The Complexity of Second-order HyperLTL

Joint work with Hadar Frenkel
(Bar-Ilan University, Ramat Gan, Israel)

Martin Zimmermann

Aalborg University

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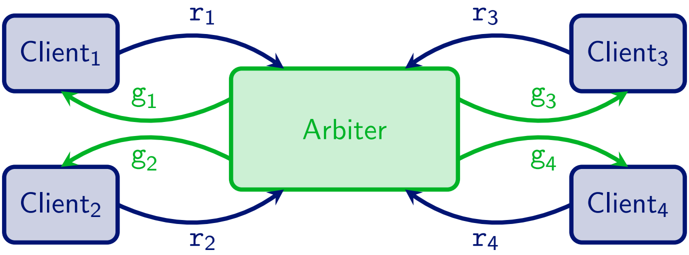
Reactive Systems

- Setting: an arbiter with n clients
- requests $\mathtt{r}_i$ from client i controlled by the environment
- grants $\mathtt{g}_i$ for client i controlled by the system



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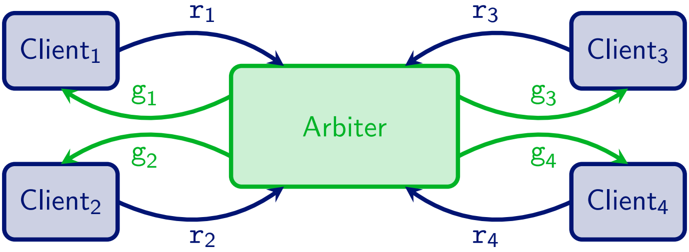

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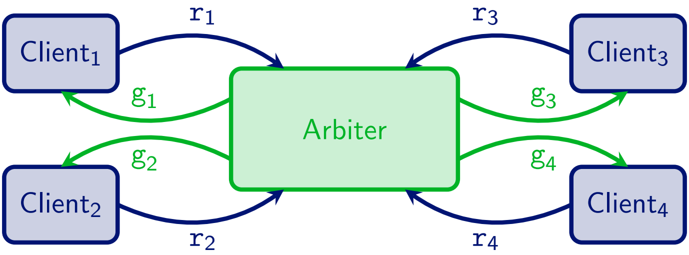

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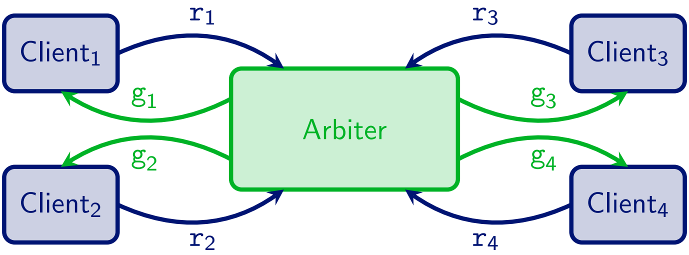

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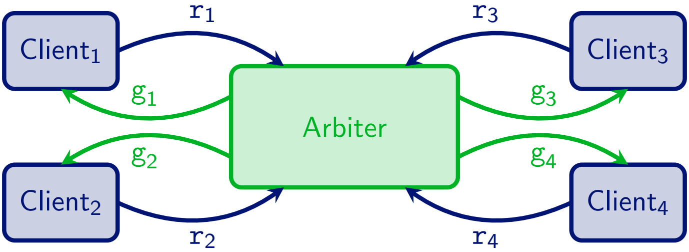

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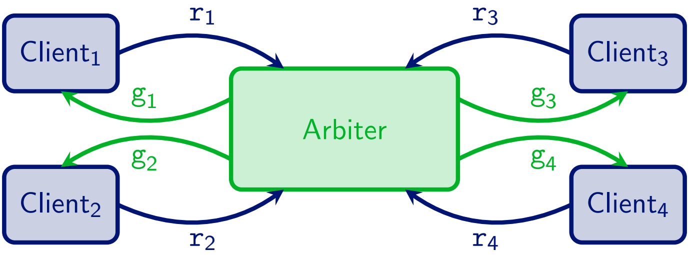

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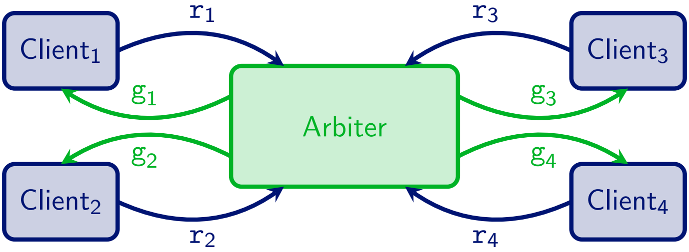

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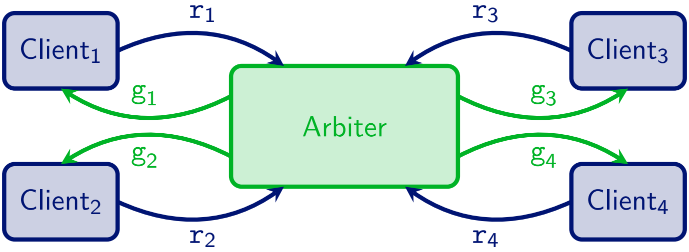

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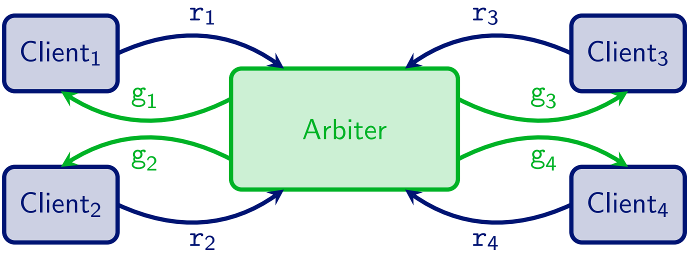

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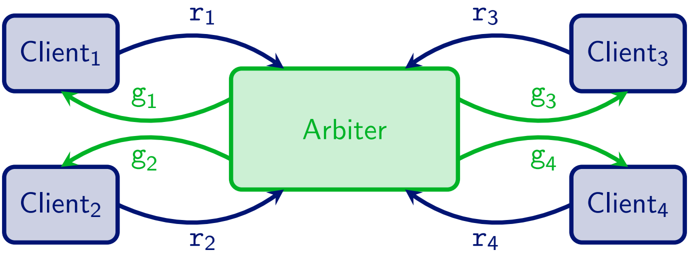

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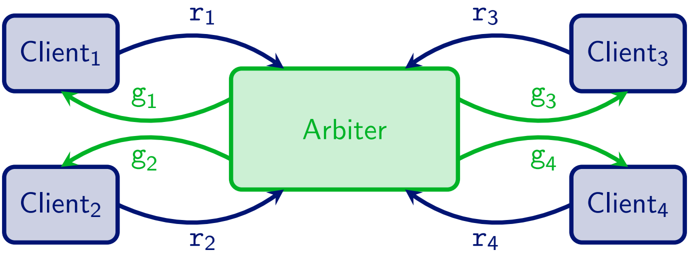

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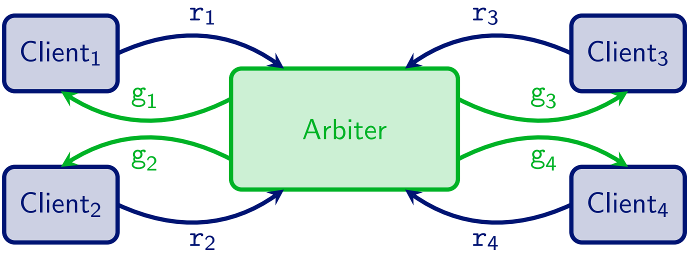

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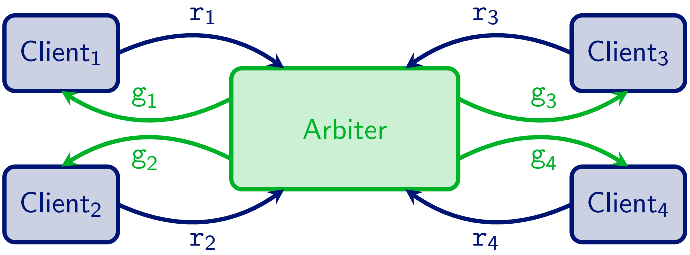

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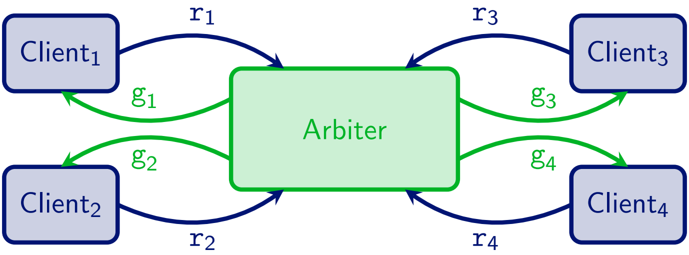

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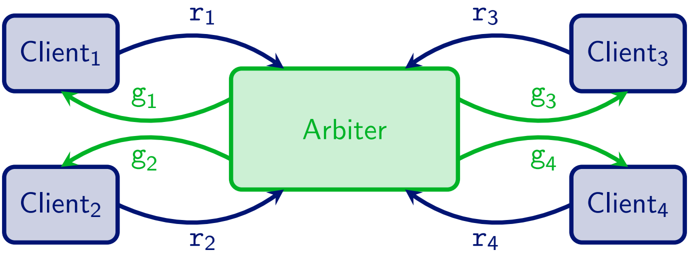

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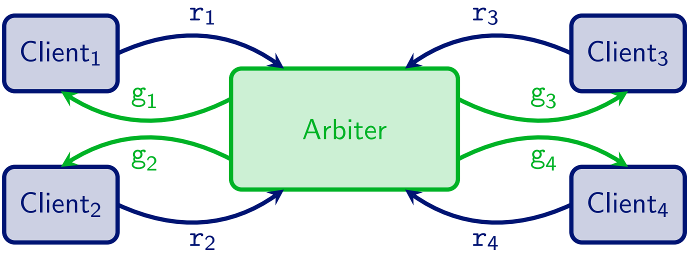

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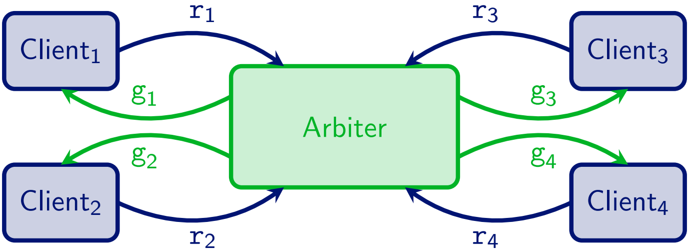

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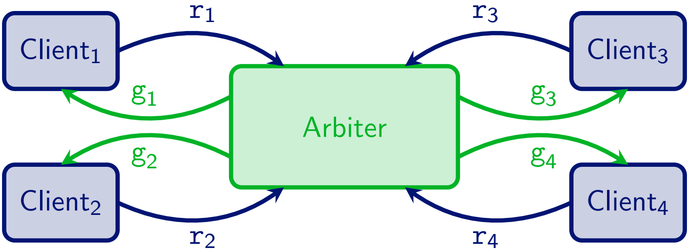

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All these properties φ are trace properties, i.e., each φ is a set of traces and $\mathfrak{T}$ satisfies φ if $\mathrm{Tr}(\mathfrak{T}) \subseteq \varphi$.

LTL in a Nutshell

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \mathbf{X}\,\varphi \mid \varphi\,\mathbf{U}\,\varphi$$

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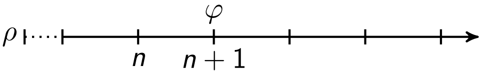

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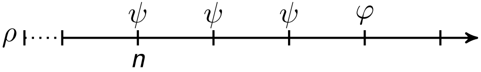

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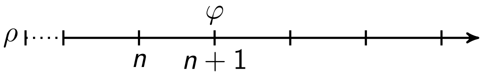

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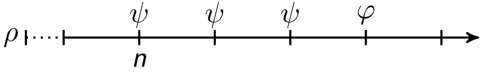

Syntactic sugar:

- $\mathbf{F}\,\varphi = \mathbf{tt}\,\mathbf{U}\,\varphi$: φ holds eventually (finally φ)
- $\mathbf{G}\,\varphi = \neg\,\mathbf{F}\,\neg\varphi$: φ holds always (generally φ)

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- No spurious grants:

$$\bigwedge_i \neg[\,(\neg \mathtt{r}_i\, \mathbf{U}(\neg \mathtt{r}_i \wedge \mathtt{g}_i))\,] \wedge \neg[\,\mathbf{F}(\mathtt{g}_i \wedge \mathbf{X}(\neg \mathtt{r}_i\, \mathbf{U}(\neg \mathtt{r}_i \wedge \mathtt{g}_i)))\,]$$

But not Everything is a Trace Property

- Input determinism: projection to the $\mathtt{r}_i$ uniquely determines the projection to the $\mathtt{g}_i$.

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- This property (and many others) cannot be expressed by considering single traces in isolation. Instead one needs to reason about pairs of traces.
- Clarkson and Schneider termed such properties hyperproperties: formally, they are sets of sets of traces.
- $\mathfrak{T}$ satisfies a hyperproperty $H \subseteq 2^{(2^{\mathrm{AP}})^\omega}$ if $\mathrm{Tr}(\mathfrak{T}) \in H$.

HyperLTL in a Nutshell

- To express hyperproperties, Clarkson et al. introduced HyperLTL, LTL + trace quantification (in prenex normal form).
- Input determinism:

$$\forall \pi.\ \forall \pi'.\ (\mathbf{G} \bigwedge_i (\mathtt{r}_i)_\pi \leftrightarrow (\mathtt{r}_i)_{\pi'}) \rightarrow (\mathbf{G} \bigwedge_i (\mathtt{g}_i)_\pi \leftrightarrow (\mathtt{g}_i)_{\pi'})$$

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- HyperLTL is able to express many information-flow properties from security and privacy.
- Model-checking is decidable and successfully implemented (for a small number of quantifier alternations).
- Much more exciting work!

But HyperLTL Cannot Express Everything

Consider common knowledge in multi-agent systems:

- An agent knows that a property φ holds if it holds on all traces that are indistinguishable in the agent's view.
- φ is common knowledge among the agents if all agents know φ, all agents know that all agents know φ, and so on.

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- Bozelli et al. proved that common knowledge is not expressible in HyperLTL.

Second-order HyperLTL

- Beutner et al. introduced second-order HyperLTL (written $\mathrm{Hyper}^2\mathrm{LTL}$), HyperLTL + quantification over sets of traces.
- Can express common knowledge, asynchronous hyperproperties, and more.

$$\forall \pi.\, \exists X.$$
$$\pi \in X \wedge \left(\forall \pi' \in X.\, \forall \pi''.\, \left(\bigvee_{i=1}^{n} \pi' \sim_i \pi''\right) \rightarrow \pi'' \in X\right) \wedge$$
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Open problem:
What is the complexity of satisfiability (SAT) and model-checking (MC) for second-order HyperLTL?

Known Results

Logic	Satisfiability	Model-checking
LTL	PSPACE-complete	PSPACE-complete
HyperLTL	Σ_1^1-complete	TOWER-complete
Hyper^{2}LTL	?	Σ_1^1-hard

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We classify the complexity of undecidable problems using arithmetic, i.e., predicate logic with signature $(+, \cdot, <, \in)$, evaluated over $(\mathbb{N}, +, \cdot, <, \in)$

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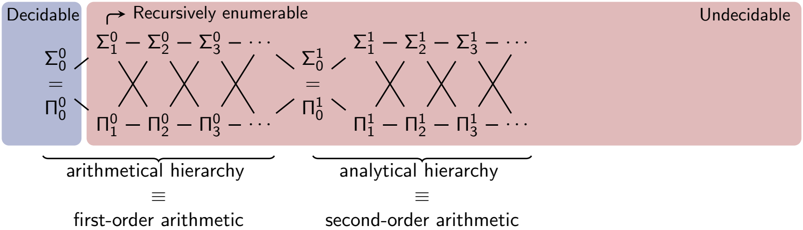

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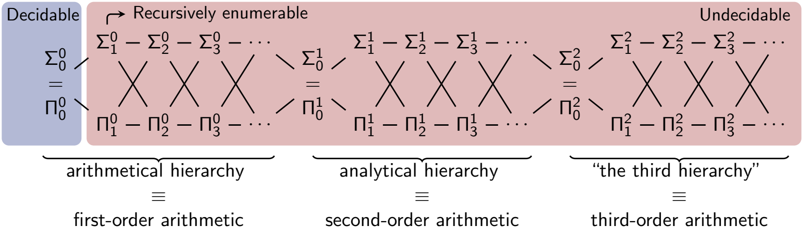

The Complexity of Second-order HyperLTL

Theorem

Second-order HyperLTL SAT and MC are polynomial-time equivalent to truth in third-order arithmetic.

The Complexity of Second-order HyperLTL

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Proof sketch

Lower bound:

- Traces can encode (characteristic sequences of) sets of natural numbers, so sets of traces encode sets of sets of natural numbers.
- Fortin et al. showed that addition and multiplication can be *implemented* in HyperLTL.

The Complexity of Second-order HyperLTL

Theorem

Second-order HyperLTL SAT and MC are polynomial-time equivalent to truth in third-order arithmetic.

Proof sketch

Upper bound:

- Sets of natural numbers can encode traces (and encoding is implementable in first-order arithmetic), so sets of sets of natural numbers can encode sets of traces.
- Semantics of temporal operators can be expressed in arithmetic.

Least Fixed Points

Recall the formula for common knowledge:

$$\forall\pi.\,\exists X.\,\overbrace{\pi \in X \wedge \left(\forall\pi' \in X.\,\forall\pi''.\,\left(\bigvee_{i=1}^{n} \pi' \sim_i \pi''\right) \rightarrow \pi'' \in X\right)}^{\psi} \wedge \forall\pi' \in X.\ \varphi(\pi')$$

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Open problem:

Does this fragment have better complexity?

"Small" Models

Lemma

Every satisfiable lfp-second-order HyperLTL formula has a countable model.

“Small” Models

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Proof sketch

- Let T be an uncountable model of φ and let $t_0 \in T$ be an arbitrary trace in T.
- Let R be the smallest subset of T that
  - contains t_0,
  - is closed under application of Skolem functions, and
  - contains witnesses for all traces being in the required fixed points.
- Then, R is a countable model of φ.

Main Theorem

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The lower bound already holds for the fragment HyperLTL, so let us consider the upper bound.

- Express the existence of a countable model, Skolem functions, and more; all encoded as sets of natural numbers.
- Use first-order arithmetic to ensure correctness.

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- Express the existence of a countable model, Skolem functions, and more; all encoded as sets of natural numbers.
- Use first-order arithmetic to ensure correctness.

So, you can add least fixed points to HyperLTL for free (at least for satisfiability).

Also in the Paper

- Model checking lfp-second-order HyperLTL
- Finite-state satisfiability
- Two semantics

Logic	Satisfiability	Finite-state satisfiability	Model-checking
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$\mathrm{Hyper}^2\mathrm{LTL}$	**T3A-equivalent**	**T3A-equivalent**	**T3A-equivalent**
$\mathrm{Hyper}^2\mathrm{LTL}_{\mathrm{mm}}$	**T3A-equivalent**	**T3A-equivalent**	**T3A-equivalent**
$\mathrm{lfp}\text{-}\mathrm{Hyper}^2\mathrm{LTL}_{\mathrm{mm}}$	**Σ^1_1-complete***	**Σ^1_1-hard/in Σ^2_2**	**Σ^1_1-hard/in Σ^2_2**

Meanwhile

- Regaud and Zimmermann: closed all gaps (arxiv.org/abs/2501.19046)
- Regaud and Zimmermann: the complexity of HyperQPTL (arxiv.org/abs/2412.07341)

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$\mathrm{Hyper}^2\mathrm{LTL}^{\curlywedge}_{\mathrm{mm}}$	**T3A-equivalent**	**T3A-equivalent**	**T3A-equivalent**
$\mathrm{Hyper}^2\mathrm{LTL}^{\curlyvee}_{\mathrm{mm}}$	**T3A-equivalent**	**T3A-equivalent**	**T3A-equivalent**
lfp-$\mathrm{Hyper}^2\mathrm{LTL}_{\mathrm{mm}}$	**Σ^2_1-complete (STD)**/ Σ^1_1-complete (CW)	**T2A-equivalent**	**T2A-equivalent**
HyperQPTL	**T2A-equivalent**	**Σ^0_1-complete**	**Tower-complete**
$\mathrm{HyperQPTL}^+$	**T3A-equivalent**	**T3A-equivalent**	**T3A-equivalent**