
The Complexity of Asynchronous HyperLTL

Joint work with Gaëtan Regaud (ENS Rennes, France)

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Decidable

Undecidable

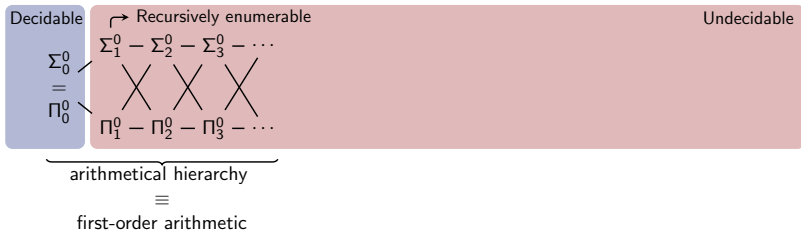
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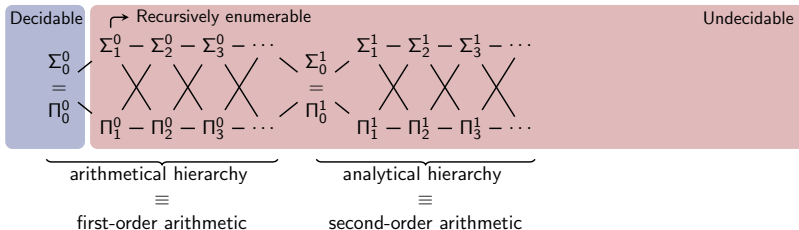
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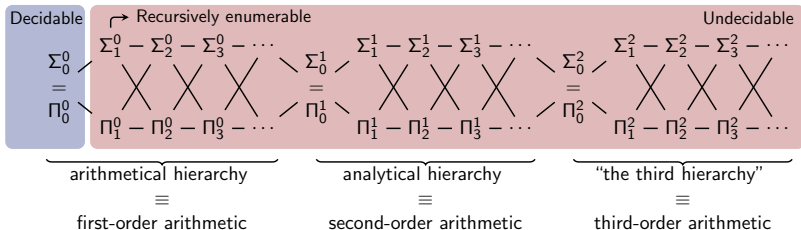
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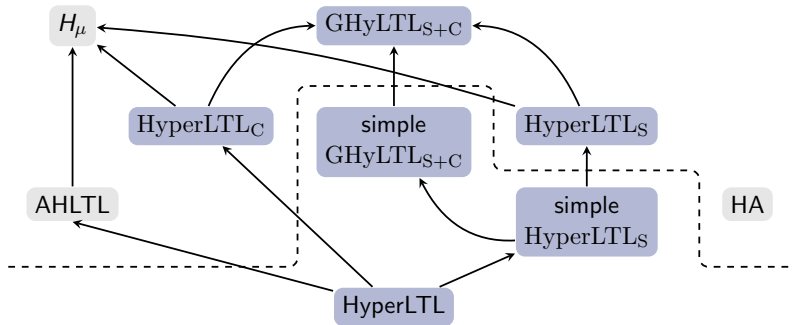
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The second generation of **asynchronous** hyperlogics:



Results for Stuttering and Contexts

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GHyLTL _{S+C}	Σ_1^1 -compl.	T2A-equiv.	T2A-equiv.
HyperLTL _S	Σ_1^1 -compl.	T2A-equiv.	T2A-equiv.
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Trajectories

Trajectories describe how time passes on different traces.

Example

$\pi:$ $\emptyset$ $\emptyset$ $\emptyset$ $\{a\}$ $\emptyset$ $\emptyset$ $\emptyset$ $\emptyset$ $\emptyset$ $\dots$

$\pi':$ $\{b\}$ $\emptyset$ $\{b\}$ $\emptyset$ $\{b\}$ $\emptyset$ $\emptyset$ $\emptyset$ $\emptyset$ $\dots$

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- With trajectory $\{\pi'\}(\{\pi, \pi'\})^\omega$, $\mathbf{F}(a_\pi \wedge b_{\pi'})$ is satisfied.
- With trajectory $\{\pi\}^3(\emptyset)^\omega$, $\mathbf{GF}(a_\pi \wedge b_{\pi'})$ is satisfied.

Asynchronous HyperLTL (AHLTL)

AHLTL extends HyperLTL with **trajectory quantification**:

$$\exists_0 \pi_0 \exists_1 \pi_1 \cdots \exists_{k-1} \pi_{k-1} Q \psi$$

where

- $\exists_0 \pi_0 \exists_1 \pi_1 \cdots \exists_{k-1} \pi_{k-1}$ is a block of trace quantifiers (i.e., $\exists_j \in \{\exists, \forall\}$) as in HyperLTL,
- Q existentially ($Q = E$) or universally ($Q = A$) quantifies a trajectory t for the variables $\pi_0, \dots, \pi_{k-1}$, and
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Fragments:

- E-AHLTL: existential trajectory quantification
- A-AHLTL: universal trajectory quantification

Baumeister, Coenen, Bonakdarpour, Finkbeiner, and Sánchez ('21)

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Bozzelli, Peron, and Sánchez ('22)

- HyperLTL satisfiability can be reduced to E-AHLTL satisfiability, so it is Σ_1^1 -hard.

Accessing Trajectories

The quantified trajectory t can only indirectly be accessed:

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- 1. If $\mathbf{G}(p_\pi \leftrightarrow \mathbf{X} \neg p_\pi)$ is satisfied w.r.t. t , then time passes on π at every position.*
- 2. Let π be mapped to a trace where the truth value of p changes at every step. Then, $\mathbf{G}(p_\pi \leftrightarrow \mathbf{X} \neg p_\pi)$ is satisfied w.r.t. t if and only if time passes on π at every position.*

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This allows us to reduce HyperLTL to AHLTL and to constrain trajectories.

Recall:

Lemma (Bozzelli, Peron, and Sánchez ('22))

E-AHLTL satisfiability is as hard as HyperLTL satisfiability, i.e., Σ_1^1 -hard.

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Theorem

- *E-AHLTL satisfiability is Σ_1^1 -complete.*
- *A-AHLTL satisfiability is Σ_1^1 -hard and in Σ_2^1 .*

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Proof sketch

- Every satisfiable AHLTL formula has a countable model.
- Express the existence of countable models in arithmetic.

“Small” Models

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Proof

- W.l.o.g. $\varphi = \forall \pi_0 \exists \pi'_0 \cdots \forall \pi_k \exists \pi'_k Q \psi$ with quantifier-free ψ .
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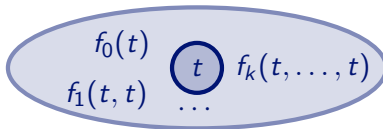
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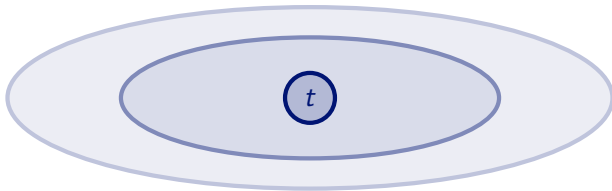
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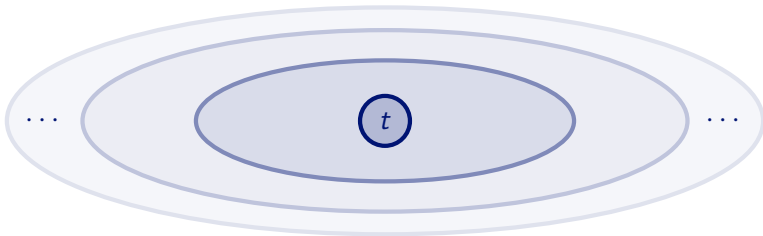
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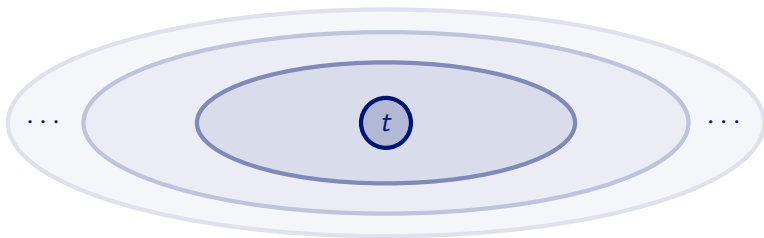
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The limit is a model of φ and countable.

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- All these objects can be existentially quantified for E-AHLTL satisfiability: $\exists T \exists \mathcal{S} \exists t \exists Tblx \psi(T, \mathcal{S}, t, Tblx)$ with first-order ψ .

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- For A-AHLTL satisfiability, the trajectory needs to be quantified universally: $\exists T \exists S \forall t \forall Tblx \psi'(T, S, t, Tblx)$ with first-order ψ' .

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- One can write a formula of second-order arithmetic that checks whether a trace comes from a given transition system.
- Assignments and trajectories can be encoded as sets of natural numbers.
- Hence, the whole semantics of AHLTL can be encoded in second-order arithmetic.

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The result for E-AHLTL follows by duality.

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AHLTL	Σ_1^1 -hard/in Σ_2^1	T2A-equiv.	T2A-equiv.
A-AHLTL	Σ_1^1 -hard/in Σ_2^1	T2A-equiv.	T2A-equiv.
E-AHLTL	Σ_1^1 -complete.	T2A-equiv.	T2A-equiv.

Future Work

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- There are more logics:

