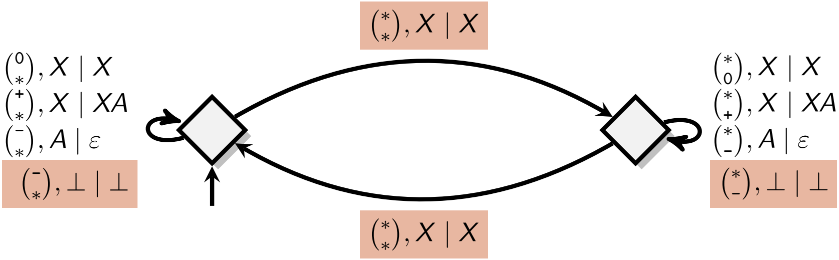

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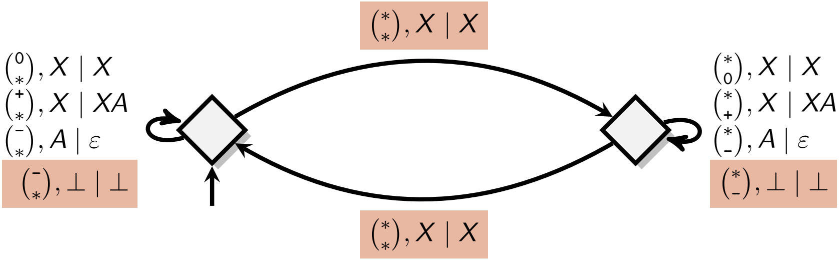

- Let us define a resolver for the ω-PDA above.
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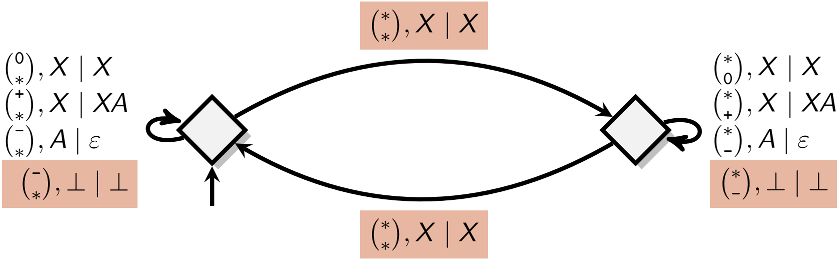

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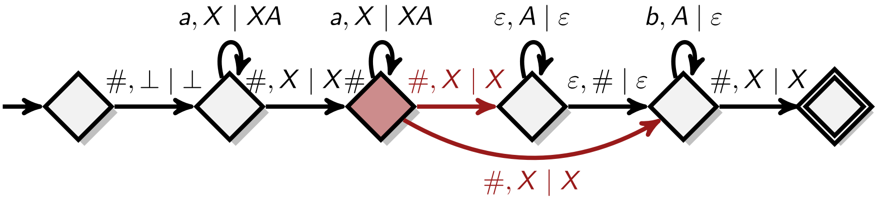
- Then, we define the resolver to guide the run
  - to the left state, if $m^0 \leq m^1$, and
  - to the right state otherwise.

And the Other One

$$L_2 = \{\# \, a^i \, \# \, a^j \, \# \, b^k \, \# \mid k \le i \text{ or } k \le j\}$$



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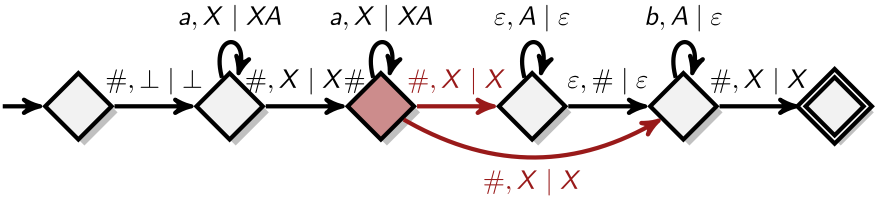

- Let us again define a resolver for the PDA.
- In the red state, after having processed $\# \, a^i \, \# \, a^j$ and the next letter being $\#$, select
  - the upper transition if $j < i$, and
  - the lower transition otherwise.

But Have we Gained Anything?

Theorem (Lehtinen, Z. 2020)

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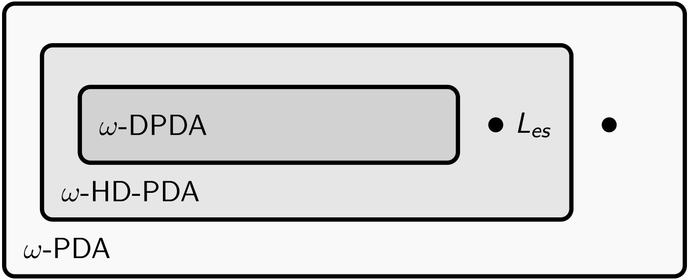
The Big Picture



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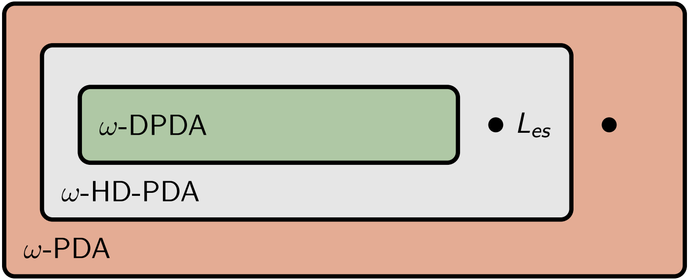

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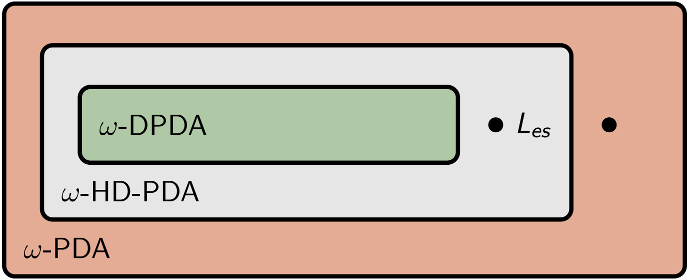

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The following problem is EXPTIME*-complete: Does protagonist win a game with history-deterministic* ω*-contextfree winning condition?*

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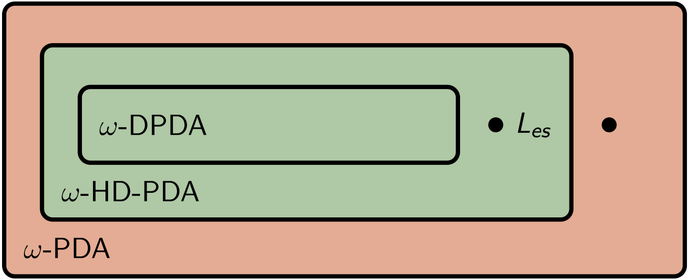

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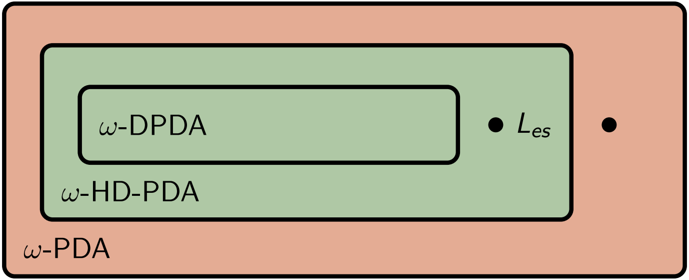

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Corollary

Universality for history-deterministic PDA is in EXPTIME.

Closure Properties

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	$\cap$	$\cup$	$\overline{}$	$\setminus$	h
ω-DPDA	✗	✗	✓	✗	✗
ω-HD-PDA	✗	✗	✗	✗	✗
ω-VPA	✓	✓	✓	✓	✗
ω-PDA	✗	✓	✗	✗	✓

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Theorem (Lehtinen, Z. 2020)

The following problems are undecidable:

1. *Given an ω-PDA $\mathcal{P}$, is $\mathcal{P}$ history-deterministic?*
2. *Is a given ω-PDA $\mathcal{P}$ equivalent to some history-deterministic PDA?*

Similar results hold for PDA on finite words.

Also in the Papers

- A comparison to unambiguous contextfree languages: incomparable (over finite words)

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- The parity-index hierarchy of history-deterministic context-free language: infinite
- A comparison to good-for-games pushdown automata: not equivalent!!!

How Hard is it to Resolve Nondeterminism?

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- *Not every history-deterministic PDA has a resolver that is implementable by a PDA with output.*

Counterexample

- Recall

$$L_2 = \{\# a^i \# a^j \# b^k \# \mid k \leq i \text{ or } k \leq j\}.$$

It has resolver that is implementable by a PDA with output (after processing $\# a^i \# a^j$, we need to check $j < i$).

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Open Problem

Does every history-deterministic PDA have a computable resolver?

Succinctness

Theorem (Valiant 1976)

There is no computable function f with the following property: For every n-state PDA accepting a deterministic contextfree language, there is an equivalent deterministic PDA with $f(n)$ states.

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Theorem (Guha, Jecker, Lehtinen, Z. 2021)

There are exponential and doubly-exponential lower bounds.

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Questions and more open problems?