

Time for Timed Monitorability

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Abstract

Monitoring is an important part of the verification toolbox, in particular in situations where exhaustive verification using, e.g., model-checking, is infeasible. The goal of online monitoring is to determine the satisfaction or violation of a specification during runtime, i.e., based on finite execution prefixes. However, not every specification is amenable to monitoring, e.g., properties for which no finite execution can witness satisfaction or violation. Monitorability is the question whether a given specification is amenable to monitoring, and has been extensively studied in discrete time.

Here, we study, for the first time, the monitorability problem for real-time specifications. For specifications given by deterministic Timed Muller Automata, we prove decidability while we show that the problem is undecidable for specifications given by nondeterministic Timed Büchi automata.

Furthermore, we refine monitorability to also determine bounds on the number of events as well as the time that must pass before monitoring the property may yield an informative verdict. We prove that for deterministic Timed Muller automata, such bounds can be effectively computed. In contrast we show that for nondeterministic Timed Büchi automata such bounds are not computable.

1 Introduction

A fundamental challenge in Runtime Verification (RV) is that many properties provide no utility when monitored in an online setting. Thus, it is of utmost importance to identify those properties that do provide useful information.

Behaviors of long-running systems are typically specified as languages of infinite words, but online monitors only observe finite (prefixes of) system executions. Thus, a monitor has to determine whether such a finite prefix already implies satisfaction or violation of the property. While many different types of monitors have been proposed, most online monitors return information in the form of *verdicts* about the finite prefix. Inherently to the problem, there are at least three verdicts [9]: $\{\top, \perp, ?\}$, where \top and \perp are *conclusive* verdicts that mean that the finite prefix guarantees that *every* possible infinite extension satisfies, respectively violates, the property, and the *inconclusive* verdict $?$ signifying that neither is the case.

For example, consider an arbiter granting access to a shared resource. The property φ_1 expressing “there is no request during the first second” is satisfied if the first request arrives after two seconds, no matter how the execution continues. Hence, the verdict for such a finite execution is \top . Similarly, the property φ_2 expressing “after any request, there are no requests for at least one second” is violated as soon as two consecutive requests are observed within one second, no matter how the execution continues. Hence, the verdict for such a finite execution is \perp . On the other hand, if requests arrive with a gap of two seconds between them, then the verdict for such a prefix (w.r.t. φ_2) is $?$, since there are infinite extensions satisfying the property and infinite extensions violating it.

As seen above, there are prefixes for which we get a conclusive verdict w.r.t. φ_2 . This distinguishes it from properties like φ_3 expressing “every request is eventually granted”, for which every finite prefix can be extended to satisfy the property and can be extended to violate it. Hence, every finite prefix yields the verdict $?$. Phrased concisely: monitoring the property φ_3 is futile.

However, for the user it is not transparent, while receiving the verdict $?$, whether in the future a conclusive verdict may be given, or whether every possible extension yields the verdict $?$. The concept of *monitorability* has been introduced to capture those properties that are amenable to monitoring. It

comes in two variants, strong monitorability (every prefix can be extended to one that yields a conclusive verdict) and weak monitorability (some prefix yields a conclusive verdict). In this language, φ_3 is not weakly monitorable while φ_1 and φ_2 are (even strongly) monitorable. Thus, before constructing and deploying a monitor for a property, it is prudent to first check whether the property is monitorable.

The problem of deciding if a property is monitorable has been studied extensively over the last 20 years (we discuss related work in Section 7), but only in the setting of discrete time. However, many properties require real-time constraints to express, e.g., deadlines like “every request is answered within 745 ms”. In particular, safety-critical systems are nearly always real-time systems with physics-based deadlines, and these systems tend to benefit the most from formal verification methods like RV. Had the property “the Therac 25 control program must wait eight seconds before switching between X-ray and electron modes” been monitored, it might have saved lives [23]. While monitoring algorithms exist for real-time properties [30, 8, 7, 20], the problem of real-time monitorability has gone unexamined.

Our Contribution. This work makes monitoring of real-time systems more useful by examining the monitorability problem for real-time properties and by introducing qualitative refinements of the verdict $?$. We consider real-time properties expressed by Timed Automata (TA) over infinite words [3]. Nondeterministic Timed Büchi Automata (TBA) are used for model checking tools like UPPAAL [22] and for monitoring temporal logics [20, 18, 13] and are strictly more expressive than deterministic Timed Muller Automata (DTMA). We prove that strong and weak monitorability are undecidable for nondeterministic TBA, but decidable for DTMA. Thus, one can algorithmically determine that it is futile to monitor properties like φ_3 , thereby increasing the applicability of monitoring of real-time systems.

Furthermore, we introduce monitorability with step-bounded horizons that strengthens monitorability by limiting the number of events in a timed-word before a conclusive verdict must be reached. A step-bounded horizon allows one to determine for a given property and $n \in \mathbb{N}$, if a conclusive verdict is possible within n steps, enabling corrective actions earlier. Again, we show that monitorability with step-bounded horizons is undecidable for nondeterministic TBA, but decidable for DTMA.

Finally, we refine monitoring of real-time properties with time-horizon verdicts. Here, the verdict $?$ is enhanced with information about the minimum time until a conclusive verdict may be reached. Intuitively, before this time is reached, the monitor will only yield the inconclusive verdict $?$, i.e., no information can be gained from querying the monitor before. This notion was introduced by Grosen et al. [20] as “time-predictive” monitoring. Here, we formally prove that time-horizon queries can be computed effectively for DTMA.

Thus, our results highlight the importance of properties being given by deterministic timed automata when checking monitorability. This is in contrast to monitoring itself, where it suffices to have nondeterministic automata for the property *and* its negation [20], which is, e.g., the case when specifying properties in Metric Interval Temporal Logic [4]. Finally, we show that it is necessary to have automata for the property and the complement, as the monitoring function is otherwise not effectively computable.

All proofs omitted due to space restrictions can be found in the appendix.

2 Preliminaries

The nonnegative integers are denoted by \mathbb{N} and the nonnegative reals by $\mathbb{R}_{\geq 0}$. An alphabet is a finite nonempty set of letters.

A timed word is a pair $\rho = (\sigma, \tau)$, where σ is a (finite or infinite) word over an alphabet Σ and τ is a sequence of non-decreasing, non-negative real numbers of the same length as σ . For convenience, we often write $(\sigma_1, \tau_1), (\sigma_2, \tau_2), \dots$ for a timed word (σ, τ) . $T\Sigma^*$ and $T\Sigma^\omega$ denote the sets of finite and infinite timed words over Σ . For $n \in \mathbb{N} \cup \{\infty\}$ we denote by $T\Sigma^{\leq n}$ the set of timed words over Σ of length at most n . Given a finite word $\rho = (\sigma_1, \tau_1), (\sigma_2, \tau_2), \dots, (\sigma_n, \tau_n)$, we denote its duration as $\tau(\rho) = \tau_n$. Slightly abusively, we write ε for the empty timed word $(\varepsilon, \varepsilon)$ and define $\tau(\varepsilon) = 0$. For a finite timed word $\rho = (\sigma_1, \tau_1), (\sigma_2, \tau_2), \dots, (\sigma_n, \tau_n)$, a finite or infinite word $\rho' = (\sigma'_1, \tau'_1), (\sigma'_2, \tau'_2), \dots$ and a timepoint $t \geq \tau(\rho)$, we define the concatenation of ρ and ρ' at t as

$$\rho \cdot_t \rho' = (\sigma_1, \tau_1), (\sigma_2, \tau_2), \dots, (\sigma_n, \tau_n), (\sigma'_1, \tau'_1 + t), (\sigma'_2, \tau'_2 + t), \dots,$$

which is a timed word. As a shorthand, we write $\rho \cdot \rho'$ for $\rho \cdot_{\tau(\rho)} \rho'$. Given two finite words ρ, ρ' and a timepoint $t \geq \tau(\rho)$, we write $\rho \sqsubseteq_t \rho'$ if there exists a ρ'' such that $\rho \cdot_t \rho'' = \rho'$.

Timed Automata A timed Büchi automaton (TBA) is a tuple $\mathcal{A} = (Q, Q_0, \Sigma, \mathcal{X}, \Delta, F)$ where Q is a finite set of locations, $Q_0 \subseteq Q$ is the set of initial locations, Σ is an alphabet, \mathcal{X} is a finite set of clocks, $F \subseteq Q$ is a set of accepting locations, and $\Delta \subseteq Q \times Q \times \Sigma \times 2^{\mathcal{X}} \times G(\mathcal{X})$ is a set of transitions, where $G(\mathcal{X})$ is the set of clock constraints over \mathcal{X} . A transition $(q, q', \alpha, \lambda, g) \in \Delta$ is an edge from q to q' with label $\alpha \in \Sigma$, where $\lambda \in 2^{\mathcal{X}}$ is a set of clocks to be reset and $g \in G(\mathcal{X})$ is a clock constraint. A clock constraint is a finite conjunction of atomic constraints of the form $x \sim n$ where $x \in \mathcal{X}$, $n \in \mathbb{N}$, and $\sim \in \{<, \leq, =, \geq, >\}$. A state of \mathcal{A} is a pair (q, v) where $q \in Q$ and $v: \mathcal{X} \rightarrow \mathbb{R}_{\geq 0}$ is a clock valuation.

A run of \mathcal{A} from a state (q_0, v_0) over an infinite word $(\sigma, \tau) \in T\Sigma^\omega$ is a sequence of steps of the form

$$(q_0, v_0) \xrightarrow{(\sigma_1, \tau_1)} (q_1, v_1) \xrightarrow{(\sigma_2, \tau_2)} (q_2, v_2) \xrightarrow{(\sigma_3, \tau_3)} \dots$$

where for every $i \geq 1$, there is a transition $(q_{i-1}, q_i, \sigma_i, \lambda_i, g_i) \in \Delta$ such that $v_i(x) = 0$ for all $x \in \lambda_i$ and $v_i(x) = v_{i-1}(x) + \tau_i - \tau_{i-1}$ otherwise, and g is satisfied by $v_{i-1} + \tau_i - \tau_{i-1}$ (where we use $\tau_0 = 0$). For a run r , $\text{inf}(r) \subseteq Q$ denotes the set of locations visited infinitely often in r . A run r is (Büchi) accepting if $\text{inf}(r) \cap F \neq \emptyset$.

A timed Muller automata (TMA) $\mathcal{A} = (Q, Q_0, \Sigma, \mathcal{X}, \Delta, \mathcal{F})$ is like a TBA, but the set F of accepting locations is replaced by a set $\mathcal{F} \subseteq 2^Q$ of sets of locations. A run r of \mathcal{A} is (Muller) accepting if $\text{inf}(r) \in \mathcal{F}$.

The language $L(\mathcal{A})$ of a timed (Büchi or Muller) automaton is the set of words $\rho \in T\Sigma^\omega$ such that \mathcal{A} has an accepting run over ρ .

An automaton is deterministic if the set of initial location is a singleton and if all edges from the same location and with the same label must have disjoint clock constraints. We use the abbreviations DTBA and DTMA to refer to the two deterministic automaton models.

Proposition 1. *The following results on TBA and TMA are due to Alur and Dill [3].*

1. Let $\mathcal{A} = (Q, Q_0, \Sigma, \mathcal{X}, \Delta, F)$ be a TBA and let $\mathcal{A}' = (Q, Q_0, \Sigma, \mathcal{X}, \Delta, \{F' \mid F \cap F' \neq \emptyset\})$, which is a TMA. Then, $L(\mathcal{A}) = L(\mathcal{A}')$. Furthermore, if \mathcal{A} is deterministic, then so is \mathcal{A}' .
2. For every TMA $\mathcal{A} = (Q, Q_0, \Sigma, \mathcal{X}, \Delta, \mathcal{F})$, there is a TBA \mathcal{A}' with $L(\mathcal{A}) = L(\mathcal{A}')$. The set of locations of \mathcal{A}' has the form $Q \times S$ for some set S satisfying $|S| \leq |\mathcal{F}| \cdot |Q| \cdot (|Q| + 1)$.
3. The class of languages accepted by DTMA is a strict subset of the class of languages accepted by TBA.

Logic We use Metric Temporal Interval Logic (MITL) to formally express properties to be monitored. The syntax of MITL formulas over a finite alphabet Σ is defined as

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \psi \mid X_I\varphi \mid \varphi U_I\psi$$

where $p \in \Sigma$ and I ranges over non-singular intervals over $\mathbb{R}_{\geq 0}$ with endpoints in $\mathbb{N} \cup \{\infty\}$. Note that we often write $\sim n$ for $I = \{d \in \mathbb{R}_{\geq 0} \mid d \sim n\}$ where $\sim \in \{<, \leq, \geq, >\}$, and $n \in \mathbb{N}$. We also define the standard syntactic sugar **true** $= p \vee \neg p$, **false** $= \neg \text{true}$, $\varphi \wedge \psi = \neg(\neg\varphi \vee \neg\psi)$, $\varphi \rightarrow \psi = \neg\varphi \vee \psi$, $F_I\varphi = \text{true} U_I\varphi$, and $G_I\varphi = \neg F_I\neg\varphi$.

The semantics of MITL is defined over infinite timed words. Given such a timed word $\rho = (\sigma_1, \tau_1), (\sigma_2, \tau_2), \dots \in T\Sigma^\omega$, a position $i \geq 1$, and an MITL formula φ , we inductively define the satisfaction relation $\rho, i \models \varphi$ as follows:

- $\rho, i \models p$ if and only if $p = \sigma_i$.
- $\rho, i \models \neg\varphi$ if and only if $\rho, i \not\models \varphi$.
- $\rho, i \models \varphi \vee \psi$ if $\rho, i \models \varphi$ or $\rho, i \models \psi$.
- $\rho, i \models X_I\varphi$ if and only if $\rho, (i+1) \models \varphi$ and $\tau_{i+1} - \tau_i \in I$.
- $\rho, i \models \varphi U_I\psi$ if and only if there exists $k \geq i$ s.t. $\rho, k \models \psi$, $\tau_k - \tau_i \in I$, and $\rho, j \models \varphi$ for all $i \leq j < k$.

We write $\rho \models \varphi$ whenever $\rho, 1 \models \varphi$, and say that ρ satisfies φ . The language $L(\varphi)$ of an MITL formula φ is the set of all infinite timed words that satisfy φ .

Proposition 2 ([4, 11]). *For each MITL formula φ there is a TBA \mathcal{A} such that $L(\varphi) = L(\mathcal{A})$.*

Example 1. *Figure 1 illustrates the above theorem by providing DTBA for the formula $F_{[0,10]}a \wedge G_{[0,20]} \neg b$ and its negation.*

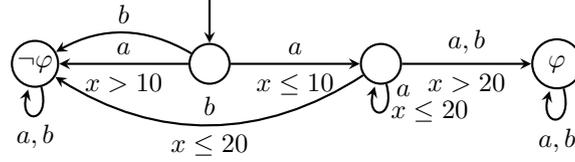


Figure 1: DTBA for the language of the MITL formula $\varphi = F_{[0,10]}a \wedge G_{[0,20]}\neg b$ and its negation: If location φ ($\neg\varphi$) is accepting then it accepts $L(\varphi)$ ($L(\neg\varphi)$, respectively).

Monitoring The monitoring problem asks to make verdicts about the satisfaction or violation of properties (over infinite timed words) after having observed only a finite prefix. Here, we follow the classical approach of considering the possible extensions of a finite observations.

Definition 1 (Observation). *An observation is a pair (ρ, t) containing a finite timed word ρ and a timepoint $t \geq \tau(\rho)$, representing the current timepoint (which might be later than the last observed event in ρ). As we use observations as inputs for algorithms, we require that all timepoints in ρ and t are rational.*

We continue with giving some intuition for the three-valued monitoring approach. Here, one is given an observation and aims to determine whether a property φ (of infinite timed words) is already satisfied, already violated, or neither.

- If all possible extensions of the observation satisfy φ , then we give the corresponding verdict \top signifying that the observation conclusively witnesses satisfaction of φ .
- If all possible extensions of the observation violate φ , then we give the corresponding verdict \perp signifying that the observation conclusively witnesses violation of φ .
- Otherwise, i.e., if there is an extension of the observation that satisfies φ and there is a extension of the observation that violates φ , then we give the inconclusive verdict $?$.

Let us formalize this intuition.

Definition 2 (Timed Monitoring Function). *Given a property $\varphi \subseteq T\Sigma^\omega$ and an observation (ρ, t) , the monitoring function V_φ is defined as*

$$V_\varphi(\rho, t) = \begin{cases} \top & \text{if } \rho \cdot_t \mu \in \varphi \text{ for all } \mu \in T\Sigma^\omega, \\ \perp & \text{if } \rho \cdot_t \mu \notin \varphi \text{ for all } \mu \in T\Sigma^\omega, \\ ? & \text{otherwise.} \end{cases}$$

In the following, we use $V_\varphi(\rho)$ as a shorthand for $V_\varphi(\rho, \tau(\rho))$.

Example 2. *Consider the specification $\varphi = F_{[0,10]}a \wedge G_{[0,20]}\neg b$ from Example 1. We have*

- $V_\varphi((a, 3), 4) = ?$,
- $V_\varphi((a, 11), 11) = \perp$,
- $V_\varphi((a, 3)(c, 7), 13) = ?$,
- $V_\varphi((a, 3)(c, 7)(c, 22), 22) = \top$, but
- $V_\varphi((a, 3)(c, 7)(b, 12), 12) = \perp$. Also,
- $V_\varphi((a, 3)(c, 7), 22) = \top$ while
- $V_\varphi((a, 3)(c, 7)) = V_\varphi((a, 3)(c, 7), 7) = ?$, i.e., the current timepoint t can yield conclusive verdicts when time is passing, even if no new events are observed.

Proposition 3 (Effectiveness of Timed Monitoring [20]). V_φ is effectively computable when given TBA accepting both φ and $T\Sigma^\omega \setminus \varphi$.

Recall that TBA are *not* closed under complement [3], so this result is not applicable to all properties accepted by TBA. However, for the important special case of properties φ specified in MITL, V_φ is effectively computable, as MITL properties are closed under complementation (as the logic allows for negations) and MITL formulas can be translated into equivalent TBA (see Proposition 2). Similarly, when φ is given by a DTMA, then V_φ is effectively computable, as DTMA are closed under complementation and can be turned into equivalent TBA [3].

However, it was previously open whether V_φ was computable if φ was given by a non-deterministic automaton, but we did not have access to an automaton for the complement $T\Sigma^\omega \setminus \varphi$. In Section 3, we answer the question negatively.

Monitorability Not every property is amenable to monitoring, e.g., for $\varphi = G_{\geq 0}F_{\geq 0}a$, we have $V_\varphi(\rho, t) = ?$ for every observation (ρ, t) . The reason is that every ρ can be extended to satisfy φ and can be extended to violate φ . In the untimed setting, much effort has been put into characterizing the monitorable properties, i.e., those for which monitoring can generate some information. Here, we consider, for the first time, monitorability in the timed setting.

Definition 3 (Timed Monitorability). Fix an observation (ρ, t) and a property $\varphi \subseteq T\Sigma^\omega$.

- φ is strongly (ρ, t) -monitorable if and only if

$$\text{for all } \rho' \in T\Sigma^* \text{ there exists } \rho'' \in T\Sigma^* \text{ such that } V_\varphi(\rho \cdot_t \rho' \cdot \rho'') \in \{\top, \perp\}.$$

- φ is weakly (ρ, t) -monitorable if and only if

$$\text{there exists } \rho' \in T\Sigma^* \text{ such that } V_\varphi(\rho \cdot_t \rho') \in \{\top, \perp\}.$$

- φ is strongly monitorable if it is strongly $(\varepsilon, 0)$ -monitorable.
- φ is weakly monitorable if it is weakly $(\varepsilon, 0)$ -monitorable.

Example 3.

1. Consider the property $\varphi_1 = F_{\geq 0}a$. For every observation (ρ, t) , we have $V_{\varphi_1}(\rho \cdot_t (a, 0), t) = \top$. Hence, φ_1 is strongly monitorable.
2. Now, consider $\varphi_2 = a \rightarrow G_{\geq 0}F_{\geq 0}a$. Then, we have $V_{\varphi_2}((b, 0), 0) = \top$, as every extension of $((b, 0), 0)$ satisfies φ_2 (as the premise is violated). Hence, φ_2 is weakly monitorable.
However, it is not strongly monitorable: Consider the observation $((a, 0), 0)$, for which every extension satisfies the premise. We have $(a, 0) \cdot_0 \rho'' \cdot (a, 0), (a, 1), (a, 2), \dots \models \varphi_2$ and $(a, 0) \cdot_0 \rho'' \cdot (b, 0), (b, 1), (b, 2), \dots \not\models \varphi_2$. Hence, $V_{\varphi_2}((a, 0) \cdot_0 \rho'') = ?$ for all ρ'' .
3. Now, consider $\varphi_3 = G_{\geq 0}F_{\geq 0}a$. Arguments as for φ_2 show that it is neither strongly nor weakly monitorable, as every finite word can be extended by $(a, 0), (a, 1), (a, 2), \dots$ to satisfy φ_3 and can be extended by $(b, 0), (b, 1), (b, 2), \dots$ to violate φ_3 .

Remark 1. The astute reader might wonder why we quantify only over words ρ' (and ρ'') in Definition 3 and not over words and timepoints to concatenate at. The reason is that both definitions are equivalent, as $\rho_1 \cdot_t \rho_2$, for finite words ρ_1 and ρ_2 and a timepoint $t \geq \tau(\rho)$, is equal to $\rho_1 \cdot \rho'_2$, where ρ'_2 is obtained from ρ_2 by incrementing all its timepoints by $t - \tau(\rho_1)$.

Let us continue by collecting some simple consequences of Definition 3.

Remark 2. Let (ρ, t) and (ρ', t') be two observations with $\rho \sqsubseteq_t \rho'$, and let $\varphi \subseteq T\Sigma^\omega$.

1. If φ is strongly (ρ, t) -monitorable, then it is also weakly (ρ, t) -monitorable. Thus, if φ is strongly monitorable, then it is also weakly monitorable.

2. If φ is strongly (ρ, t) -monitorable, then it is also strongly (ρ', t') -monitorable.
3. If φ is not weakly (ρ, t) -monitorable, then it is also not weakly (ρ', t') -monitorable.

In the following we study the decidability of monitorability, i.e., we consider the following decision problems where properties are given by timed automata:

1. Given a property φ , is φ strongly monitorable?
2. Given a property φ , is φ weakly monitorable?
3. Given a property φ and an observation (ρ, t) , is φ strongly (ρ, t) -monitorable?
4. Given a property φ and an observation (ρ, t) , is φ weakly (ρ, t) -monitorable?

By definition, if Problem 3 is decidable for a class of properties, then Problem 1 is also decidable for the same class of properties. Hence, if Problem 1 is undecidable for a class of properties, then Problem 3 is also undecidable for the same class of properties. A similar relation holds between Problem 2 and Problem 4.

3 Monitoring and Monitorability for TBA

In this section, we prove that the monitoring function is not computable and that the strong types of monitorability are undecidable when the property is given by a TBA. Note that this shows that the positive results for monitoring in the literature [20, 18, 13], which require automata both for the property and its complement, are tight in that sense: Only giving an automaton for the property, but not for its complement, is not sufficient to compute the monitoring function. We start by investigating the computability of the monitoring function.

Theorem 1 (Ineffectiveness of Timed Monitoring). *The function “Given a TBA \mathcal{A} and an observation (ρ, t) , return $V_{L(\mathcal{A})}(\rho, t)$ ” is not computable.*

Proof. For every property $\varphi \subseteq T\Sigma^\omega$, we have $V_\varphi(\varepsilon, 0) = \top$ if and only if $\varphi = T\Sigma^\omega$. Hence, universality of a TBA \mathcal{A} reduces to checking whether $V_{L(\mathcal{A})}(\varepsilon, 0) = \top$. As universality for timed automata is undecidable [5], the monitoring function cannot be computable. \square

Note that the specification automaton \mathcal{A} is part of the input in the problem considered in Theorem 1. We leave it open whether $V_{L(\mathcal{A})}$ is computable for every fixed \mathcal{A} , i.e., in the setting where only the observation is the input.

Next, we turn our attention to deciding monitorability for TBA.

Theorem 2. *Strong and weak monitorability are undecidable for properties given by TBA.*

Proof. We reduce the (undecidable [3]) universality problem for non-deterministic timed automata (over finite words) to the problem of strong monitorability, following Diekert, Muscholl, and Walukiewicz [16].

Let $\mathcal{A} = (Q, Q_0, \Sigma, \mathcal{X}, \Delta, F)$ be a such a timed automaton, i.e., a finite run is accepting if it ends in a location in F . We add a new letter $\# \notin \Sigma$ to obtain $\Sigma_\# = \Sigma \cup \{\#\}$ and construct a TBA $\mathcal{A}' = (Q', Q_0, \Sigma_\#, \mathcal{X}, \Delta', F')$ from \mathcal{A} such that \mathcal{A}' is strongly monitorable if and only if $L(\mathcal{A}) = T\Sigma^*$. To this end, we introduce three new locations r, s, t , i.e., $Q' = Q \cup \{r, s, t\}$. Next, we define Δ' (see Figure 2) by copying the transitions from Δ and by adding the following transitions, where we write $q \xrightarrow{a} q'$ to denote $(q, q', a, \emptyset, \text{true})$:

- $\{q \xrightarrow{\#} r \mid q \in Q \setminus F\}$: from every non-accepting location of \mathcal{A} , there is a $\#$ -transition to r .
- $\{r \xrightarrow{a} s, s \xrightarrow{a} s \mid a \in \Sigma\}$: for every $a \neq \#$ there is an a -transition from r to s and an a -labeled self-loop on s .
- $\{s \xrightarrow{\#} r, r \xrightarrow{\#} r\}$: there is a $\#$ -transition from s to r and a $\#$ -labeled self-loop on r .
- $\{q \xrightarrow{\#} t \mid q \in F\}$: from every accepting location there is a $\#$ -transition to t .

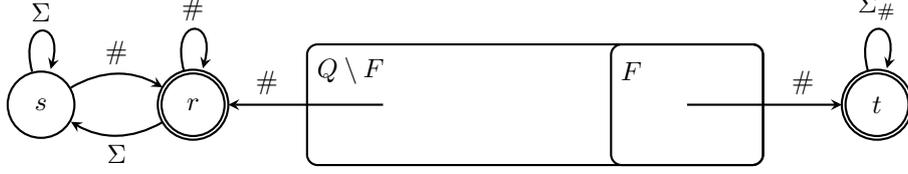


Figure 2: The TBA \mathcal{A}' constructed for the proof of Theorem 2.

- $\{t \xrightarrow{a} t \mid a \in \Sigma_{\#}\}$: for every letter in $\Sigma_{\#}$, there is a self-loop on t labeled with that letter.

We define the accepting locations of \mathcal{A}' as $F' = Q \cup \{r, t\}$.

Figure 2 shows the TBA \mathcal{A}' with the locations of the (finite-word) timed automaton \mathcal{A} partitioned into F and $Q \setminus F$. In the figure, the new locations r, s , and t and accompanying transitions are shown separately from the original automaton. Double circles denote accepting locations. Intuitively, processing a $\#$ from an accepting location of \mathcal{A} takes the run to the accepting sink t . On the other hand, processing a $\#$ from a non-accepting location of \mathcal{A} takes the run to a component where the run continuation is accepting if and only if it processes infinitely many $\#$'s.

It remains to show that the language $L(\mathcal{A}')$ is strongly monitorable if and only if $L(\mathcal{A}) = T\Sigma^*$. One direction is trivial: If $L(\mathcal{A}) = T\Sigma^*$, then $L(\mathcal{A}') = T\Sigma_{\#}^*$. Hence, $L(\mathcal{A}')$ is strongly monitorable. On the other hand, if $L(\mathcal{A}) \neq T\Sigma^*$, then there exists a finite timed word $\rho \notin L(\mathcal{A})$. Hence, every run prefix of \mathcal{A}' processing $\rho \cdot_{\tau(\rho)} (\#, \tau(\rho))$ must be in location r . Now, choose some $a \in \Sigma$. Then, for all $\rho'' \in T\Sigma_{\#}^*$, we have:

- $\rho \cdot_{\tau(\rho)} (\#, \tau(\rho)) \cdot \rho'' \cdot (a, 0), (a, 1), (a, 2), \dots \notin L(\mathcal{A}')$ (as it contains only finitely many $\#$).
- $\rho \cdot_{\tau(\rho)} (\#, \tau(\rho)) \cdot \rho'' \cdot (\#, 0), (\#, 1), (\#, 2), \dots \in L(\mathcal{A}')$ (as it contains infinitely many $\#$).

Hence, $V_{L(\mathcal{A}')}(\rho \cdot_{\tau(\rho)} (\#, \tau(\rho)) \cdot \rho'') = ?$ for all ρ'' . Thus, $L(\mathcal{A}')$ is not strongly monitorable.

Now, we consider weak monitorability of TBA and reduce the (undecidable [3]) universality problem for TBA to it. As an intermediate step, we consider a problem about Brzozowski derivatives of properties of infinite timed words: Let $\rho \in T\Sigma^*$ be a finite timed word and $\varphi \subseteq T\Sigma^{\omega}$ be such a property. We say that ρ is a universal prefix for φ if the Brzozowski derivative

$$\{\mu \in T\Sigma^{\omega} \mid \rho \cdot \mu \in \varphi\}$$

of φ and ρ is equal to $T\Sigma^{\omega}$. Note that if φ is equal to $T\Sigma^{\omega}$, then every prefix is universal for φ . However, the property $(a, 0) \cdot T\Sigma^{\omega}$ has a universal prefix (e.g., $(a, 0)$), but is not universal.

Further, we say that φ is weakly \top -monitorable (cp. Definition 5) if and only if there is a ρ such that $V_{\varphi}(\rho) = \top$, i.e., in comparison to (standard) weak monitorability, we only consider the verdict \top . Note that φ has a universal prefix if and only if it is weakly \top -monitorable.

So, it remains to first reduce universality of TBA to the existence of a universal prefix for languages of TBA and then to reduce weak \top -monitorability to weak monitorability.

We begin with reducing TBA universality to the existence of universal prefixes. Intuitively, we add edges allowing to “restart” a run which allow every prefix to simulate the empty prefix. Then, every prefix behaves like empty prefix, which is universal if and only if the automaton is universal.

Let $\mathcal{A} = (Q, Q_0, \Sigma, \mathcal{X}, \Delta, F)$ be a TBA. We add a new letter $\# \notin \Sigma$ to obtain $\Sigma_{\#} = \Sigma \cup \{\#\}$ and construct a TBA $\mathcal{A}' = (Q', Q_0, \Sigma_{\#}, \mathcal{X}, \Delta', F')$ from \mathcal{A} such that $L(\mathcal{A}) = T\Sigma^{\omega}$ if and only if \mathcal{A}' has a universal prefix. To this end, we introduce two new locations r and s , i.e., $Q' = Q \cup \{r, s\}$. Next, we define Δ' (see Figure 3) by copying the transitions from Δ and by adding the following transitions, where we write $q \xrightarrow{a} q'$ to denote $(q, q', a, \emptyset, \text{true})$:

- $\{q \xrightarrow{a} r \mid q \in Q_0, a \in \Sigma_{\#}\}$: from every initial location of \mathcal{A} , there is an a -transition to r for every $a \in \Sigma_{\#}$.
- $\{r \xrightarrow{a} s, s \xrightarrow{a} s \mid a \in \Sigma\}$: for every $a \neq \#$ there is an a -transition from r to s and an a -labeled self-loop on s .

- $\{s \xrightarrow{\#} r, r \xrightarrow{\#} r\}$: there is a $\#$ -transition from s to r and a $\#$ -labeled self-loop on r .
- $\{(q, q_0, \#, \mathcal{X}, \text{true}) \mid q \in Q, q \in Q_0\}$: from every location there is a $\#$ -transition to each initial location, which resets all clocks.

We define the accepting locations of \mathcal{A}' as $F' = Q \cup \{r\}$.

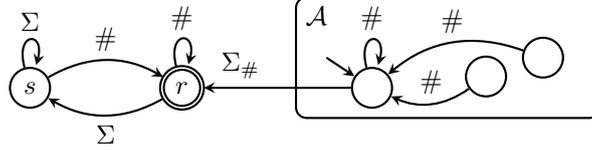


Figure 3: The TBA \mathcal{A}' constructed for the proof of Theorem 2.

Let $L(\mathcal{A}) = T\Sigma^\omega$ and fix some $\mu' \in T\Sigma_{\#}^\omega$. If μ' contains infinitely many $\#$'s, then it is accepted by \mathcal{A}' using the new states r and s . On the other hand, if μ' contains only finitely many $\#$'s, then it has the form

$$\mu' = \rho_0 \cdot (\#, t_0) \cdot \rho_1 \cdot (\#, t_1) \cdots (\#, t_{n-2}) \cdot \rho_{n-1} \cdot (\#, t_{n-1}) \cdot \mu''$$

for some $n \geq 0$, where the ρ_i and μ'' are $\#$ -free. As each ρ_i is a prefix of some word in $L(\mathcal{A}) = T\Sigma^\omega$ and μ'' is in $L(\mathcal{A}) = T\Sigma^\omega$, one can construct an accepting run of \mathcal{A}' on μ' . Thus, $L(\mathcal{A}') = T\Sigma_{\#}^\omega$, i.e., ε is a universal prefix of \mathcal{A}' .

Now for the other direction. If a word ρ is a universal prefix of \mathcal{A}' , then for all $\mu \in T\Sigma^\omega$, the word $\rho \cdot (\#, 0) \cdot \mu$ is accepted by \mathcal{A}' . Hence, there is an accepting run of \mathcal{A}' on $\rho \cdot (\#, 0) \cdot \mu$. As all $\#$ -transitions lead to an initial state of \mathcal{A} and reset the clocks, and as μ does not contain any $\#$'s, the suffix of the run on μ must also be an accepting run of \mathcal{A} , i.e., we have $\mu \in L(\mathcal{A})$. Thus, $L(\mathcal{A}) = T\Sigma^\omega$. This concludes the first step of our proof.

In the second and last step of our proof we reduce weak \top -monitorability to (standard) weak monitorability. Intuitively, we manipulate the automaton so that it can never give the verdict \perp while preserving the existence of observations that yield the verdict \top .

Let $\mathcal{A} = (Q, Q_0, \Sigma, \mathcal{X}, \Delta, F)$ be a TBA. We add a new letter $\# \notin \Sigma$ to obtain $\Sigma_{\#} = \Sigma \cup \{\#\}$ and construct a TBA $\mathcal{A}' = (Q', Q'_0, \Sigma_{\#}, \mathcal{X}, \Delta', F')$ from \mathcal{A} such that \mathcal{A} is weakly \top -monitorable if and only if \mathcal{A}' is weakly monitorable.

Now Q' essentially contains two copies of Q : $Q_1 = Q \times \{1\}$ and $Q_2 = Q \times \{2\}$ as well as a new additional accepting location r . The transition relation of \mathcal{A}' is defined as follows (see also Figure 4), we write $q \xrightarrow{a} q'$ to denote $(q, q', a, \emptyset, \text{true})$:

- $\{(q, 1) \xrightarrow{\#} (q, 1), (q, 1) \xrightarrow{\#} r \mid q \in Q\}$: locations of Q_1 have a $\#$ -labeled self-loop and a $\#$ -transition to r .
- $\{((q, 1), (q', 2), a, \lambda, g) \mid (q, q', a, \lambda, g) \in \Delta\}$: locations of Q_1 have their Σ -labeled transitions redirected to Q_2 .
- $\{(q, 2), (q', 2), a, \lambda, g) \mid (q, q', a, \lambda, g) \in \Delta\}$: locations of Q_2 have copies of the Σ -labeled transitions from \mathcal{A} .
- $\{(q, 2) \xrightarrow{\#} (q, 1) \mid q \in Q\}$: locations of Q_2 have $\#$ -labeled transitions directed back to Q_1 .
- $\{r \xrightarrow{\#} r\}$: The new location r has a $\#$ -labeled self-loop.

The set Q'_0 of initial locations of \mathcal{A}' is $Q_0 \times \{1\}$ and the set F' of accepting locations of \mathcal{A}' is $F \times \{2\} \cup \{r\}$.

Now, let $\rho \in T\Sigma$ be a timed word witnessing weak \top -monitorability of \mathcal{A} , i.e., $\rho \cdot \mu \in L(\mathcal{A})$ for every $\mu \in T\Sigma^\omega$. We define $\rho' = \rho$. In \mathcal{A} , processing ρ' leads to a location in Q_2 . We argue that ρ' witnesses weak monitorability of \mathcal{A}' . To this end, consider any $\mu' \in T\Sigma_{\#}^\omega$ and let μ be obtained from μ' by removing all occurrences of $\#$. If μ is infinite, we may simply mimic the accepting run of \mathcal{A} on μ from ρ in \mathcal{A}' . Clearly, this will also be accepting. If μ is finite, μ' must have a suffix of the form $(\#^\omega, \tau)$, where τ is a sequence of timepoints. This suffix is accepted by utilizing the $\#$ transitions to r , from where the suffix $(\#^\omega, \tau)$ can be accepted.

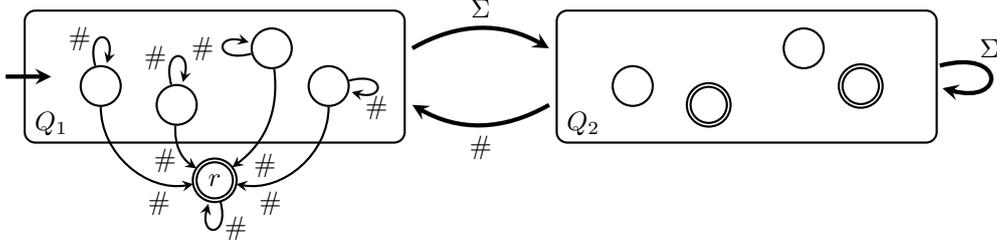


Figure 4: The TBA \mathcal{A}' constructed for the proof of Theorem 2.

For the opposite direction, let $\rho' \in T\Sigma\#$ be a timed word witnessing weak monitorability of \mathcal{A}' . That is, either $\rho' \cdot \mu' \in L(\mathcal{A}')$ for all μ' or $\rho' \cdot \mu' \notin L(\mathcal{A}')$ for all μ' . However, note that for any finite ρ' , we have $\rho' \cdot (\#, 0), (\#, 1), (\#, 2), \dots \in L(\mathcal{A}')$, due to the $\#$ -labeled transition to r , from where $(\#, 0), (\#, 1), (\#, 2), \dots$ is accepted. Thus, we have $\rho' \cdot \mu' \in L(\mathcal{A}')$ for all μ' .

Now, let $\rho \in T\Sigma$ be obtained from ρ' by stripping all occurrences of $\#$ in ρ' . We show that ρ is a word witnessing weak \top -monitorability of \mathcal{A} . Note that any run of \mathcal{A}' on ρ' reaching a location in Q_1 or Q_2 can be mimicked by a run of \mathcal{A} on ρ reaching the corresponding location in Q . Let $\mu \in T\Sigma^\omega$. Then $\rho' \cdot \mu \in L(\mathcal{A}')$. Clearly, this must be due to an accepting run where a suffix of the run processing μ stays in Q_2 . Hence, $\rho \cdot \mu \in L(\mathcal{A})$, as we may mimic, for the prefix ρ , the prefix processing ρ' of the accepting run of \mathcal{A}' processing $\rho' \cdot \mu$, and, for the suffix μ , we simply copy the run staying in Q_2 . \square

Due to Remark 2, we also obtain the undecidability of strong and weak (ρ, t) -monitorability.

Corollary 1. *Strong and weak (ρ, t) -monitorability is undecidable for properties given by TBA, even if (ρ, t) is fixed.*

4 Monitorability for DTMA

In this section, we show that (strong and weak) monitorability is decidable for properties given by DTMA. The key difference to TBA, for which we have shown that strong monitorability is undecidable, is that they are trivially closed under complement [3], which we rely on in our proof. This again demonstrates the importance of having automata for the property and its complement, as noted throughout this paper.

Let us begin by introducing some notation and definitions. Fix some TBA or TMA $\mathcal{A} = (Q, Q_0, \Sigma, \mathcal{X}, \Delta, \text{Acc})$. We write $(q_0, v_0) \xrightarrow{\rho}_{\mathcal{A}} (q_n, v_n)$ for a finite timed word $\rho = (\sigma, \tau) \in T\Sigma^*$ to denote the existence of a finite sequence

$$(q_0, v_0) \xrightarrow{(\sigma_1, \tau_1)} (q_1, v_1) \xrightarrow{(\sigma_2, \tau_2)} \dots \xrightarrow{(\sigma_n, \tau_n)} (q_n, v_n)$$

of states, where for all $1 \leq i \leq n$ there is a transition $(q_{i-1}, q_i, \sigma_i, \lambda_i, g_i)$ such that $v_i(c) = 0$ for all c in λ_i and $v_{i-1}(c) + (\tau_i - \tau_{i-1})$ otherwise, and g is satisfied by the valuation $v_{i-1} + (\tau_i - \tau_{i-1})$, where we use $\tau_0 = 0$.

Definition 4 (Non-Empty Language States, Reach Set, Pre^*). *Let \mathcal{A} be a TBA or TMA.*

- *The set of non-empty language states of \mathcal{A} is*

$$S_{\mathcal{A}}^{ne} = \{s \mid s \text{ is a state of } \mathcal{A} \text{ and } L(\mathcal{A}, s) \neq \emptyset\}.$$

Here, $L(\mathcal{A}, s)$ is the set of infinite timed words accepted by a run starting in the state s .

- *Given an observation (ρ, t) , the set of states in which a run over ρ can end starting from the initial states of \mathcal{A} after time t has passed is*

$$\mathcal{T}_{\mathcal{A}}(\rho, t) = \bigcup_{q_0 \in Q_0} \{(q, v + (t - \tau(\rho))) \mid (q_0, v_0) \xrightarrow{\rho}_{\mathcal{A}} (q, v)\},$$

where v_0 is the clock valuation mapping every clock to 0. We call $\mathcal{T}_{\mathcal{A}}(\rho, t)$ the reach set of (ρ, t) in \mathcal{A} .

- The Pre^* of a set S of states is

$$Pre_{\mathcal{A}}^*(S) = \{(q, v) \mid (q, v) \xrightarrow{\rho}_{\mathcal{A}} (q', v') \text{ for some } (q', v' + (t - \tau(\rho))) \in S, \rho \in T\Sigma^*, \text{ and } t \geq \tau(\rho)\}.$$

A symbolic state is a pair (q, Z) of a location q and a zone Z . A zone is a finite conjunction of clock constraints of the form $x_1 \sim n$ or $x_1 - x_2 \sim n$, where x_1, x_2 are clocks, $\sim \in \{<, \leq, =, \geq, >\}$, and $n \in \mathbb{Q}$. Such a zone describes a convex set of clock valuations. Zones may be efficiently represented using so-called Difference-bounded Matrices (DBM) [10].

Proposition 4 ([20]). *There are zone-based algorithms for the following problems:*

- Given a TBA, compute its nonempty language states.
- Given a TBA or TMA \mathcal{A} and an observation (ρ, t) , compute the reach set $\mathcal{T}_{\mathcal{A}}(\rho, t)$.
- Given a TBA or TMA \mathcal{A} and a finite union S of symbolic states, compute $Pre_{\mathcal{A}}^*(S)$.

We continue by showing that the non-empty language states can also be computed for TMA, relying on an analysis of Alur and Dill's translation of TMA into equivalent TBA [3]. Here, we just present the parts of their construction we need to prove our results.

Lemma 1. *There is a zone-based algorithm that, given a TMA, computes its nonempty language states.*

Proof. Fix a TMA $\mathcal{A} = (Q, Q_0, \Sigma, \mathcal{X}, \Delta, \mathcal{F})$. The translation from TMA to TBA relies on the fact that $L(\mathcal{A}) = \bigcup_{F \in \mathcal{F}} L(\mathcal{A}_F)$, where \mathcal{A}_F is the TMA $(Q, Q_0, \Sigma, \mathcal{X}, \Delta, \{F\})$. Thus, it suffices to translate each such \mathcal{A}_F into an equivalent TBA \mathcal{A}'_F with set $Q \times \{0, 1, \dots, |F|\}$ of locations and the same clocks as \mathcal{A} . Then, the disjoint union of the \mathcal{A}'_F for $F \in \mathcal{F}$ is a TBA that is equivalent to \mathcal{A} .

The TBA \mathcal{A}'_F constructed by Alur and Dill satisfy the following property: $L(\mathcal{A}'_F, ((q, 0), v)) = L(\mathcal{A}_F, (q, v))$ for all locations q of \mathcal{A} and all clock valuations. Thus, we have

$$S_{\mathcal{A}}^{ne} = \bigcup_{F \in \mathcal{F}} \{(q, v) \mid ((q, 0), v) \in S_{\mathcal{A}'_F}^{ne}\}.$$

Hence, we can symbolically compute the non-empty language states of \mathcal{A} by symbolically computing the non-empty states of the \mathcal{A}'_F and then project the states of the form $(q, 0)$ to q , but leaving the zones unchanged. \square

Using this result, we can decide monitorability.

Theorem 3. *Strong and weak (ρ, t) -monitorability are decidable for properties given by DTMA.*

Proof. We assume w.l.o.g. that \mathcal{A} is complete in the sense that every word has a run, which then must be unique due to determinism. This can always be achieved by adding a fresh sink location and by rerouting all missing transitions to it (while preserving determinism).

Given \mathcal{A} , we compute the non-empty language states $S_{\mathcal{A}}^{ne}$. The complement of the non-empty language states is the empty language states, i.e., the set of states for which there is no accepting run starting there. Let us denote this set as $S_{\mathcal{A}}^{\emptyset} = \overline{S_{\mathcal{A}}^{ne}}$. Using backwards reachability, we compute all the possibly-empty language states, i.e., states from which there is at least one non-accepting run, i.e., $S_{\mathcal{A}}^{pe} = Pre_{\mathcal{A}}^*(S_{\mathcal{A}}^{\emptyset})$. Note that $S_{\mathcal{A}}^{\emptyset} \subseteq S_{\mathcal{A}}^{pe}$. If a state s is not in $S_{\mathcal{A}}^{pe}$, then all states reachable from s have at least one accepting run.

The sets $S_{\mathcal{A}}^{ne}$, $S_{\mathcal{A}}^{\emptyset}$ and $S_{\mathcal{A}}^{pe}$ are illustrated in Fig. 5.

Given an observation (ρ, t) , we can compute the reach-set of \mathcal{A} over (ρ, t) as $\mathcal{T}_{\mathcal{A}}(\rho, t)$, which is a singleton set, as \mathcal{A} is deterministic and complete. Thus, we identify $\mathcal{T}_{\mathcal{A}}(\rho, t)$ with the unique state in it.

Now, we have $\mathcal{T}_{\mathcal{A}}(\rho, t) \in S_{\mathcal{A}}^{pe}$ if and only if there exists $\rho' \in T\Sigma^*$ such that for all $\mu \in T\Sigma^{\omega}$ we have $\rho \cdot_t \rho' \cdot \mu \notin L(\mathcal{A})$. If the latter condition is violated, then monitoring any extension of (ρ, t) will not provide the verdict \perp , i.e., $V_{L(\mathcal{A})}(\rho' \cdot_t \rho) \neq \perp$ for all $\rho' \in T\Sigma^*$.

Since DTMA are complementable by changing the acceptance condition, but not the set of locations and clocks, we can compute $S_{\mathcal{A}}^{pe}$ for the complement automaton $\overline{\mathcal{A}}$ and provide the dual characterization for the \top -verdict: $\mathcal{T}_{\mathcal{A}}(\rho, t) \in S_{\mathcal{A}}^{pe}$ if and only if there exists $\rho' \in T\Sigma^*$ such that for all $\mu \in T\Sigma^{\omega}$ we

have $\rho \cdot_t \rho' \cdot \mu \in L(\mathcal{A})$. Again, if the latter condition is violated, then monitoring any extension of (ρ, t) will not provide the verdict \top , i.e., $V_{L(\mathcal{A})}(\rho' \cdot_t \rho') \neq \top$ for all $\rho' \in T\Sigma^*$.

With these characterizations, we can conclude whether the \top or \perp verdict are still possible for some extensions, which allows us to decide weak (ρ, t) -monitorability: $L(\mathcal{A})$ is weakly (ρ, t) -monitorable if and only if $\mathcal{T}_{\mathcal{A}}(\rho, t) \in S_{\mathcal{A}}^{pe} \cup S_{\mathcal{A}}^{ne}$. Note that this characterization crucially depends on the fact that both \mathcal{A} and $\bar{\mathcal{A}}$ have the same states (locations and clock valuations).

For strong (ρ, t) -monitorability, we need to compute the states that can leave the possibly empty states via a finite run: \mathcal{A} is strongly (ρ, t) -monitorable if and only if $\mathcal{T}_{\mathcal{A}}(\rho, t) \notin \text{Pre}_{\mathcal{A}}^*(S_{\mathcal{A}}^{pe}) \cap \text{Pre}_{\mathcal{A}}^*(S_{\mathcal{A}}^{ne})$.

Due to Lemma 4 and Lemma 1, both characterizations are effectively decidable. \square

Due to Remark 2, we also obtain the decidability of strong and weak monitorability.

Corollary 2. *Strong and weak monitorability are decidable for properties given by DTMA.*

5 Monitorability with Step-Bounded Horizons

Strong monitorability of a property ensures that at any time during monitoring, the observation (ρ, t) made so far can be extended by some finite ρ' such that a conclusive verdict can be made after $\rho \cdot_t \rho'$. Thus, in the setting of strong monitorability, it is always meaningful to keep monitoring. In contrast, for weakly monitorable properties, the ability to make a conclusive verdict after some future (finite) monitoring is not guaranteed. Ideally, we would like to refine the rather uninformative verdict $?$ with guaranteed minimum bounds on the number of further events before a positive or negative verdict can be made. In case both these bounds are ∞ , further monitoring is useless as the verdicts reported will always be $?$. Here it is prudent to distinguish between the two definitive verdicts, as they have different decidability properties.

Definition 5 (Weak Monitorability with Step-Bounded Horizons). *Fix a property $\varphi \subseteq T\Sigma^\omega$, an observation (ρ, t) , and a bound $n \in \mathbb{N}$.*

- φ is bounded weakly \top - (ρ, t) -monitorable with respect to n if and only if

$$\text{there exists } \rho' \in T\Sigma^{\leq n} \text{ such that } V_\varphi(\rho \cdot_t \rho') = \top.$$

- φ is bounded weakly \perp - (ρ, t) -monitorable with respect to n if and only if

$$\text{there exists } \rho' \in T\Sigma^{\leq n} \text{ such that } V_\varphi(\rho \cdot_t \rho') = \perp.$$

- φ is bounded weakly (ρ, t) -monitorable with respect to n if and only if

$$\text{there exists } \rho' \in T\Sigma^{\leq n} \text{ such that } V_\varphi(\rho \cdot_t \rho') \in \{\top, \perp\}.$$

- φ is bounded weakly \top -monitorable with respect to n if it is bounded weakly \top - $(\varepsilon, 0)$ -monitorable with respect to n .

- φ is bounded weakly \perp -monitorable with respect to n if it is bounded weakly \perp - $(\varepsilon, 0)$ -monitorable with respect to n .



Figure 5: Representation of all states of an automaton \mathcal{A} . On the left, only the empty language states $S_{\mathcal{A}}^0$ and non-empty language states $S_{\mathcal{A}}^{ne}$ are shown. On the right, the possibly empty language states $S_{\mathcal{A}}^{pe}$ are added in gray. The arrows are examples of possible transitions and the dashed arrows are examples of impossible transitions. A state in $S_{\mathcal{A}}^0$ has no accepting run, thus cannot reach $S_{\mathcal{A}}^{ne}$. A state in $S_{\mathcal{A}}^{ne}$ might reach a state in $S_{\mathcal{A}}^0$. A state outside $S_{\mathcal{A}}^{pe}$ cannot reach $S_{\mathcal{A}}^0$.

- φ is bounded weakly monitorable with respect to n if it is bounded weakly $(\varepsilon, 0)$ -monitorable with respect to n .

In the following, we show that bounded weak monitorability and bounded weak \top -monitorability are undecidable, but that bounded weak \perp -monitorability is decidable.

Theorem 4. *Bounded weak \top -monitorability is undecidable for properties given by TBA.*

Proof. Let φ be a property given by a TBA \mathcal{A} . Then universality of \mathcal{A} is equivalent to bounded weak \top -monitorability of φ for the bound $n = 0$. As universality for timed automata is undecidable [3], 0-bounded weak \top -monitorability for TBA properties is undecidable. \square

Next, we show that weak \perp - (ρ, t) -monitorability behaves differently.

Theorem 5. *Bounded weak \perp - (ρ, t) -monitorability is decidable for properties given by TBA.*

Proof. Let (ρ, t) be an observation with $\rho = (\sigma_1, t_1), (\sigma_2, t_2), \dots, (\sigma_m, t_m)$ and let the property be given by the TBA \mathcal{A} .

Let $\pi = q_0 \xrightarrow{\sigma_1, \lambda_1, g_1} q_1 \cdots q_{m+n-1} \xrightarrow{\sigma_{m+n}, \lambda_{m+n}, g_{m+n}} q_{m+n}$ be a syntactic path of length $m + n$ of the TBA \mathcal{A} induced by transitions $(q_i, q_{i+1}, \sigma_i, \lambda_i, g_i)$. Also, let $\tau = (\tau_1, \dots, \tau_{m+n})$ be variables ranging over global time (i.e., $\mathbb{R}_{\geq 0}$), where τ_i denotes the time the i -th transition is taken. Then, there is a zone $Z_\pi(\tau)$ over τ that captures when (σ, τ) is a timed word realizing π . In this case, we may also express the resulting clock valuation after realizing (σ, τ) on π as $v_\pi(\tau) = (v_\pi^1, \dots, v_\pi^k)$, where k is the number of clocks of \mathcal{A} and $v_\pi^i = \tau_{m+n} - \tau_{\ell(i)}$, with $\ell(i)$ being the index of the last transition when the clock x_i was reset. Now, if $(q_{m+n}, v_\pi(\tau)) \notin S_{\mathcal{A}}^{ne}$, then the run of (σ, τ) following π cannot be extended to an accepting run of \mathcal{A} .

The following formula expresses the existence of a timed word $\rho' = (\sigma', \tau')$ of length n^1 such that all runs on $\rho \cdot_t \rho'$ end in a state outside of $S_{\mathcal{A}}^{ne}$, which characterizes $V_{L(\mathcal{A})}(\rho \cdot_t \rho') = \perp$:

$$\exists \tau_1 \dots \exists \tau_{m+n} \bigvee_{\sigma_{m+1} \cdots \sigma_{m+n} \in \Sigma^n} \bigwedge_{\pi \in \text{SP}} \left(\bigwedge_{i=1}^m \tau_i = t_i \wedge \tau_m \leq t \leq \tau_{m+1} \wedge Z_\pi(\tau) \right) \rightarrow v_\pi(\tau) \notin S_{\mathcal{A}}^{ne},$$

where SP is the set of syntactic paths of the form $\pi = q_0 \xrightarrow{g_1, \sigma_1, \lambda_1} q_1 \cdots \xrightarrow{g_n, \sigma_n, \lambda_n} q_n$. The formula is in the first order theory of real-closed fields, which is decidable [29]. Thus, bounded weak \perp -monitorability is decidable for properties given by TBA. \square

Remark 3. *It follows from Theorem 5 that for properties where both the property and its complement are given by TBA, bounded weak monitorability is decidable. Hence, if the property is given by a DTMA, then bounded weak monitorability is decidable. Moreover, one can even compute the tightest bound n using standard reachability techniques (i.e., breadth-first zone-based search) for timed automata.*

Lastly, we prove bounded weak monitorability undecidable for properties given by TBA.

Theorem 6. *Bounded weak monitorability is undecidable for properties given by TBA.*

Proof. In the second step of the undecidability proof of weak monitorability (see Theorem 2), we have shown how to reduce weak \top -monitorability to weak monitorability. The same construction reduces bounded weak \top -monitorability to bounded weak monitorability, as the length of the witnesses is only decreased by removing $\#$'s. \square

Due to Remark 2, we also obtain results for the remaining cases.

Corollary 3. *Bounded weak \top - (ρ, t) -monitorability and bounded weak (ρ, t) -monitorability are undecidable (even for fixed (ρ, t)) while bounded weak \perp -monitorability is decidable.*

¹We can restrict ourselves w.l.o.g. to words of length exactly n instead of words of length at most n , as witnesses for weak monitorability are closed under extensions.

6 Refined Monitoring with Time-Horizon Verdicts

In the previous section, we have shown that the uninformative verdict $?$ can be refined by checking whether within a bounded number of new observations, a definitive verdict may be given. In this section, we again refine the uninformative verdict $?$ by computing lower bounds on the time that needs to pass (independently of the number of events observed during that time) before a definitive verdict may be given. Again, if this is infinite, then the monitoring process can be stopped, as no amount of waiting will yield a definitive verdict. This refinement was introduced and briefly studied as “time predictive monitoring” by Grosen et al. [20]. Here we revisit (and rename) this notion by proving computability of the time-bounded monitorability for properties given by a DTMA.

Recall that time passing without any new observed events can nevertheless yield definitive verdicts (see, e.g., the last two items in Example 2). Thus, in a practical setting, one is interested in intermittently querying the monitoring function even if no events are observed. Here, we give lower bounds on the time one should let pass before the next such query is made, thereby reducing the computational overhead of these queries.

Definition 6 (Refined Monitoring with Time-Horizon Verdicts). *Given an observation (ρ, t) and a property $\varphi \subseteq T\Sigma^\omega$, the refined monitoring function P_φ is defined as*

$$P_\varphi(\rho, t) = \begin{cases} \top & \text{if } V_\varphi(\rho, t) = \top, \\ \perp & \text{if } V_\varphi(\rho, t) = \perp, \\ (P_\varphi^\top(\rho, t), P_\varphi^\perp(\rho, t)) & \text{otherwise,} \end{cases}$$

with

- $P_\varphi^\top(\rho, t) = \inf \{t' \mid \rho' \in T\Sigma^* \text{ such that } V_\varphi(\rho \cdot_t \rho', t') = \top \text{ and } t' \geq \tau(\rho \cdot_t \rho')\}$ and
- $P_\varphi^\perp(\rho, t) = \inf \{t' \mid \rho' \in T\Sigma^* \text{ such that } V_\varphi(\rho \cdot_t \rho', t') = \perp \text{ and } t' \geq \tau(\rho \cdot_t \rho')\},$

where we use the convention $\inf \emptyset = \infty$.

Example 4. Consider the MITL property $\varphi = F_{[20,40]}b$. Monitoring the finite timed word $\rho = (a, 5.1), (c, 21.0), (c, 30.4), (b, 35.1)$ will result in three $?$ verdicts followed by the verdict \top when $(b, 35.1)$ is read. However, we may offer significantly more information, e.g., when reading $(a, 5.1)$ it is clear that at least 14.9 time-units must elapse before we can give the verdict \top , and at least 34.9 time-units must elapse before we can give the verdict \perp . Hence, $P_\varphi((a, 5.1), 5.1) = (14.9, 34.9)$.

Remark 4. $V_\varphi(\rho, t) = \top$ implies $P_\varphi^\top(\rho, t) = 0$ and $P_\varphi^\perp(\rho, t) = \infty$ and $V_\varphi(\rho, t) = \perp$ implies $P_\varphi^\top(\rho, t) = \infty$ and $P_\varphi^\perp(\rho, t) = 0$, but the converse is in general not true. Consider, for example, the MITL property $\varphi = F_{\geq 0}a$ and the observation $\rho = (b, 1)$. Then, $P_\varphi^\top(\rho, 1) = 0$ (witnessed by $\rho' = (a, 0)$ for which we have $V_{\rho \cdot_t \rho'} = \top$) and $P_\varphi^\perp = \infty$, but $V_\varphi(\rho, 1) = ?$.

Time-bounded monitorability for an observation (ρ, t) refines weak (ρ, t) -monitorability.

Lemma 2. *Let (ρ, t) be an observation and φ a property. Then, φ is weakly (ρ, t) -monitorable if and only if at least one of the values $P_\varphi^\top(\rho, t)$ and $P_\varphi^\perp(\rho, t)$ is finite.*

Proof. Let φ be weakly (ρ, t) -monitorable, i.e., there is a $\rho' \in T\Sigma^*$ such that $V_\varphi(\rho \cdot_t \rho') \in \{\top, \perp\}$, say it is \top (the other case is analogous). Then, ρ' witnesses $P_\varphi^\top(\rho, t) \leq \tau(\rho')$, i.e., $P_\varphi^\top(\rho, t)$ is finite. On the other hand, if (say) $P_\varphi^\top(\rho, t)$ is finite (the other case is analogous), then there is a ρ' such that $V_\varphi(\rho \cdot_t \rho') = \top$. Hence, φ is weakly (ρ, t) -monitorable. \square

Theorem 7 (Refined Monitoring is Effective). *P_φ is effectively computable, if φ is given by a DTMA.*

Proof. When online-monitoring φ over some observation (ρ, t) , we compute the reach-set $\mathcal{T}_A(\rho, t)$ and check if it is a subset of the non-empty language states S_A^{ne} or a subset of the empty language states S_A^\emptyset [20]. Thus, $P_\varphi^\top(\rho, t)$ is the infimum of the time duration of all paths from $\mathcal{T}_A(\rho, t)$ to S_A^\emptyset . This is a time-optimal reachability problem.

Asarin and Maler [6] introduced the concept of time-optimal strategies for timed game automata. Since timed game automata trivially generalize timed automata, we can adapt the computation of the

optimal time bound to obtain the minimal possible time to reach an empty language state from the reach-set. This gives us the value $P_{\mathcal{A}}^{\perp}(\rho, t)$.

Since DTMA are closed under complement, the same procedure for the complement automaton gives us the value $P_{\mathcal{A}}^{\top}(\rho, t)$. \square

As explained in the proof of Theorem 7, monitoring is implemented by keeping track of the reach-set of the observation and checking at each update whether it is contained in the non-empty language states or in the empty language states. But, as explained in the introduction of this section, one should not only update the reach-set when a new event is observed, but also intermittently when time has passed.

For this special case where only time passes, which is covered in Definition 6 by considering $\rho' = \varepsilon$, we do not need to solve the expensive time-optimal reachability problem described in the proof of Theorem 7. Instead, we rely on simple zone operations to compute $\delta_{\varphi}(\rho, t) = \inf\{d \mid V_{\varphi}(\rho, t + d) \in \{\top, \perp\}\}$. Now, having observed ρ and the current time being t such that $V_{\varphi}(\rho, t) = ?$, querying the monitoring function again without a new observation before time $t + \delta_{\varphi}(\rho, t)$ will not yield a definitive verdict and can thus be avoided.

7 Related Work

A formal notion of monitorability was first introduced by Pnueli and Zaks in their work on monitoring Property Specification Language (PSL) [26]. In that work, the authors defined strong monitorability given a finite prefix, called σ -monitorability, on which we base our (ρ, t) -monitorability. From this, Bauer, Leuker, and Schallhart defined the most common definition of monitorability, that is strong σ -monitorability from the initial state [9]. They proved that safety and guarantee properties are a proper subset of the class of strongly monitorable properties. Later, Chen et al. [12] and Peled and Havelund [25] noticed that there exist properties that are not strongly monitorable but still have utility to monitor, and proposed equivalent definitions of weak monitorability. Mascle et al. showed the monitorability of LTL properties can be improved by considering robust semantics [24]. The term *strong monitorability* has been used before in the context of partially observable stochastic systems modeled as Hidden Markov Models. Sistla, Žefran, and Feng first used the term, contrasted with (standard) monitorability [28] in their work extending the results of Gondi, Patel, and Sistla on monitoring ω -regular properties of stochastic systems [19].

The complexity of monitorability problems has been studied in other untimed settings. Diekert and Leuker proposed a topological definition of strong monitorability, showing that problem is equivalent to showing that the boundary in the Cantor topology has an empty interior [15]. Diekert, Muscholl, and Walukiewicz later proved that deciding monitorability for (untimed) Büchi automata is PSPACE-complete [16]. Agrawal and Bonakdarpour proposed a definition of monitorability for hyperproperties and determined the monitorable classes for their three-valued specification language, HyperLTL [14]. Francalanza, Aceto, and Ingólfssdóttir characterized monitorable properties of the branching-time μ -Hennessy-Milner Logic [17]. This was later extended by Aceto et al. who introduced a hierarchy of monitorable fragments of the language [1].

Attempts have been made to unify these different notions of monitorability. Peled and Havelund introduced a classification for properties centered around monitorability [25]. Kauffman, Havelund, and Fischmeister defined a common notation for strong and weak monitorability for different verdict domains [21]. Aceto, Achilleos, and Francalanza provided syntactic characterizations of monitorability for classical notions of monitorability as well as for a variant of the modal μ -calculus, recHML [2].

8 Conclusion

In this work, we have studied, for the first time, monitorability for timed properties specified by either (possibly nondeterministic) TBA or by deterministic TMA. In general, we proved monitorability decidable for specifications given by deterministic automata and undecidable for specifications given by nondeterministic automata. The notable exception here is bounded weak \perp -monitorability, which is even decidable for nondeterministic TBA.

Also, we provided refinements of monitoring and monitorability making the verdict $?$ more informative by providing bounds on the number of events or the amount of time that needs to pass before

a conclusive verdict may be given. In practical settings, this is crucial information and also allows to optimize the monitoring process in the real-time setting.

Our decidability proof for monitorability of DTMA relies on the fact that DTMA can be complemented without changing the state space. On the other hand, monitorability is undecidable if the property is given by a TBA. Thus, another question for further research is to consider strong monitorability when given TBA for the property *and* for its negation. This is a very natural setting, as the specification logic MITL is closed under negation and can be translated into TBA. Also, monitoring often requires TBA for both the property and its negation [20, 18, 13].

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