The Complexity of Second-order HyperLTL

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Abstract

We determine the complexity of second-order HyperLTL satisfiability and model-checking: Both are as hard as truth in third-order arithmetic.

1 Introduction

The introduction of hyperlogics [2] for the specification and verification of hyperproperties [3], properties that relate multiple system executions, has been one of the major success stories of formal verification during the last decade. Logics like HyperLTL, the extension of LTL with trace quantification, and HyperCTL*, the extension of CTL* with trace quantification, are natural specification languages for information-flow properties, have a decidable model-checking problem [5], and hence found many applications.

However, while expressive enough to express common information-flow properties, they are unable to express other important hyperproperties, e.g., common knowledge in multi-agent systems and asynchronous properties (witnessed by a plethora of asynchronous extensions of HyperLTL). These examples all have in common that they are *second-order* properties, i.e., they naturally require quantification over sets of traces while HyperLTL only allows quantification over traces.

In light of this situation, Beutner et al. [1] introduced Hyper^{2}LTL, HyperLTL extended with second-order quantification of traces. They show that the logic is indeed able to capture common knowledge, asynchronous extensions of HyperLTL, and many other applications. However, they also note that this expressiveness comes at a steep price: model-checking Hyper^{2}LTL is highly undecidable, i.e., Σ_1^1-hard. Thus, their main result is a partial model-checking algorithm for a fragment of Hyper^{2}LTL where second-order quantification is replaced by inflationary (deflationary) fixedpoints of HyperLTL definable operators. The algorithm over- and underapproximates these fixedpoints and then invokes a HyperLTL model-checking algorithm on these approximations. A prototype implementation of the algorithm is able to model-check properties capturing common knowledge, asynchronous hyperproperties, and distributed computing.

However, one question has been left open: Just how complex is Hyper^{2}LTL verification?

Complexity Classes for Undecidable Problems The complexity of undecidable problems is typically captured in terms of the arithmetical and analytical hierarchy, where decision problems (encoded as subsets of $\mathbb{N}$) are classified based on their definability by formulas of higher-order arithmetic, namely by the type of objects one can quantify over and by the number of alternations of such quantifiers. We refer to Roger's textbook [9] for fully formal definitions and refer to Figure 1 for a visualization. We recall the following classes: Σ_1^0 contains the sets of natural numbers of the form

$$\{x \in \mathbb{N} \mid \exists x_0.\ \cdots \exists x_k.\ \psi(x, x_0, \ldots, x_k)\}$$

where quantifiers range over natural numbers and ψ is a quantifier-free arithmetic formula. Note that this is exactly the class of recursively enumerable sets. The notation Σ_1^0 signifies that there is a single block of existential quantifiers (the subscript 1) ranging over natural numbers (type 0 objects, explaining the superscript 0). Analogously, Σ_1^1 is induced by arithmetic formulas with existential quantification of type 1 objects (sets of natural numbers) and arbitrary (universal and existential) quantification of type 0 objects. So, Σ_1^0 is part of the first level of the arithmetical hierarchy while Σ_1^1 is part of the first level of the analytical

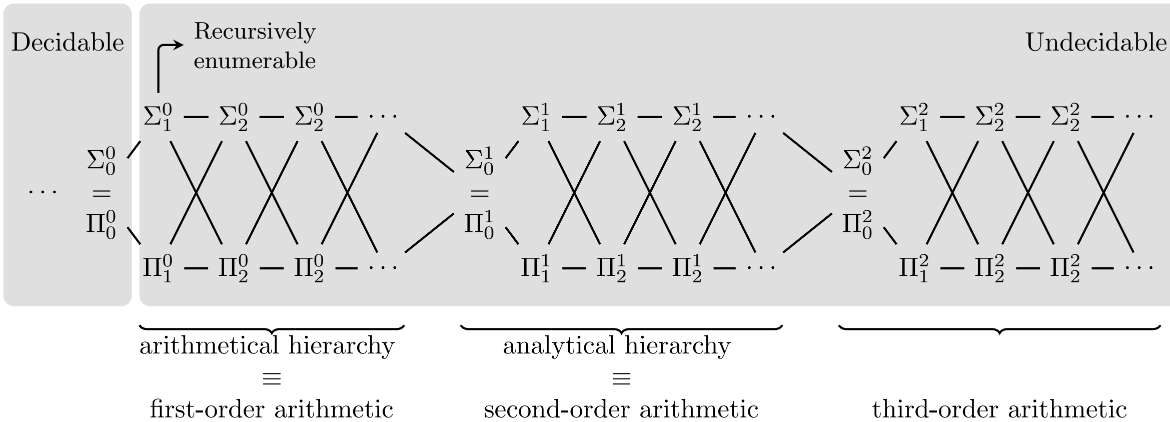

Figure 1: The arithmetical hierarchy, the analytical hierarchy, and beyond.

hierarchy. In general, level Σ^0_n (level Π^0_n) of the arithmetical hierarchy is induced by formulas with at most n alternations between existential and universal type 0 quantifiers, starting with an existential (universal) quantifier. Similar hierarchies can be defined for arithmetic of any fixed order by limiting the alternations of the highest-order quantifiers and allowing arbitrary lower-order quantification. In this work, the highest order we are concerned with is three, i.e., quantification over sets of sets of natural numbers.

HyperLTL satisfiability is Σ^1_1-complete [7], HyperLTL finite-state satisfiability is Σ^0_1-complete [4], and, as mentioned above, Hyper^{2}LTL model-checking is Σ^1_1-hard [1], but no upper bounds are known.

Another yardstick is truth for order k arithmetic, i.e., the question whether a given sentence of order k arithmetic evaluates to true. In the following, we are in particular interested in the case $k = 3$, i.e., we consider formulas with arbitrary quantification over type 0 objects, type 1 objects, and type 2 objects (sets of sets of natural numbers). Note that these formulas span the whole third hierarchy, as we allow arbitrary nesting of existential and universal third-order quantification.

Our Contribution In this work, we determine the exact complexity of Hyper^{2}LTL satisfiability and model-checking, as well as some variants of satisfiability.

An important stepping stone is the investigation of the cardinality of models of Hyper^{2}LTL. It is known that every satisfiable HyperLTL sentence has a countable model, and that some have no finite models [6]. This restricts the order of arithmetic that can be simulated in HyperLTL and explains in particular the Σ^1_1-completeness of HyperLTL satisfiability [7]. We show that (unsurprisingly) second-order quantification allows to write formulas that only have uncountable models by generalizing the lower bound construction of HyperLTL to Hyper^{2}LTL. Note that the cardinality of the continuum is a trivial upper bound on the size of models, as they are sets of traces.

With this tool at hand, we are able to show that Hyper^{2}LTL satisfiability is as hard as truth in third-order arithmetic, i.e., much harder than HyperLTL satisfiability. This in itself is not surprising, as second-order quantification is expected to increase the complexity considerably. But what might be surprising is that the problem is not Σ^2_1-complete, i.e., at the same position of the third hierarchy that HyperLTL satisfiability occupies in one full hierarchy below (see Figure 1).

Furthermore, we also show that Hyper^{2}LTL finite-state satisfiability is as hard as truth in third-order arithmetic, and therefore as hard as general satisfiability. This should be contrasted with the situation for HyperLTL described above, where finite-state satisfiability is Σ^0_1-complete (i.e., recursively enumerable) and thus much simpler than general satisfiability, which is Σ^1_1-complete.

Finally, our techniques for Hyper^{2}LTL satisfiability also shed light on the complexity of Hyper^{2}LTL model-checking, which we show to be as hard as truth in third-order arithmetic as well, i.e., all three problems we consider have the same complexity. Again, this has be contrasted with the situation for HyperLTL, where

Table 1: List of our results (in gray) and comparison to other logics. "T3OA-complete" stands for "as hard as truth in third-order arithmetic".

	satisfiability	finite-state satisfiability	model-checking
LTL	PSPACE-complete	PSPACE-complete	PSPACE-complete
HyperLTL	Σ_1^1-complete	Σ_1^0-complete	TOWER-complete
Hyper^{2}LTL	T3OA-complete	T3OA-complete	T3OA-complete

model-checking is decidable, albeit TOWER-complete.

One could rightfully expect that quantification over arbitrary sets of traces is the culprit behind the formidable complexity of Hyper^{2}LTL. In fact, Beutner et al. noticed that many of the applications of Hyper^{2}LTL described above only require limited forms of set quantification. However, we show that two natural fragments of Hyper^{2}LTL obtained by restricting the range of second-order quantifiers do retain the same complexity for all three decision problems.

Table 1 lists our results and compares them to the corresponding results for LTL and HyperLTL.

2 Preliminaries

We denote the nonnegative integers by $\mathbb{N}$. An alphabet is a nonempty finite set. The set of infinite words over an alphabet Σ is denoted by Σ^ω. Throughout this paper, we fix a finite set AP of atomic propositions. A trace over AP is an infinite word over the alphabet 2^{AP}. Given $\mathrm{AP}' \subseteq \mathrm{AP}$, the AP'-projection of a trace $t(0)t(1)t(2)\cdots$ over AP is the trace $(t(0) \cap \mathrm{AP}')(t(1) \cap \mathrm{AP}')(t(2) \cap \mathrm{AP}')\cdots$ over AP'.

A transition system $\mathfrak{T} = (V, E, I, \lambda)$ consists of a finite set V of vertices, a set $E \subseteq V \times V$ of (directed) edges, a set $I \subseteq V$ of initial vertices, and a labeling $\lambda \colon V \to 2^{\mathrm{AP}}$ of the vertices by sets of atomic propositions. We assume that every vertex has at least one outgoing edge. A path ρ through $\mathfrak{T}$ is an infinite sequence $\rho(0)\rho(1)\rho(2)\cdots$ of vertices with $\rho(0) \in I$ and $(\rho(n), \rho(n+1)) \in E$ for every $n \geq 0$. The trace of ρ is defined as $\lambda(\rho) = \lambda(\rho(0))\lambda(\rho(1))\lambda(\rho(2))\cdots$. The set of traces of $\mathfrak{T}$ is $\mathrm{Tr}(\mathfrak{T}) = \{\lambda(\rho) \mid \rho \text{ is a path through } \mathfrak{T}\}$.

2.1 Hyper^{2}LTL

Let $\mathcal{V}_1$ be a set of first-order trace variables (i.e., ranging over traces) and $\mathcal{V}_2$ be a set of second-order trace variables (i.e., ranging over sets of traces) such that $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$. We typically use π (possibly with decorations) to denote first-order variables and X (possibly with decorations) to denote second-order variables. Also, we assume the existence of two distinguished second-order variables X_a and X_d that refer to the set $(2^{\mathrm{AP}})^\omega$ of all traces and the universe of discourse (a fixed set of traces, often that of a given transition system over which the formula is evaluated, e.g., in model-checking), respectively.

Then, the formulas of Hyper^{2}LTL are given by the grammar

$$\varphi ::= \exists X.\ \varphi \mid \forall X.\ \varphi \mid \exists \pi \in X.\ \varphi \mid \forall \pi \in X.\ \varphi \mid \psi$$
$$\psi ::= \mathtt{p}_\pi \mid \neg\psi \mid \psi \vee \psi \mid \mathbf{X}\,\psi \mid \psi\,\mathbf{U}\,\psi$$

where $\mathtt{p}$ ranges over AP, π ranges over $\mathcal{V}_1$, and X ranges over $\mathcal{V}_2$. Conjunction, implication, and equivalence are defined as usual, and the temporal operators eventually $\mathbf{F}$ and always $\mathbf{G}$ are derived as $\mathbf{F}\,\psi = \neg\psi\,\mathbf{U}\,\psi$ and $\mathbf{G}\,\psi = \neg\,\mathbf{F}\,\neg\psi$. A *sentence* is a formula without free (first- and second-order) variables, which are defined as usual. We measure the size of a formula by its number of distinct subformulas.

The semantics of Hyper^{2}LTL is defined with respect to a *variable assignment*, a partial mapping $\Pi\colon \mathcal{V}_1 \cup \mathcal{V}_2 \to (2^{\mathrm{AP}})^\omega \cup 2^{(2^{\mathrm{AP}})^\omega}$ such that

- if $\Pi(\pi)$ for $\pi \in \mathcal{V}_1$ is defined, then $\Pi(\pi) \in (2^{\mathrm{AP}})^\omega$ and
- if $\Pi(X)$ for $X \in \mathcal{V}_2$ is defined, then $\Pi(X) \in 2^{(2^{\mathrm{AP}})^\omega}$.

Given a variable assignment Π, a variable $\pi \in \mathcal{V}_1$, and a trace t, we denote by $\Pi[\pi \mapsto t]$ the assignment that coincides with Π everywhere but at π, which is mapped to t. Similarly, given a variable assignment Π, a variable $X \in \mathcal{V}_2$, and a set T of traces we denote by $\Pi[X \mapsto T]$ the assignment that coincides with Π everywhere but at X, which is mapped to T. Furthermore, $\Pi[j, \infty)$ denotes the variable assignment mapping every $\pi \in \mathcal{V}_1$ in Π's domain to $\Pi(\pi)(j)\Pi(\pi)(j+1)\Pi(\pi)(j+2)\cdots$, the suffix of $\Pi(\pi)$ starting at position j (note that the assignment of variables $X \in \mathcal{V}_2$ is not updated, as this is not necessary for our application).

For a variable assignment Π we define

- $\Pi \models \mathtt{p}_\pi$ if $\mathtt{p} \in \Pi(\pi)(0)$,
- $\Pi \models \neg\psi$ if $\Pi \not\models \psi$,
- $\Pi \models \psi_1 \vee \psi_2$ if $\Pi \models \psi_1$ or $\Pi \models \psi_2$,
- $\Pi \models \mathbf{X}\,\psi$ if $\Pi[1, \infty) \models \psi$,
- $\Pi \models \psi_1 \,\mathbf{U}\, \psi_2$ if there is a $j \geq 0$ such that $\Pi[j, \infty) \models \psi_2$ and for all $0 \leq j' < j$ we have $\Pi[j', \infty) \models \psi_1$,
- $\Pi \models \exists \pi \in X.\ \varphi$ if there exists a trace $t \in \Pi(X)$ such that $\Pi[\pi \mapsto t] \models \varphi$,
- $\Pi \models \forall \pi \in X.\ \varphi$ if for all traces $t \in \Pi(X)$ we have $\Pi[\pi \mapsto t] \models \varphi$,
- $\Pi \models \exists X.\ \varphi$ if there exists a set $T \subseteq (2^{\mathrm{AP}})^\omega$ such that $\Pi[X \mapsto T] \models \varphi$, and
- $\Pi \models \forall X.\ \varphi$ if for all sets $T \subseteq (2^{\mathrm{AP}})^\omega$ we have $\Pi[X \mapsto T] \models \varphi$.

The variable assignment with empty domain is denoted by $\Pi_\emptyset$. We say that a set T of traces satisfies a Hyper^{2}LTL sentence φ, written $T \models \varphi$, if $\Pi_\emptyset[X_a \mapsto (2^{\mathrm{AP}})^\omega, X_d \mapsto T] \models \varphi$, i.e., if we assign the set of all traces to X_a and the set T to the universe of discourse X_d. In this case, we say that T is a model of φ. A transition system $\mathfrak{T}$ satisfies φ, written $\mathfrak{T} \models \varphi$, if $\mathrm{Tr}(\mathfrak{T}) \models \varphi$. Slightly sloppily, we again say that $\mathfrak{T}$ satisfies φ in this case. Although Hyper^{2}LTL sentences are required to be in prenex normal form, they are closed under Boolean combinations, which can easily be seen by transforming such a formula into an equivalent formula in prenex normal form. Thus, in examples and proofs we will often use Boolean combinations of Hyper^{2}LTL formulas.

Remark 1. *HyperLTL is the fragment of Hyper^{2}LTL obtained by disallowing second-order quantification and only allowing first-order quantification of the form $\exists \pi \in X_d$ and $\forall \pi \in X_d$, i.e., one can only quantify over traces from the universe of discourse. Hence, we typically simplify our notation to $\exists \pi$ and $\forall \pi$ in HyperLTL formulas.*

To conclude, we highlight that second-order quantification in Hyper^{2}LTL ranges over arbitrary sets of traces (not necessarily from the universe of discourse) and that first-order quantification ranges over elements in such sets, i.e., (possibly) again over arbitrary traces. To disallow this, we introduce *closed-world* semantics for Hyper^{2}LTL. Here, we only consider formulas that do not use the variable X_a and change the semantics of the set quantifiers as follows:

- $\Pi \models \exists X.\ \varphi$ if there exists a set $T \subseteq \Pi(X_d)$ such that $\Pi[X \mapsto T] \models \varphi$, and
- $\Pi \models \forall X.\ \varphi$ if for all sets $T \subseteq \Pi(X_d)$ we have $\Pi[X \mapsto T] \models \varphi$.

We say that $T \subseteq (2^{\mathrm{AP}})^\omega$ satisfies φ under closed-world semantics, if $\Pi_\emptyset[X_d \mapsto T] \models \varphi$. Hence, under closed-world semantics, second-order quantifiers only range over subsets of the universe of discourse. Consequently, first-order quantifiers also range over traces from the universe of discourse.

Lemma 1. *Every Hyper^{2}LTL formula φ can in polynomial time be translated into a Hyper^{2}LTL formula φ' such that for all sets T of traces we have $T \models \varphi$ under closed-world semantics if and only if $T \models \varphi'$ (under standard semantics).*

Proof. Second-order quantification over subsets of the universe of discourse can easily be mimicked by guarding classical quantifiers ranging over arbitrary sets. Here, we rely on the formula $\forall \pi \in X.\ \exists \pi' \in X_d.\ \mathbf{G} \bigwedge_{\mathtt{p} \in \mathrm{AP}} \mathtt{p}_\pi \leftrightarrow \mathtt{p}_{\pi'}$, which expresses that every trace in X is also in X_d.

Now, given a Hyper^{2}LTL sentence φ, let φ' be the Hyper^{2}LTL sentence obtained by recursively replacing

- each existential second-order quantifier $\exists X.\ \psi$ in φ by $\exists X.\ (\forall \pi \in X.\ \exists \pi' \in X_d.\ \mathbf{G} \bigwedge_{\mathtt{p} \in \mathrm{AP}} \mathtt{p}_\pi \leftrightarrow \mathtt{p}_{\pi'}) \wedge \psi$ and
- each universal second-order quantifier $\forall X.\ \psi$ in φ by $\forall X.\ (\forall \pi \in X.\ \exists \pi' \in X_d.\ \mathbf{G} \bigwedge_{\mathtt{p} \in \mathrm{AP}} \mathtt{p}_\pi \leftrightarrow \mathtt{p}_{\pi'}) \rightarrow \psi$,

and then bringing the resulting sentence into prenex normal form, which can be done as no quantifier is under the scope of a temporal operator. □

Thus, all complexity upper bounds for standard semantics also hold for closed-world semantics and all lower bounds for closed-world semantics also hold for standard semantics.

2.2 Arithmetic

We consider formulas of arithmetic, i.e., predicate logic with signature $(+, \cdot, <, \in)$, evaluated over the structure $(\mathbb{N}, +, \cdot, <, \in)$. A type 0 object is a natural number $n \in \mathbb{N}$, a type 1 object is a subset of $\mathbb{N}$, and a type 2 object is a set of subsets of $\mathbb{N}$. Our benchmark is third-order arithmetic, i.e., predicate logic with quantification over type 0, type 1, and type 2 objects. In the following, we use lower-case roman letters (possibly with decorations) for first-order variables, upper-case roman letters (possibly with decorations) for second-order variables, and upper-case calligraphic roman letters (possibly with decorations) for third-order variables. Note that every fixed natural number is definable in first-order arithmetic, so we freely us them as syntactic sugar.

Truth of third-order arithmetic is the following decision problem: given a sentence φ of third-order arithmetic, does $(\mathbb{N}, +, \cdot, <, \in)$ satisfy φ?

3 The Cardinality of Hyper^{2}LTL Models

A Hyper^{2}LTL sentence is satisfiable if it has a model. In this section, we investigate the cardinality of models of satisfiable Hyper^{2}LTL sentences. We begin by stating a (trivial) upper bound, which follows from the fact that models are sets of traces. Here, $\mathfrak{c}$ denotes the cardinality of the continuum (equivalently, the cardinality of $(2^{\mathrm{AP}})^\omega$ for every finite AP).

Proposition 1. *Every satisfiable Hyper^{2}LTL sentence has a model of cardinality* $\mathfrak{c}$.

Next, we show that this trivial upper bound is tight.

Remark 2. *There is a very simple, albeit equally unsatisfactory, way to obtain the desired lower bound: Consider* $\forall \pi \in X_a.\ \exists \pi' \in X_d.\ \mathbf{G} \bigwedge_{p \in \mathrm{AP}} p_\pi \leftrightarrow p_{\pi'}$ *expressing that every trace in the set of all traces is also in the universe of discourse, i.e.,* $(2^{\mathrm{AP}})^\omega$ *is its only model. However, this crucially relies on the fact that* X_a *is, by definition, interpreted as the set of all traces. In fact the formula does not even use second-order quantification.*

In the following, we construct a sentence that has only uncountable models, and which retains that property under closed-world semantics (which in particular means it cannot use X_a). This should be compared with HyperLTL, where every satisfiable sentence has a countable model [6]. Unsurprisingly, the addition of (even restricted) second-order quantification increases the cardinality of minimal models, even without cheating.

Example 1. *We begin by recalling a construction of Finkbeiner and Zimmermann giving a satisfiable HyperLTL sentence* ψ *that has no finite models [6]. The sentence intuitively posits the existence of a unique trace for every natural number* n*. Our lower bound for Hyper^{2}LTL builds upon that construction.*

Fix $\mathrm{AP} = \{x\}$ *and consider the conjunction* $\psi = \psi_1 \wedge \psi_2 \wedge \psi_3$ *of the following three formulas:*

1. $\psi_1 = \forall\pi.\ \neg\boldsymbol{x}_\pi\, \mathbf{U}(\boldsymbol{x}_\pi \wedge \mathbf{X}\,\mathbf{G}\,\neg\boldsymbol{x}_\pi)$: *every trace in a model is of the form* $\emptyset^n\{\boldsymbol{x}\}\emptyset^\omega$ *for some* $n \in \mathbb{N}$, *i.e., every model is a subset of* $\{\emptyset^n\{\boldsymbol{x}\}\emptyset^\omega \mid n \in \mathbb{N}\}$.
2. $\psi_2 = \exists\pi.\ \boldsymbol{x}_\pi$: *the trace* $\emptyset^0\{\boldsymbol{x}\}\emptyset^\omega$ *is in every model.*
3. $\psi_3 = \forall\pi.\ \exists\pi'.\ \mathbf{F}(\boldsymbol{x}_\pi \wedge \mathbf{X}\,\boldsymbol{x}_{\pi'})$: *if* $\emptyset^n\{\boldsymbol{x}\}\emptyset^\omega$ *is in a model for some* $n \in \mathbb{N}$, *then also* $\emptyset^{n+1}\{\boldsymbol{x}\}\emptyset^\omega$.

Then, ψ *has exactly one model (over* AP*), namely* $\{\emptyset^n\{\boldsymbol{x}\}\emptyset^\omega \mid n \in \mathbb{N}\}$.

Traces of the form $\emptyset^n\{\mathtt{x}\}\emptyset^\omega$ indeed encode natural numbers and ψ expresses that every model contains the encodings of all natural numbers and nothing else. But we can of course also encode sets of natural numbers with traces as follows: a trace t over a set of atomic propositions containing $\mathtt{x}$ encodes the set $\{n \in \mathbb{N} \mid \mathtt{x} \in t(n)\}$. In the following, we show that second-order quantification allows us to express the existence of the encodings of all subsets of natural numbers by requiring that for every subset $S \subseteq \mathbb{N}$ (encoded by the set $\{\emptyset^n\{\mathtt{x}\}\emptyset^\omega \mid n \in S\}$ of traces) there is a trace t encoding S, which means $\mathtt{x}$ is in $t(n)$ if and only if S contains a trace in which $\mathtt{x}$ holds at position n. This equivalence can be expressed in Hyper^{2}LTL. For technical reasons, we do not capture the equivalence directly but instead use encodings of both the natural numbers that are in S and the natural numbers that are not in S.

Theorem 1. *There is a satisfiable Hyper*^{2}*LTL sentence that only has models of cardinality* $\mathfrak{c}$.

Proof. We prove that there is a satisfiable Hyper^{2}LTL sentence $\varphi_{allSets}$ whose unique model has cardinality $\mathfrak{c}$. To this end, we fix AP $= \{\mathtt{+}, \mathtt{-}, \mathtt{s}, \mathtt{x}\}$ and consider the conjunction $\varphi_{allSets} = \varphi_0 \wedge \cdots \wedge \varphi_4$ of the following formulas:

- $\varphi_0 = \forall\pi \in X_d.\ \bigvee_{\mathtt{p}\in\{\mathtt{+},\mathtt{-},\mathtt{s}\}} \mathbf{G}(\mathtt{p}_\pi \wedge \bigwedge_{\mathtt{p}'\in\{\mathtt{+},\mathtt{-},\mathtt{s}\}\setminus\{\mathtt{p}\}} \neg\mathtt{p}'_\pi)$: In each trace of a model, either one of the propositions in $\{\mathtt{+}, \mathtt{-}, \mathtt{s}\}$ holds at every position and the other two propositions in $\{\mathtt{+}, \mathtt{-}, \mathtt{s}\}$ hold at none of the positions. Consequently, we speak in the following about type $\mathtt{p}$ traces for $\mathtt{p} \in \{\mathtt{+}, \mathtt{-}, \mathtt{s}\}$.
- $\varphi_1 = \forall\pi \in X_d.\ (\mathtt{+}_\pi \vee \mathtt{-}_\pi) \rightarrow \neg\mathtt{x}_\pi\, \mathbf{U}(\mathtt{x}_\pi \wedge \mathbf{X}\,\mathbf{G}\,\neg\mathtt{x}_\pi)$: Type $\mathtt{p}$ traces for $\mathtt{p} \in \{\mathtt{+}, \mathtt{-}\}$ in the model have the form $\{\mathtt{p}\}^n\{\mathtt{x}, \mathtt{p}\}\{\mathtt{p}\}^\omega$.
- $\varphi_2 = \bigwedge_{\mathtt{p}\in\{\mathtt{+},\mathtt{-}\}} \exists\pi \in X_d.\ \mathtt{p}_\pi \wedge \mathtt{x}_\pi$: for both $\mathtt{p} \in \{\mathtt{+}, \mathtt{-}\}$, the type $\mathtt{p}$ trace $\{\mathtt{p}\}^0\{\mathtt{x}, \mathtt{p}\}\{\mathtt{p}\}^\omega$ is in every model.
- $\varphi_3 = \bigwedge_{\mathtt{p}\in\{\mathtt{+},\mathtt{-}\}} \forall\pi \in X_d.\ \exists\pi' \in X_d.\ \mathtt{p}_\pi \rightarrow (\mathtt{p}_{\pi'} \wedge \mathbf{F}(\mathtt{x}_\pi \wedge \mathbf{X}\,\mathtt{x}_{\pi'}))$: for both $\mathtt{p} \in \{\mathtt{+}, \mathtt{-}\}$, if the type $\mathtt{p}$ trace $\{\mathtt{p}\}^n\{\mathtt{x}, \mathtt{p}\}\{\mathtt{p}\}^\omega$ is in a model for some $n \in \mathbb{N}$, then also $\{\mathtt{p}\}^{n+1}\{\mathtt{x}, \mathtt{p}\}\{\mathtt{p}\}^\omega$.

  Note that the formulas $\varphi_1, \varphi_2, \varphi_3$ are similar to the formulas ψ_1, ψ_2, ψ_3 from Example 1. Hence, every model of the first four conjuncts contains $\{\{\mathtt{+}\}^n\{\mathtt{x}, \mathtt{+}\}\{\mathtt{+}\}^\omega \mid n \in \mathbb{N}\}$ and $\{\{\mathtt{-}\}^n\{\mathtt{x}, \mathtt{-}\}\{\mathtt{-}\}^\omega \mid n \in \mathbb{N}\}$ as subsets, and no other type $\mathtt{+}$ or type $\mathtt{-}$ traces.

  Now, consider an arbitrary set T of traces over AP (recall that second-order quantification ranges over arbitrary sets, not only over subsets of the universe of discourse). We say that T is contradiction-free if there is no $n \in \mathbb{N}$ such that $\{\mathtt{+}\}^n\{\mathtt{x}, \mathtt{+}\}\{\mathtt{+}\}^\omega \in T$ and $\{\mathtt{-}\}^n\{\mathtt{x}, \mathtt{-}\}\{\mathtt{-}\}^\omega \in T$. Furthermore, a trace t over AP is consistent with a contradiction-free T if

  (C1) $\{\mathtt{+}\}^n\{\mathtt{x}, \mathtt{+}\}\{\mathtt{+}\}^\omega \in T$ implies $\mathtt{x} \in t(n)$ and
  (C2) $\{\mathtt{-}\}^n\{\mathtt{x}, \mathtt{-}\}\{\mathtt{-}\}^\omega \in T$ implies $\mathtt{x} \notin t(n)$.

  Note that T does not necessarily specify the truth value of $\mathtt{x}$ in every position of t, i.e., in those positions $n \in \mathbb{N}$ where neither $\{\mathtt{+}\}^n\{\mathtt{x}, \mathtt{+}\}\{\mathtt{+}\}^\omega$ nor $\{\mathtt{-}\}^n\{\mathtt{x}, \mathtt{-}\}\{\mathtt{-}\}^\omega$ are in T. Nevertheless, for every trace t over $\{\mathtt{x}\}$ there is a contradiction-free T such that the $\{\mathtt{x}\}$-projection of every trace t' over AP that is consistent with T is equal to t.

- Hence, we define φ_4 as the formula

$$\forall X.\ \overbrace{[\forall \pi \in X.\ \forall \pi' \in X.\ (\texttt{+}_\pi \wedge \texttt{-}_{\pi'}) \rightarrow \neg \mathbf{F}(\texttt{x}_\pi \wedge \texttt{x}_{\pi'})]}^{X \text{ is contradiction-free}} \rightarrow$$
$$\exists \pi'' \in X_d.\ \forall \pi''' \in X.\ \texttt{s}_{\pi''} \wedge \underbrace{(\texttt{+}_{\pi'''} \rightarrow \mathbf{F}(\texttt{x}_{\pi'''} \wedge \texttt{x}_{\pi''}))}_{\text{(C1)}} \wedge \underbrace{(\texttt{-}_{\pi'''} \rightarrow \mathbf{F}(\texttt{x}_{\pi'''} \wedge \neg\texttt{x}_{\pi''}))}_{\text{(C2)}},$$

expressing that for every contradiction-free set of traces T, there is a type $\texttt{s}$ trace t'' in the model (note that π'' is required to be in X_d) that is consistent with T.

While $\varphi_{allSets}$ is not in prenex normal form, it can easily be turned into an equivalent formula in prenex normal form (at the cost of readability). Now, the set

$$T_{allSets} = \{\{\texttt{+}\}^n\{\texttt{x},\texttt{+}\}\{\texttt{+}\}^\omega \mid n \in \mathbb{N}\} \cup \{\{\texttt{-}\}^n\{\texttt{x},\texttt{-}\}\{\texttt{-}\}^\omega \mid n \in \mathbb{N}\} \cup$$
$$\{(t(0) \cup \{\texttt{s}\})(t(1) \cup \{\texttt{s}\})(t(2) \cup \{\texttt{s}\}) \cdots \mid t \in (2^{\{\texttt{x}\}})^\omega\}$$

of traces satisfies $\varphi_{allSets}$. On the other hand, every model of $\varphi_{allSets}$ must indeed contain $T_{allSets}$ as a subset, as $\varphi_{allSets}$ requires the existence of all of its traces in the model. Finally, due to φ_0 and φ_1, a model cannot contain any traces that are not in $T_{allSets}$, i.e., $T_{allSets}$ is the unique model of $\varphi_{allSets}$.

To conclude, we just remark that

$$\{(t(0) \cup \{\texttt{s}\})(t(1) \cup \{\texttt{s}\})(t(2) \cup \{\texttt{s}\}) \cdots \mid t \in (2^{\{\texttt{x}\}})^\omega\} \subseteq T_{allSets}$$

has indeed cardinality $\mathfrak{c}$, as $(2^{\{\texttt{x}\}})^\omega$ has cardinality $\mathfrak{c}$. □

As alluded to above, we could restrict the second-order quantifier in φ_4 (the only one in $\varphi_{allSets}$) to subsets of the universe of discourse, as the set $T = \{\{\texttt{+}\}^n\{\texttt{x},\texttt{+}\}\{\texttt{+}\}^\omega \mid n \in \mathbb{N}\} \cup \{\{\texttt{-}\}^n\{\texttt{x},\texttt{-}\}\{\texttt{-}\}^\omega \mid n \in \mathbb{N}\}$ of traces (which is a subset of every model) is already *rich* enough to encode every subset of $\mathbb{N}$ by an appropriate contradiction-free subset of T. Thus, $\varphi_{allSets}$ has the unique model $T_{allSets}$ even under closed-world semantics.

Corollary 1. *There is a satisfiable Hyper^{2}LTL sentence that only has models of cardinality $\mathfrak{c}$ under closed-world semantics.*

4 The Complexity of Hyper^{2}LTL Satisfiability

The Hyper^{2}LTL satisfiability problem asks, given a Hyper^{2}LTL sentence φ, whether φ is satisfiable. In this section, we determine tight bounds on the complexity of the Hyper^{2}LTL satisfiability problem and some of its variants.

Recall that in Section 3 we encoded sets of natural numbers as traces over a set of propositions containing $\texttt{x}$ and encoded natural numbers as singleton sets. Hence, sets of traces can encode sets of sets of natural numbers, i.e., type 2 objects. Using these encodings, we show that Hyper^{2}LTL and truth in third-order arithmetic have the same complexity.

An important ingredient in our proof is the implementation of addition and multiplication in temporal logic following Fortin et al. [8]: Let $\mathrm{AP}_{arith} = \{\texttt{arg1}, \texttt{arg2}, \texttt{res}, \texttt{add}, \texttt{mult}\}$ and let $T_{(+,\cdot)}$ be the set of all traces $t \in (2^{\mathrm{AP}_{arith}})^\omega$ such that

- there are unique $n_1, n_2, n_3 \in \mathbb{N}$ with $\texttt{arg1} \in t(n_1)$, $\texttt{arg2} \in t(n_2)$, and $\texttt{res} \in t(n_3)$, and
- either $\texttt{add} \in t(n)$, $\texttt{mult} \notin t(n)$ for all n, and $n_1 + n_2 = n_3$, or $\texttt{mult} \in t(n)$, $\texttt{add} \notin t(n)$ for all n, and $n_1 \cdot n_2 = n_3$.

Proposition 2 (Theorem 5.5 of [8])**.** *There is a satisfiable HyperLTL sentence $\varphi_{(+,\cdot)}$ such that the AP_{arith}-projection of every model of $\varphi_{(+,\cdot)}$ is $T_{(+,\cdot)}$.*

Now, we are able to settle the complexity of the Hyper^{2}LTL satisfiability problem.

Theorem 2. *The Hyper^{2}LTL satisfiability problem is polynomial-time equivalent to truth in third-order arithmetic.*

Proof. We begin with the lower bound by reducing truth in third-order arithmetic to Hyper^{2}LTL satisfiability: we present a polynomial-time translation from sentences φ of third-order arithmetic to Hyper^{2}LTL sentences φ' such that $(\mathbb{N}, +, \cdot, <, \in) \models \varphi$ if and only if φ' is satisfiable.

Given a third-order sentence φ, we define

$$\varphi' = \varphi_{allSets} \wedge \exists X_{arith}.\ (\varphi'_{(+,\cdot)} \wedge hyp(\varphi))$$

where $\varphi_{allSets}$ is the Hyper^{2}LTL sentence from the proof of Theorem 1 enforcing every subset of $\mathbb{N}$ to be encoded in a model, $\varphi'_{(+,\cdot)}$ is the Hyper^{2}LTL formula obtained from the HyperLTL formula $\varphi_{(+,\cdot)}$ by replacing each quantifier $\exists\pi$ ($\forall\pi$, respectively) by $\exists\pi \in X_{arith}$ ($\forall\pi \in X_{arith}$, respectively), and where $hyp(\varphi)$ is defined inductively as follows:

- For third-order variables $\mathcal{Y}$, $hyp(\exists\mathcal{Y}.\ \psi) = \exists X_{\mathcal{Y}}.\ (\forall\pi \in X_{\mathcal{Y}}.\ \mathtt{s}_\pi) \wedge hyp(\psi)$.
- For third-order variables $\mathcal{Y}$, $hyp(\forall\mathcal{Y}.\ \psi) = \forall X_{\mathcal{Y}}.\ (\forall\pi \in X_{\mathcal{Y}}.\ \mathtt{s}_\pi) \rightarrow hyp(\psi)$.
- For second-order variables Y, $hyp(\exists Y.\ \psi) = \exists\pi_Y \in X_d.\ \mathtt{s}_{\pi_Y} \wedge hyp(\psi)$.
- For second-order variables Y, $hyp(\forall Y.\ \psi) = \forall\pi_Y \in X_d.\ \mathtt{s}_{\pi_Y} \rightarrow hyp(\psi)$.
- For first-order variables y, $hyp(\exists y.\ \psi) = \exists\pi_y \in X_d.\ \mathtt{s}_{\pi_y} \wedge [(\neg\mathtt{x}_{\pi_y})\,\mathbf{U}(\mathtt{x}_{\pi_y} \wedge \mathbf{X}\,\mathbf{G}\,\neg\mathtt{x}_{\pi_y})] \wedge hyp(\psi)$.
- For first-order variables y, $hyp(\forall y.\ \psi) = \forall\pi_y \in X_d.\ (\mathtt{s}_{\pi_y} \wedge [(\neg\mathtt{x}_{\pi_y})\,\mathbf{U}(\mathtt{x}_{\pi_y} \wedge \mathbf{X}\,\mathbf{G}\,\neg\mathtt{x}_{\pi_y})]) \rightarrow hyp(\psi)$.
- $hyp(\psi_1 \vee \psi_2) = hyp(\psi_1) \vee hyp(\psi_2)$.
- $hyp(\neg\psi) = \neg hyp(\psi)$.
- For second-order variables Y and third-order variables $\mathcal{Y}$, $hyp(Y \in \mathcal{Y}) = \exists\pi \in X_{\mathcal{Y}}.\ \mathbf{G}(\mathtt{x}_{\pi_Y} \leftrightarrow \mathtt{x}_\pi)$.
- For first-order variables y and second-order variables Y, $hyp(y \in Y) = \mathbf{F}(\mathtt{x}_{\pi_y} \wedge \mathtt{x}_{\pi_Y})$.
- For first-order variables y, y', $hyp(y < y') = \mathbf{F}(\mathtt{x}_{\pi_y} \wedge \mathbf{X}\,\mathbf{F}\,\mathtt{x}_{\pi_{y'}})$.
- For first-order variables y_1, y_2, y, $hyp(y_1 + y_2 = y) = \exists\pi \in X_{arith}.\ \mathtt{add}_\pi \wedge \mathbf{F}(\mathtt{arg1}_\pi \wedge \mathtt{x}_{\pi_{y_1}}) \wedge \mathbf{F}(\mathtt{arg2}_\pi \wedge \mathtt{x}_{\pi_{y_2}}) \wedge \mathbf{F}(\mathtt{res}_\pi \wedge \mathtt{x}_{\pi_y})$.
- For first-order variables y_1, y_2, y, $hyp(y_1 \cdot y_2 = y) = \exists\pi \in X_{arith}.\ \mathtt{mult}_\pi \wedge \mathbf{F}(\mathtt{arg1}_\pi \wedge \mathtt{x}_{\pi_{y_1}}) \wedge \mathbf{F}(\mathtt{arg2}_\pi \wedge \mathtt{x}_{\pi_{y_2}}) \wedge \mathbf{F}(\mathtt{res}_\pi \wedge \mathtt{x}_{\pi_y})$.

Note that φ' is not in prenex normal form, but can easily be brought into prenex normal form, as there are no quantifiers under the scope of a temporal operator.

Now, an induction shows that $(\mathbb{N}, +, \cdot, <, \in)$ satisfies φ if and only if $T_{allSets}$ satisfies φ'. As $T_{allSets}$ is the unique model of $\varphi_{allSets}$, it is also the unique model of φ, i.e., φ' is satisfiable if and only if $T_{allSets}$ satisfies φ'. Altogether we obtain the desired equivalence between $(\mathbb{N}, +, \cdot, <, \in) \models \varphi$ and φ' being satisfiable.

For the upper bound, we conversely reduce Hyper^{2}LTL satisfiability to truth in third-order arithmetic: we present a polynomial-time translation from Hyper^{2}LTL sentences φ to sentences φ' of third-order arithmetic such that φ is satisfiable if and only if $(\mathbb{N}, +, \cdot, <, \in) \models \varphi'$.

Let $pair\colon \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ denote Cantor's pairing function defined as $pair(i,j) = \frac{1}{2}(i+j)(i+j+1) + j$, which is a bijection. Furthermore, fix some bijection $e\colon \mathrm{AP} \rightarrow \{0, 1, \ldots, |\mathrm{AP}| - 1\}$. Then, we encode a trace $t \in (2^{\mathrm{AP}})^\omega$ by the set $S_t = \{pair(j, e(\mathtt{p})) \mid j \in \mathbb{N} \text{ and } \mathtt{p} \in t(j)\} \subseteq \mathbb{N}$. As *pair* is a bijection, we have that $t \neq t'$ implies $S_t \neq S_{t'}$. While not every subset of $\mathbb{N}$ encodes some trace t, the first-order formula

$$\varphi_{isTrace}(Y) = \forall x.\ \forall y.\ y \geq |\mathrm{AP}| \rightarrow pair(x,y) \notin Y$$

checks if a set does encode a trace. Here, we use *pair* as syntactic sugar, which is possible as the definition of *pair* only uses addition and multiplication.

As (certain) sets of natural numbers encode traces, sets of (certain) sets of natural numbers encode sets of traces. This is sufficient to reduce Hyper^{2}LTL to third-order arithmetic, which allows the quantification over sets of sets of natural numbers. Before we present the translation, we need to introduce some more auxiliary formulas:

- Let $\mathcal{Y}$ be a third-order variable (i.e., $\mathcal{Y}$ ranges over sets of sets of natural numbers). Then, the formula

  $$\varphi_{onlyTraces}(\mathcal{Y}) = \forall Y.\ Y \in \mathcal{Y} \to \varphi_{isTrace}(Y)$$

  checks if a set of sets of natural numbers only contains sets encoding a trace.

- Further, the formula

  $$\varphi_{allTraces}(\mathcal{Y}) = \varphi_{onlyTraces}(\mathcal{Y}) \wedge \forall Y.\ \pi_{isTrace}(Y) \to Y \in \mathcal{Y}$$

  checks if a set of sets of natural numbers contains exactly the sets encoding a trace.

Now, we are ready to define our encoding of Hyper^{2}LTL in third-order arithmetic. Given a Hyper^{2}LTL sentence φ, let

$$\varphi' = \exists \mathcal{Y}_a.\ \exists \mathcal{Y}_d.\ \varphi_{allTraces}(\mathcal{Y}_a) \wedge \varphi_{onlyTraces}(\mathcal{Y}_d) \wedge ar(\varphi)(0)$$

where $ar(\varphi)$ is defined inductively as presented below. Note that φ' requires $\mathcal{Y}_a$ to contain exactly the encodings of all traces (i.e., it corresponds to the distinguished Hyper^{2}LTL variable X_a in the following translation) and $\mathcal{Y}_d$ is an existentially quantified set of trace encodings (i.e., it corresponds to the distinguished Hyper^{2}LTL variable X_d in the following translation).

In the inductive definition of $ar(\varphi)$, we will employ a free first-order variable i to denote the position at which the formula is to be evaluated to capture the semantics of the temporal operators. As seen above, in φ', this free variable is set to zero in correspondence with the Hyper^{2}LTL semantics.

- $ar(\exists X.\ \psi) = \exists \mathcal{Y}_X.\ \varphi_{onlyTraces}(\mathcal{Y}_X) \wedge ar(\psi)$. Here, the free variable of $ar(\exists X.\ \psi)$ is the free variable of $ar(\psi)$.

- $ar(\forall X.\ \psi) = \forall \mathcal{Y}_X.\ \varphi_{onlyTraces}(\mathcal{Y}_X) \to ar(\psi)$. Here, the free variable of $ar(\forall X.\ \psi)$ is the free variable of $ar(\psi)$.

- $ar(\exists \pi \in X.\ \psi) = \exists Y_\pi.\ Y_\pi \in \mathcal{Y}_X \wedge ar(\psi)$. Here, the free variable of $ar(\exists \pi \in X.\ \psi)$ is the free variable of $ar(\psi)$.

- $ar(\forall \pi \in X.\ \psi) = \forall Y_\pi.\ Y_\pi \in \mathcal{Y}_X \to ar(\psi)$. Here, the free variable of $ar(\forall \pi \in X.\ \psi)$ is the free variable of $ar(\psi)$.

- $ar(\psi_1 \vee \psi_2) = ar(\psi_1) \vee ar(\psi_2)$. Here, we require the free variables of $ar(\psi_1)$ and $ar(\psi_2)$ are the same (which can always be achieved by variable renaming), which is then also the free variable of $ar(\psi_1 \vee \psi_2)$.

- $ar(\neg\psi) = \neg ar(\psi)$. Here, the free variable of $ar(\neg\psi)$ is the free variable of $\neg ar(\psi)$.

- $ar(\mathbf{X}\,\psi) = (i' = i+1) \wedge ar(\psi)$, where i' is the free variable of $ar(\psi)$ and i is the free variable of $ar(\mathbf{X}\,\psi)$.

- $ar(\psi_1\,\mathbf{U}\,\psi) = \exists i_1.\ i_1 \geq i \wedge ar(\psi_1) \wedge \forall i_2.\ (i \leq i_2 \wedge i_2 < i_1) \to ar(\psi_2)$, where i_1 is the free variable of $ar(\psi_1)$, i_2 is the free variable of $ar(\psi_2)$, and i is the free variable of $ar(\psi_1\,\mathbf{U}\,\psi_2)$.

- $ar(\mathtt{p}_\pi) = pair(i, e(\mathtt{p})) \in Y_\pi$, where $e\colon \mathrm{AP} \to \{0, 1, \ldots, |\mathrm{AP}| - 1\}$ is the encoding of propositions by natural numbers introduced above. Note that i is the free variable of $ar(a_\pi)$.

Now, an induction shows that $\Pi_\emptyset[X_a \to (2^{\mathrm{AP}})^\omega, X_d \mapsto T] \models \varphi$ if and only if $(\mathbb{N}, +, \cdot, <, \in)$ satisfies $ar(\varphi)$ when the variable $\mathcal{Y}_a$ is interpreted by the encoding of $(2^{\mathrm{AP}})^\omega$ and $\mathcal{Y}_d$ is interpreted by the encoding of T. Hence, φ is indeed satisfiable if and only if $(\mathbb{N}, +, \cdot, <, \in)$ satisfies φ'. $\square$

Again, let us also consider the lower bound under closed-world semantics. Recall that we have constructed from a sentence φ of third-order arithmetic a Hyper^{2}LTL sentence φ' such that $(\mathbb{N}, +, \cdot, <, \in) \models \varphi$ if and only if φ' is satisfiable. Furthermore, if φ' is satisfiable, then it has the unique model $T_{allSets}$. The unique model $T_{(+,\cdot)}$ of the conjunct $\varphi_{(+,\cdot)}$ of φ', is not a subset of $T_{allSets}$, i.e., the construction presented above is not correct under closed-world semantics. However, by slightly modifying the construction of $\varphi_{allSets}$ so that it also allows for the traces in $T_{(+,\cdot)}$ in the model, we obtain from φ' a formula that is satisfied by $T_{allSets} \cup T_{(+,\cdot)}$ if and only if $(\mathbb{N}, +, \cdot, <, \in) \models \varphi$. We leave the details to the reader.

Thus, the lower bound holds even under closed-world semantics. Together with Lemma 1 we obtain the following corollary.

Corollary 2. *The Hyper^{2}LTL satisfiability problem under closed-world semantics is polynomial-time equivalent to truth in third-order arithmetic.*

The Hyper^{2}LTL finite-state satisfiability problem asks, given a Hyper^{2}LTL sentence φ, whether there is a finite transition system satisfying φ. Note that we do not ask for a finite set T of traces satisfying φ, and that the set of traces of the finite transition system may still be infinite or even uncountable. Nevertheless, the problem is potentially simpler, as there are only countably many finite transition systems (and their sets of traces are much simpler). Nevertheless, we show that the finite-state satisfiability problem is as hard as the general satisfiability problem, as Hyper^{2}LTL allows the quantification over arbitrary (sets of) traces, i.e., restricting the universe of discourse to the traces of a finite transition system does not restrict second-order quantification at all. This has to be contrasted with the finite-state satisfiability problem for HyperLTL (defined analogously), which is recursively enumerable, as HyperLTL model-checking of finite transition systems is decidable [2].

Theorem 3. *The Hyper^{2}LTL finite-state satisfiability problem is polynomial-time equivalent to truth in third-order arithmetic.*

Proof. For the lower bound, we reduce truth in third-order arithmetic to Hyper^{2}LTL finite-state satisfiability: we present a polynomial-time translation from sentences φ of third-order arithmetic to Hyper^{2}LTL sentences φ' such that $(\mathbb{N}, +, \cdot, <, \in) \models \varphi$ if and only if φ' is satisfied by a finite transition system.

So, let φ be a sentence of third-order arithmetic. Recall that in the proof of Theorem 2, we have shown how to construct from φ the Hyper^{2}LTL sentence φ' such that the following three statements are equivalent:

- $(\mathbb{N}, +, \cdot, <, \in) \models \varphi$.
- φ' is satisfiable.
- φ' is satisfied by $T_{allSets}$.

As there is a finite transition system $\mathfrak{T}_{allSets}$ with $\mathrm{Tr}(\mathfrak{T}_{allSets}) = T_{allSets}$, the lower bound follows from Theorem 2.

For the upper bound, we conversely reduce Hyper^{2}LTL finite-state satisfiability to truth in third-order arithmetic: we present a polynomial-time translation from Hyper^{2}LTL sentences φ to sentences φ'' of third-order arithmetic such that φ is satisfied by a finite transition system if and only if $(\mathbb{N}, +, \cdot, <, \in) \models \varphi''$.

Recall that in the proof of Theorem 2, we have constructed a Hyper^{2}LTL sentence

$$\varphi' = \exists \mathcal{Y}_a.\ \exists \mathcal{Y}_d.\ \varphi_{allTraces}(\mathcal{Y}_a) \wedge \varphi_{onlyTraces}(\mathcal{Y}_d) \wedge ar(\varphi)(0)$$

where $\mathcal{Y}_a$ represents the distinguished Hyper^{2}LTL variable X_a, $\mathcal{Y}_d$ represents the distinguished Hyper^{2}LTL variable X_d, and where $ar(\varphi)$ is the encoding of φ in Hyper^{2}LTL.

To encode the general satisfiability problem it was sufficient to express that $\mathcal{Y}_d$ only contains traces. Here, we now require that $\mathcal{Y}_d$ contains exactly the traces of some finite transition system, which can easily be expressed in second-order arithmetic[1] as follows.

[1] With a little more effort, and a little less readability, first-order suffices for this task, as finite transition systems can be encoded by natural numbers.

We begin with a formula $\varphi_{isTS}(n, E, I, \ell)$ expressing that the second-order variables E, I, and ℓ encode a transition system with set $\{0, 1, \ldots, n-1\}$ of vertices. Our encoding will make extensive use of the pairing function introduced in the proof of Theorem 2. Formally, we define $\varphi_{isTS}(n, E, I, \ell)$ as the conjunction of the following formulas (where all quantifiers are first-order and we use *pair* as syntactic sugar):

- $\forall y.\ y \in E \rightarrow \exists v.\ \exists v'.\ (v < n \wedge v' < n \wedge y = pair(v, v'))$: edges are pairs of vertices.
- $\forall v.\ v < n \rightarrow \exists v'.\ (v' < n \wedge pair(v, v') \in E)$: every vertex has a successor.
- $\forall v.\ v \in I \rightarrow v < n$: the set of initial vertices is a subset of the set of all vertices.
- $\forall y.\ y \in \ell \rightarrow \exists v.\ \exists p.\ (v < n \wedge p < |\mathrm{AP}| \wedge y = pair(v, p))$: the labeling of v by p is encoded by the pair (v, p).

Next, we define $\varphi_{isPath}(P, n, E, I)$, expressing that the second-order variable P encodes a path through the transition system encoded by n, E, and I, as the conjunction of the following formulas:

- $\forall j.\ \exists v.\ (v < n \wedge pair(j, v) \in P \wedge \neg\exists v'.\ (v' \neq v \wedge pair(j, v') \in P))$: the fact that at position j the path visits vertex v is encoded by the pair (j, v). Exactly one vertex is visited at each position.
- $\exists v.\ v \in I \wedge pair(0, v) \in P$: the path starts in an initial vertex.
- $\forall j.\ \exists v.\ \exists v'.\ pair(j, v) \in P \wedge pair(j+1, v') \in P \wedge pair(v, v') \in E$: successive vertices in the path are indeed connected by an edge.

Finally, we define $\varphi_{traceOf}(T, P, \ell)$, expressing that the second-order variable T encodes the trace (using the encoding from the proof of Theorem 2) of the path encoded by the second-order variable P, as the following formula:

- $\forall j.\ \forall p.\ pair(j, p) \in T \leftrightarrow (\exists v.\ (j, v) \in P \wedge (v, p) \in \ell)$: a proposition holds in the trace at position j if and only if it is in the labeling of the j-th vertex of the path.

Now, we define the sentence φ'' as

$$\begin{aligned}
&\exists \mathcal{Y}_a.\ \exists \mathcal{Y}_d.\ \varphi_{allTraces}(\mathcal{Y}_a) \wedge \varphi_{onlyTraces}(\mathcal{Y}_d) \wedge \\
&\quad [\underbrace{\exists n.\ \exists E.\ \exists I.\ \exists \ell.\ \varphi_{isTS}(n, E, I, \ell)}_{\text{there exists a transition system } \mathfrak{T}} \wedge \overbrace{(\forall T.\ T \in \mathcal{Y}_d \rightarrow \exists P.\ (\varphi_{isPath}(P, n, E, I) \wedge \varphi_{traceOf}(T, P, \ell)))}^{\mathcal{Y}_d \text{ contains only traces of paths through } \mathfrak{T}} \\
&\quad \wedge \underbrace{(\forall P.\ (\varphi_{isPath}(P, n, E, I) \rightarrow \exists T.\ T \in \mathcal{Y}_d \wedge \varphi_{traceOf}(T, P, \ell)))}_{\mathcal{Y}_d \text{ contains all traces of paths through } \mathfrak{T}.}] \\
&\quad \wedge ar(\varphi)(0)
\end{aligned}$$

holds in $(\mathbb{N}, +, \cdot, <, \in)$ if and only if φ is satisfied by a finite transition system. □

Again, let us also consider the case of closed-world semantics. There is no finite transition system $\mathfrak{T}$ with $\mathrm{Tr}(\mathfrak{T}) = T_{(+,\cdot)}$. But the topological closure $\overline{T_{(+,\cdot)}}$ of $T_{(+,\cdot)}$, which contains all traces of $T_{(+,\cdot)}$, is also the unique model of some HyperLTL sentence [8]. Using these facts, we can show that the lower bound also works for closed-world semantics. To this end, we again need to modify $\varphi_{allSets}$ to allow the traces in $\overline{T_{(+,\cdot)}}$, modify $ar(\varphi)$ to ignore the traces in $\overline{T_{(+,\cdot)}} \setminus T_{(+,\cdot)}$, and then consider the model $T_{allSets} \cup \overline{T_{(+,\cdot)}}$, which can be represented by a finite transition system. We leave the details to the reader.

With Lemma 1, we obtain the following corollary.

Corollary 3. *The Hyper^{2}LTL finite-state satisfiability problem under closed-world semantics is polynomial-time equivalent to truth in third-order arithmetic.*

Let us also just remark that the proof of Theorem 3 can easily be adapted to show that other natural variations of the satisfiability problem are also polynomial-time equivalent to truth in third-order arithmetic, e.g., satisfiability by countable transition systems, satisfiability by finitely branching transition systems, etc. In fact, as long as a class $\mathcal{C}$ of transition systems is definable in third-order arithmetic, the Hyper^{2}LTL satisfiability problem restricted to transition systems in $\mathcal{C}$ is reducible to truth in third-order arithmetic. On the other hand, Hyper^{2}LTL satisfiability restricted to transition systems in $\mathcal{C}$ is polynomial-time reducible to truth in third-order arithmetic, for any nonempty class $\mathcal{C}$ of transition systems, as $T_{allSets}$ is definable in Hyper^{2}LTL: one can just posit the existence of all traces in $T_{allSets}$ and does not need to have them contained in the models of the formula (in standard semantics).

5 The Complexity of Hyper^{2}LTL Model-Checking

The Hyper^{2}LTL model-checking problem asks, given a finite transition system $\mathfrak{T}$ and a Hyper^{2}LTL sentence φ, whether $\mathfrak{T} \models \varphi$. Beutner et al. [1] have shown that Hyper^{2}LTL model-checking is Σ^1_1-hard, but there is no known upper bound in the literature. We improve the lower bound considerably, i.e., also to truth in third-order arithmetic, and then show that this bound is tight. This is the first upper bound on the problem's complexity.

Theorem 4. *The Hyper^{2}LTL model-checking problem is polynomial-time equivalent to truth in third-order arithmetic.*

Proof. For the lower bound, we reduce truth in third-order arithmetic to the Hyper^{2}LTL model-checking problem: we present a polynomial-time translation from sentences φ of third-order arithmetic to pairs $(\mathfrak{T}, \varphi')$ of a finite transition system $\mathfrak{T}$ and a Hyper^{2}LTL sentence φ' such that $(\mathbb{N}, +, \cdot, <, \in) \models \varphi$ if and only if $\mathfrak{T} \models \varphi'$.

In the proof of Theorem 3 we have, given a sentence φ of third-order arithmetic, constructed a Hyper^{2}LTL sentence φ' such that $(\mathbb{N}, +, \cdot, <, \in) \models \varphi$ if and only if $\mathfrak{T}_{allSets}$ satisfies φ', where $\mathfrak{T}_{allSets}$ is a finite transition system that is independent of φ. Thus, we obtain the lower bound by mapping φ to φ' and $\mathfrak{T}_{allSets}$.

For the upper bound, we reduce the Hyper^{2}LTL model-checking problem to truth in third-order arithmetic: we present a polynomial-time translation from pairs $(\mathfrak{T}, \varphi)$ of a finite transition system and a Hyper^{2}LTL sentence φ to sentences φ' of third-order arithmetic such that $\mathfrak{T} \models \varphi$ if and only if $(\mathbb{N}, +, \cdot, <, \in) \models \varphi'$.

In the proof of Theorem 3, we have constructed, from a Hyper^{2}LTL sentence φ, a sentence φ' of third-order arithmetic that expresses the existence of a finite transition system that satisfies φ. We obtain the desired upper bound by modifying φ' to replace the existential quantification of the transition system by hardcoding $\mathfrak{T}$ instead. □

Again, the lower bound proof can easily be extended to closed-world semantics, as argued in the proof of Theorem 3.

Corollary 4. *The Hyper^{2}LTL model-checking problem under closed-world semantics is polynomial-time equivalent to truth in third-order arithmetic.*

6 Hyper^{2}LTL$_{fp}$

As we have seen, unrestricted second-order quantification makes Hyper^{2}LTL very expressive and therefore algorithmically infeasible. But restricted forms of second-order quantification are sufficient for many application areas. Hence, Beutner et al. [1] introduced Hyper^{2}LTL$_{fp}$, a fragment of Hyper^{2}LTL in which second-order quantification ranges over smallest/largest sets that satisfy a given guard. For example, the formula $\exists (X, \curlyvee, \varphi_1).\ \varphi_2$ expresses that there is a set T of traces that satisfies both φ_1 and φ_2, and T is *a* smallest set that satisfies φ_1 (i.e., φ_1 is the guard). This fragment is expressive enough to express common knowledge, asynchronous hyperproperties, and causality in reactive systems [1].

The formulas of $\text{Hyper}^2\text{LTL}_{fp}$ are given by the grammar

$$\varphi ::= \exists(X, \curlyvee\!\!\!\curlywedge, \varphi).\ \varphi \mid \forall(X, \curlyvee\!\!\!\curlywedge, \varphi).\ \varphi \mid \exists\pi \in X.\ \varphi \mid \forall\pi \in X.\ \varphi \mid \psi$$
$$\psi ::= \mathtt{p}_\pi \mid \neg\psi \mid \psi \vee \psi \mid \mathbf{X}\,\psi \mid \psi\,\mathbf{U}\,\psi$$

where $\mathtt{p}$ ranges over AP, π ranges over $\mathcal{V}_1$, X ranges over $\mathcal{V}_2$, and $\curlyvee\!\!\!\curlywedge \in \{\curlyvee, \curlywedge\}$, i.e., the only modification concerns the syntax of second-order quantification.

Accordingly, the semantics of $\text{Hyper}^2\text{LTL}_{fp}$ is similar to that of Hyper^2LTL but for the second-order quantifiers, for which we define (for $\curlyvee\!\!\!\curlywedge \in \{\curlyvee, \curlywedge\}$)

- $\Pi \models \exists(X, \curlyvee\!\!\!\curlywedge, \varphi_1).\ \varphi_2$ if there exists a set $T \in \mathrm{sol}(\Pi, (X, \curlyvee\!\!\!\curlywedge, \varphi_1))$ such that $\Pi[X \mapsto T] \models \varphi_2$, and
- $\Pi \models \forall(X, \curlyvee\!\!\!\curlywedge, \varphi_1).\ \varphi_2$ if for all sets $T \in \mathrm{sol}(\Pi, (X, \curlyvee\!\!\!\curlywedge, \varphi_1))$ we have $\Pi[X \mapsto T] \models \varphi_2$,

where $\mathrm{sol}(\Pi, (X, \curlyvee\!\!\!\curlywedge, \varphi_1))$ is the set of all minimal/maximal models of the formula φ_1, which is defined as follows:

$$\mathrm{sol}(\Pi, (X, \curlyvee, \varphi_1)) = \{T \subseteq (2^{\mathrm{AP}})^\omega \mid \Pi[X \mapsto T] \models \varphi_1 \text{ and for all } T' \subsetneq T \text{ we have } \Pi[X \mapsto T'] \not\models \varphi_1\}$$
$$\mathrm{sol}(\Pi, (X, \curlywedge, \varphi_1)) = \{T \subseteq (2^{\mathrm{AP}})^\omega \mid \Pi[X \mapsto T] \models \varphi_1 \text{ and for all } T' \supsetneq T \text{ we have } \Pi[X \mapsto T'] \not\models \varphi_1\}$$

Note that $\mathrm{sol}(\Pi, (X, \curlyvee, \varphi_1))$ may be empty or may contain multiple sets, which then have to be pairwise incomparable.

The notions of satisfaction and models are defined as for Hyper^2LTL.

Proposition 3 (Proposition 1 of [1])**.** *Every $\text{Hyper}^2\text{LTL}_{fp}$ formula φ can in polynomial-time*[2] *be translated into a Hyper^2LTL formula φ' such that for all sets T of traces we have $T \models \varphi$ if and only if $T \models \varphi'$.*

Thus, every complexity upper bound for Hyper^2LTL also holds for $\text{Hyper}^2\text{LTL}_{fp}$ and every lower bound for $\text{Hyper}^2\text{LTL}_{fp}$ also holds for Hyper^2LTL. In the following, we show that lower bounds can also be transferred in the other direction, i.e., from Hyper^2LTL to $\text{Hyper}^2\text{LTL}_{fp}$. Thus, contrary to the design goal of $\text{Hyper}^2\text{LTL}_{fp}$, it is in general not more feasible than full Hyper^2LTL.

We begin again by studying the cardinality of models of $\text{Hyper}^2\text{LTL}_{fp}$ sentences, which will be the key technical tool for our complexity results. Again, as such formulas are evaluated over sets of traces, whose cardinality is bounded by $\mathfrak{c}$, there is a trivial upper bound. Our main result is that this bound is tight even for the restricted setting of $\text{Hyper}^2\text{LTL}_{fp}$.

Theorem 5. *There is a satisfiable $\text{Hyper}^2\text{LTL}_{fp}$ sentence that only has models of cardinality $\mathfrak{c}$.*

Proof. We adapt the proof of Theorem 1 to $\text{Hyper}^2\text{LTL}_{fp}$. Recall that we have constructed the formula $\varphi_{allSets} = \varphi_0 \wedge \cdots \wedge \varphi_4$ whose unique model is uncountable. The subformulas $\varphi_0, \ldots, \varphi_3$ of $\varphi_{allSets}$ are first-order, so let us consider φ_4. Recall that φ_4 has the form

$$\forall X.\ [\forall\pi \in X.\ \forall\pi' \in X.\ (\mathtt{+}_\pi \wedge \mathtt{-}_{\pi'}) \rightarrow \neg\,\mathbf{F}(\mathtt{x}_\pi \wedge \mathtt{x}_{\pi'})] \rightarrow$$
$$\exists\pi'' \in X_d.\ \forall\pi''' \in X.\ \mathtt{s}_{\pi''} \wedge (\mathtt{+}_{\pi'''} \rightarrow \mathbf{F}(\mathtt{x}_{\pi'''} \wedge \mathtt{x}_{\pi''})) \wedge (\mathtt{-}_{\pi'''} \rightarrow \mathbf{F}(\mathtt{x}_{\pi'''} \wedge \neg\mathtt{x}_{\pi''})),$$

expressing that for every contradiction-free set of traces T, there is a type $\mathtt{s}$ trace t'' in the model that is consistent with T. Here, X ranges over arbitrary sets T of traces. However, this is not necessary. Consider the formula

$$\varphi_4' = \forall(X, \curlywedge, \varphi_{conFree}).\ \exists\pi'' \in X_d.\ \forall\pi''' \in X.\ \mathtt{s}_{\pi''} \wedge (\mathtt{+}_{\pi'''} \rightarrow \mathbf{F}(\mathtt{x}_{\pi'''} \wedge \mathtt{x}_{\pi''})) \wedge (\mathtt{-}_{\pi'''} \rightarrow \mathbf{F}(\mathtt{x}_{\pi'''} \wedge \neg\mathtt{x}_{\pi''})),$$

with

$$\varphi_{conFree} = \forall\pi \in X.\ \forall\pi' \in X.\ (\mathtt{+}_\pi \wedge \mathtt{-}_{\pi'}) \rightarrow \neg\,\mathbf{F}(\mathtt{x}_\pi \wedge \mathtt{x}_{\pi'})$$

[2] The polynomial-time claim is not made in [1], but follows from the construction when using appropriate data structures for formulas.

expressing that X is contradiction-free. In φ'_4 the set variable X only ranges over *maximal* contradiction-free sets of traces, i.e., those that contain for each n either $\{\texttt{+}\}^n\{\texttt{x},\texttt{+}\}\{\texttt{+}\}^\omega$ or $\{\texttt{-}\}^n\{\texttt{x},\texttt{-}\}\{\texttt{-}\}^\omega$.

But even with the restriction to such maximal sets, φ'_4 still requires that a model of $\varphi'_{allSets} = \varphi_0 \wedge \cdots \varphi_3 \wedge \varphi'_4$ contains the encoding of every subset of $\mathbb{N}$ by a type $\texttt{s}$ trace, as every subset of $\mathbb{N}$ is captured by a maximal contradiction-free set of traces. □

Now, let us describe how we settle the complexity of Hyper^{2}LTL$_{fp}$ satisfiability and model-checking: Recall that Hyper^{2}LTL allows set quantification over arbitrary sets of traces while Hyper^{2}LTL$_{fp}$ restricts quantification to minimal/maximal sets of traces that satisfy a guard formula. By using the guard $\varphi'_{allSets}$ (using fresh propositions) the minimal sets satisfying the guard are uncountable. Thus, we can obtain every possible set over a set AP$'$ as a minimal set satisfying the guard.

Formally, let us fix some set AP$'$ not containing the propositions $\texttt{+}, \texttt{-}, \texttt{s}, \texttt{x}$ used to construct $\varphi'_{allSets}$ and let $\mathrm{AP} \subseteq \mathrm{AP}' \cup \{\texttt{+}, \texttt{-}, \texttt{s}, \texttt{x}\}$. Then, due to Theorem 5, we have

$$\{T \mid T \text{ is the AP}'\text{-projection of some } T \in \mathrm{sol}(\Pi, (X, \curlyvee, \varphi'_{allSets}))\}$$

is equal to $(2^{\mathrm{AP}'})^\omega$. Hence, we can use guarded quantification to simulate general quantification. This allows us to easily transfer all lower bounds for Hyper^{2}LTL to Hyper^{2}LTL$_{fp}$.

Theorem 6. *Hyper^{2}LTL$_{fp}$ satisfiability, finite-state satisfiability, and model-checking are polynomial-time equivalent to truth in third-order arithmetic.*

Proof. The upper bounds follow immediately from the analogous upper bounds for Hyper^{2}LTL and Proposition 3, while the lower bounds are obtained by adapting the reductions presented in the proofs of Theorem 2, Theorem 3, and Theorem 4 by replacing

- each existential second-order quantifier $\exists X$ by $\exists(X, \curlyvee, \varphi'_{allSets})$ and
- each universal second-order quantifier $\forall X$ by $\forall(X, \curlyvee, \varphi'_{allSets})$.

Here, we just have to assume that the propositions in $\varphi'_{allSets}$ do not appear in the formula we are modifying, which can always be achieved by renaming propositions, if necessary. As explained above, the modified formulas with restricted quantification are equivalent to the original Hyper^{2}LTL formulas constructed in the proofs of Theorem 2, Theorem 3, and Theorem 4, which implies the desired lower bounds. □

Let us conclude by mentioning without proof (and even without definition, for that matter) that these results can also be generalized to Hyper^{2}LTL$_{fp}$ under closed-world semantics.

7 Conclusion

We have investigated and settled the complexity of satisfiability and model-checking for Hyper^{2}LTL. All are as hard as truth in third-order arithmetic, and therefore (not surprisingly) much harder than the corresponding problems for HyperLTL, which are "only" Σ^1_1-complete and TOWER-complete, respectively. This shows that the addition of second-order quantification increases the already high complexity significantly.

All our results already hold for restricted forms of second-order quantification, i.e., for closed-world semantics and for Hyper^{2}LTL$_{fp}$, a fragment of Hyper^{2}LTL proposed by Beutner et al. to make model-checking more feasible. Our results show that Hyper^{2}LTL$_{fp}$ does (in general) not achieve this goal. However, Beutner et al. presented a further syntactic restriction of Hyper^{2}LTL$_{fp}$ that guarantees quantification over unique sets. In fact, in this fragment, quantification degenerates to a fixed-point computation of a set of traces. They show that this fixed-point can be approximated to obtain a partial model-checking algorithm. In future work, we investigate the complexity and expressiveness of this fragment.

Another interesting question for future work is the addition of second-order quantification to HyperCTL*.

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References

[1] Raven Beutner, Bernd Finkbeiner, Hadar Frenkel, and Niklas Metzger. Second-order hyperproperties. In Constantin Enea and Akash Lal, editors, *CAV 2023, Part II*, volume 13965 of *LNCS*, pages 309–332. Springer, 2023.

[2] Michael R. Clarkson, Bernd Finkbeiner, Masoud Koleini, Kristopher K. Micinski, Markus N. Rabe, and César Sánchez. Temporal logics for hyperproperties. In Martín Abadi and Steve Kremer, editors, *POST 2014*, volume 8414 of *LNCS*, pages 265–284. Springer, 2014.

[3] Michael R. Clarkson and Fred B. Schneider. Hyperproperties. *J. Comput. Secur.*, 18(6):1157–1210, 2010.

[4] Bernd Finkbeiner and Christopher Hahn. Deciding hyperproperties. In Josée Desharnais and Radha Jagadeesan, editors, *CONCUR 2016*, volume 59 of *LIPIcs*, pages 13:1–13:14. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2016.

[5] Bernd Finkbeiner, Markus N. Rabe, and César Sánchez. Algorithms for Model Checking HyperLTL and HyperCTL*. In Daniel Kroening and Corina S. Pasareanu, editors, *CAV 2015, Part I*, volume 9206 of *LNCS*, pages 30–48. Springer, 2015.

[6] Bernd Finkbeiner and Martin Zimmermann. The First-Order Logic of Hyperproperties. In *STACS 2017*, volume 66 of *LIPIcs*, pages 30:1–30:14. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2017.

[7] Marie Fortin, Louwe B. Kuijer, Patrick Totzke, and Martin Zimmermann. HyperLTL satisfiability is Σ_1^1-complete, HyperCTL* satisfiability is Σ_1^2-complete. In Filippo Bonchi and Simon J. Puglisi, editors, *MFCS 2021*, volume 202 of *LIPIcs*, pages 47:1–47:19. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2021.

[8] Marie Fortin, Louwe B. Kuijer, Patrick Totzke, and Martin Zimmermann. HyperLTL satisfiability is highly undecidable, HyperCTL* is even harder. *arXiv*, 2303.16699, 2023. Journal version of [7]. Under submission.

[9] Hartley Rogers. *Theory of Recursive Functions and Effective Computability.* MIT Press, Cambridge, MA, USA, 1987.