Game-based Model-Checking of HyperLTL

Martin Zimmermann

Aalborg University

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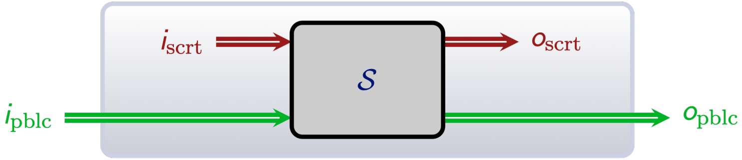
TU Dortmund

Reactive Systems



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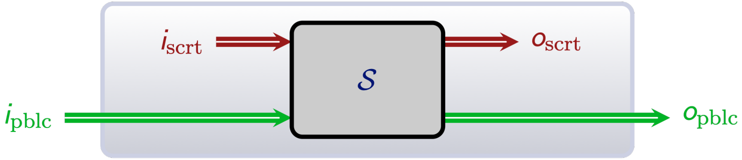

Trace-based view on $\mathcal{S}$: observe execution traces, i.e., infinite sequences over 2^{AP} for some set AP of atomic propositions.

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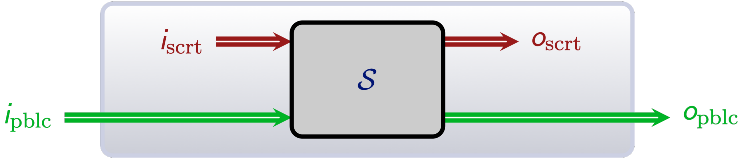

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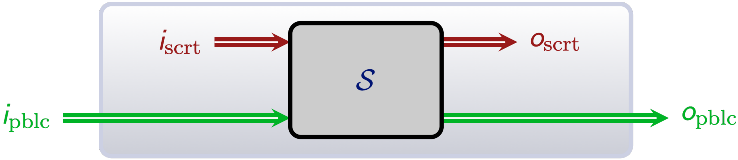

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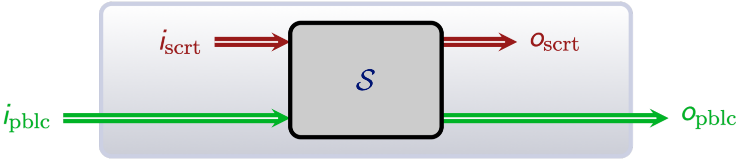

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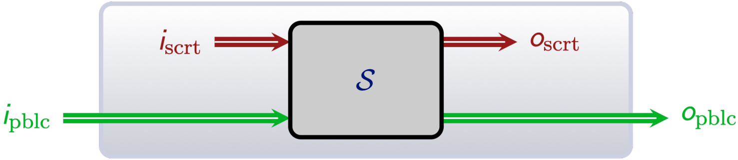

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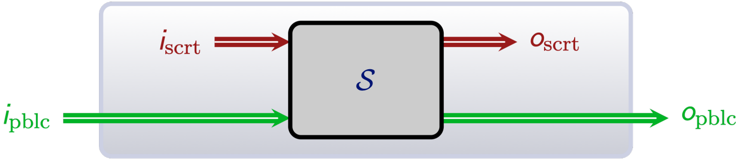

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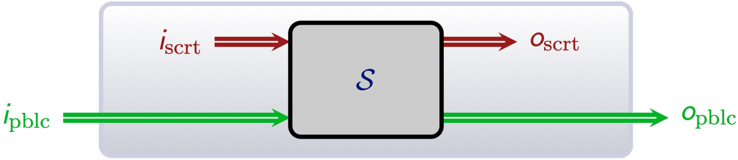

Typical specifications:

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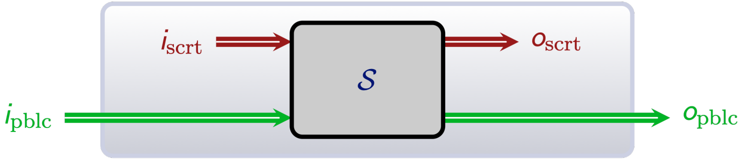

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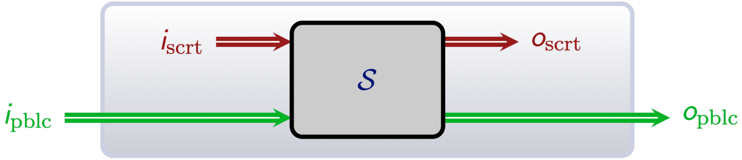

Typical specifications:

- $\mathcal{S}$ terminates
- $\mathcal{S}$ terminates within a uniform time bound

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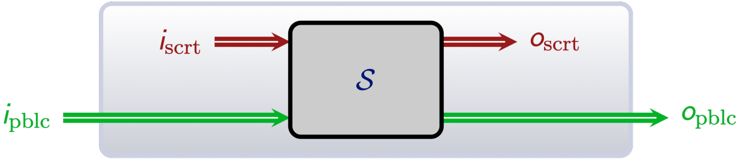

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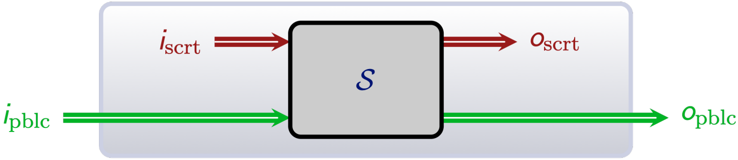

Typical specifications:

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- Noninterference for nondeterministic systems: for all traces t, t' of $\mathcal{S}$ there exists a trace t'' of $\mathcal{S}$ such that t'' and t coincide on their projection to public inputs and outputs and t'' and t' coincide on their projection to secret inputs.

Trace Properties vs. Hyperproperties

Definition

A trace property $T \subseteq (2^{\mathrm{AP}})^\omega$ is a set of traces. A system $\mathcal{S}$ satisfies T, if $\mathrm{Traces}(\mathcal{S}) \subseteq T$.

Example: The set of traces where `term` holds at least once.

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Definition

A hyperproperty $H \subseteq 2^{(2^{\mathrm{AP}})^\omega}$ is a set of sets of traces. A system $\mathcal{S}$ satisfies H if $\mathrm{Traces}(\mathcal{S}) \in H$.

Example: The set $\{T \subseteq T_n \mid n \in \mathbb{N}\}$ where T_n is the trace property containing the traces where `term` holds at least once within the first n positions.

LTL in One Slide

Syntax

$$\varphi ::= a \mid \neg\varphi \mid \varphi \vee \varphi \mid \mathbf{X}\,\varphi \mid \varphi\,\mathbf{U}\,\varphi \qquad \text{where } a \in \mathrm{AP}$$

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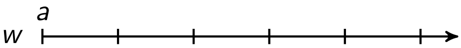

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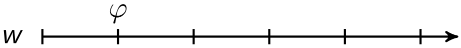

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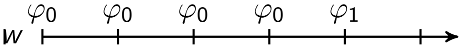

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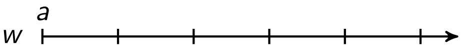

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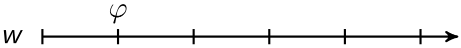

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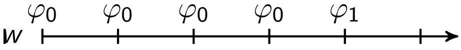

Syntactic Sugar

- $\mathbf{F}\,\psi = \mathbf{tt}\,\mathbf{U}\,\psi$
- $\mathbf{G}\,\psi = \neg\,\mathbf{F}\,\neg\psi$

HyperLTL

HyperLTL = LTL + trace quantification

$$\varphi ::= \exists\pi.\ \varphi \mid \forall\pi.\ \varphi \mid \psi$$
$$\psi ::= a_\pi \mid \neg\psi \mid \psi \vee \psi \mid \mathbf{X}\,\psi \mid \psi\,\mathbf{U}\,\psi$$

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where $a \in \mathrm{AP}$ and $\pi \in \mathcal{V}$ (trace variables).

- Prenex normal form, but
- closed under boolean combinations.

Examples

- Noninterference:

$$\forall\pi\forall\pi'.\ \mathbf{G}((i_{\mathsf{pblc}})_\pi \leftrightarrow (i_{\mathsf{pblc}})_{\pi'}) \rightarrow \mathbf{G}((o_{\mathsf{pblc}})_\pi \leftrightarrow (o_{\mathsf{pblc}})_{\pi'})$$

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- Noninterference for nondeterministic systems:

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- $\mathcal{S}$ terminates within a uniform time bound. Not expressible in HyperLTL.

Applications

- Uniform framework for information-flow control
  - Does a system leak information?
- Symmetries in distributed systems
  - Are clients treated symmetrically?
- Error resistant codes
  - Do codes for distinct inputs have at least Hamming distance d?
- Software doping
  - Think emission scandal in the automotive industry
- Network verification
  - Latency and congestion of computer networks

There are prototype tools for model checking, satisfiability checking, runtime verification, and synthesis.

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The HyperLTL model-checking problem:

Given a finite transition system $\mathcal{S}$ and φ, does $\mathrm{Traces}(\mathcal{S}) \models \varphi$?

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Theorem (Clarkson et al. '14, Rabe '16, Mascle & Z. '20)
The HyperLTL model-checking problem is TOWER-complete, even for a fixed transition system with 5 states and formulas without nested operators.

Model-Checking

Proof:

- Consider $\varphi = \exists\pi_1. \forall\pi_2. \ldots \exists\pi_{k-1}. \forall\pi_k. \psi$.
- Rewrite as $\exists\pi_1. \neg\exists\pi_2. \neg \ldots \exists\pi_{k-1}. \neg\exists\pi_k. \neg\psi$.

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- We construct, by induction over the quantifier prefix, non-determinstic Büchi automata accepting exactly the variable assignments satisfying the subformulas of φ.
- Then, we obtain an automaton $\mathcal{A}$ with $L(\mathcal{A}) \neq \emptyset$ iff $\mathrm{Traces}(\mathcal{S}) \models \varphi$.

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  - For $\neg\theta$ complement automaton for θ.

Skolem Functions

$\mathcal{S}$ satisfies a formula of the form $\forall\pi.\ \exists\pi'.\ \psi$ iff there is a (Skolem) function $f\colon \mathrm{Traces}(\mathcal{S}) \to \mathrm{Traces}(\mathcal{S})$ such that the assignment

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- In general, if $\mathcal{S} \models \varphi$, then Skolem functions for the existentially quantified variables in φ explain why $\mathcal{S} \models \varphi$.
- Dually, if $\mathcal{S} \not\models \varphi$, then $\mathcal{S} \models \neg\varphi$ and Skolem functions for the existentially quantified variables in $\neg\psi$ are a "counterexample" for $\mathcal{S} \not\models \varphi$.

Computable Skolem Functions

To interpret and algorithmically handle Skolem functions, we represent them by finite automata with output (transducers).

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Example

Consider $\mathcal{S}$ with $\mathrm{Traces}(\mathcal{S}) = (2^{\{a\}})^\omega$, which satisfies

$$\varphi = \forall\pi.\ \exists\pi'.\ (\mathbf{X}\, a_\pi) \leftrightarrow a_{\pi'}.$$

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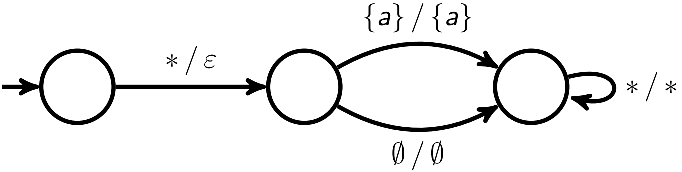
The following transducer represents a Skolem function for π':



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$$f(t) = \begin{cases} \{a\}\emptyset^\omega & \text{if } t \text{ contains an } a \text{ somewhere,} \\ \emptyset^\omega & \text{if } t \text{ does not contain an } a \text{ anywhere.} \end{cases}$$

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- However, this, and any other Skolem function, is not representable by a transducer.

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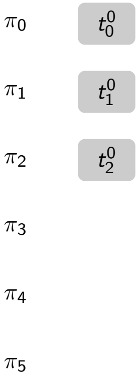

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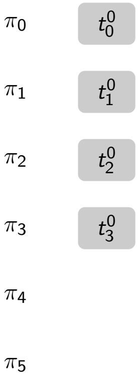

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Given $\mathcal{S}$ and φ such that $\mathcal{S} \models \varphi$, is $\mathcal{S} \models \varphi$ witnessed by Skolem functions representable by finite transducers?

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$$\forall\pi_0.\ \exists\pi_1.\ \forall\pi_2.\ \exists\pi_3.\ \forall\pi_4.\ \exists\pi_5.\ \psi$$

- The Skolem function for π_1 may only depend on the trace assigned to π_0, but not those assigned to π_2 and π_4.

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- The information is hierarchical $\Rightarrow$ solving games with ω-regular winning conditions is decidable.

Problem 2: Order of Moves

$$\varphi = \forall \pi.\ \exists \pi'.\ (\mathbf{X}\, a_\pi) \leftrightarrow a_{\pi'}$$

- To pick the first letter of the trace for π', the player needs to know the second letter of the trace for π.

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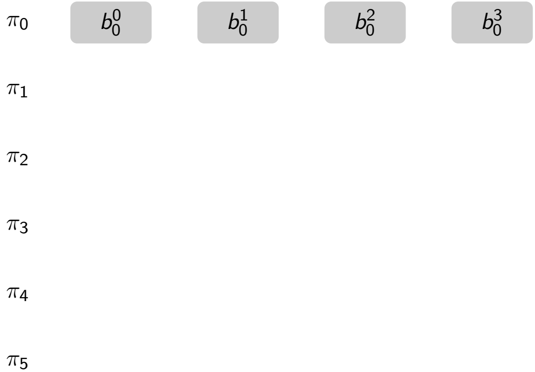

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Results

Theorem (Winter & Z. '24)

There is a block size (effectively computable from $\mathcal{S}$ and φ) such that the following are equivalent:

1. *The coalition of players for the existentially quantified variables in φ has a collection of winning strategies.*
2. *$\mathcal{S} \models \varphi$ is witnessed by Skolem functions implemented by transducers.*

Furthermore, the game is effectively solvable and the transducers can be effectively computed.

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- In the example above, the prophecy is the language of words containing an a somewhere.

What about arbitrary quantifier prefixes?

Theorem (Winter & Z. '25)

Given $\mathcal{S}$ and φ, there is an effectively computable and solvable imperfect information game such that the following are equivalent:

1. *The coalition of players for the existentially quantified variables in φ has a collection of winning strategies.*
2. $\mathcal{S} \models \varphi$.

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- Future work:
  - More expressive logics
  - Infinite-state systems
  - Complexity analysis