Prophecies all the Way:
Game-based Model-Checking for HyperQPTL beyond $\forall^*\exists^*$

Sarah Winter (IRIF, Université Paris Cité, Paris, France)
Martin Zimmermann (Aalborg University, Aalborg, Denmark)

Abstract

Model-checking HyperLTL, a temporal logic expressing properties of sets of traces with applications to information-flow based security and privacy, has a decidable, but TOWER-complete, model-checking problem. In the classical algorithm, the complexity manifests itself with a need for the complementation of automata over infinite words. To overcome this aforementioned need, a game-based alternative for the $\forall^*\exists^*$-fragment was recently presented.

Here, we employ imperfect information-games to extend the game-based approach to full HyperQPTL, i.e., we allow arbitrary quantifier prefixes and quantification over propositions, which allows us to express every ω-regular hyperproperty. As a byproduct of our game-based algorithm, we obtain finite-state implementations of Skolem functions via transducers with lookahead that explain satisfaction or violation of HyperQPTL properties.

1 Introduction

Hyperlogics like HyperLTL and HyperCTL* [6] extend their classical counterparts LTL [20] and CTL* [10] by trace quantification and are thereby able to express so-called hyperproperties [7], properties which relate multiple execution traces of a system. These are crucial for expressing information-flow properties capturing security and privacy requirements [6]. For example, generalized noninterference [19] is captured by the HyperLTL formula

$$\varphi_{\mathrm{GNI}} = \forall \pi.\ \forall \pi'.\ \exists \pi''.\ \mathbf{G}\left(\bigwedge\nolimits_{\mathtt{p}\in L_{\mathrm{in}}\cup L_{\mathrm{out}}} \mathtt{p}_\pi \leftrightarrow \mathtt{p}_{\pi''}\right) \wedge \mathbf{G}\left(\bigwedge\nolimits_{\mathtt{p}\in H_{\mathrm{in}}} \mathtt{p}_{\pi'} \leftrightarrow \mathtt{p}_{\pi''}\right)$$

expressing that for all traces π and π' there exists a trace π'' that agrees with the low-security inputs (propositions in L_{in}) and low-security outputs (propositions in L_{out}) of π and the high-security inputs (propositions in H_{in}) of π'. Intuitively, it is satisfied if every input-output behavior observable by a low-security user of a system is compatible with any sequence of high-security inputs, i.e., the low security input-output behavior does not leak information about the high-security inputs, which should indeed be unobservable (directly and indirectly) by a low-security user.

These expressive logics offer a uniform approach to the specification, analysis, and verification of hyperproperties with intuitive syntax and semantics, and a decidable [6], albeit Tower-complete [22, 18], model-checking problem. The classical model-checking algorithm [6] is automata-based and even the case of $\forall^*\exists^*$-formulas like φ_{GNI} requires the complementation of ω-automata, a famously challenging problem to implement.

To eliminate the need for complementation, it is beneficial to again draw upon the deep connections between logic and games. For the sake of simplicity let us consider a HyperLTL formula of the form $\forall \pi.\ \exists \pi'.\psi$ with quantifier-free ψ. It is straightforward to capture the semantics of HyperLTL by a model-checking game between two players called Verifier (trying to prove $\mathfrak{T} \models \varphi$) and Falsifier (trying to prove $\mathfrak{T} \not\models \varphi$). In the model-checking game, Falsifier picks a trace of $\mathfrak{T}$ to interpret π and then Verifier picks a trace of $\mathfrak{T}$ to interpret π'. Verifier wins if these two traces satisfy the formula ψ, otherwise Falsifier wins. The game can easily be shown sound (if Verifier wins, then $\mathfrak{T} \models \varphi$) and complete (if $\mathfrak{T} \models \varphi$ then Verifier wins), as winning strategies are Skolem functions for π' and vice versa. But the game has one major drawback: both players have in general uncountably many moves.

To obtain a game that can be handled algorithmically, Coenen et al. introduced the following variation of the model-checking game [8] (which we will call the alternating game): Instead of picking the traces in one move, the players construct them alternatingly one vertex at a time. Together with a deterministic ω-automaton that is equivalent to ψ, this yields a finite game with ω-regular winning condition that is sound, but possibly incomplete ($\mathfrak{T} \models \varphi$ does not always imply that Verifier wins the alternating game).

The problem boils down to the informedness of Verifier: In the model-checking game, she has knowledge about the full trace assigned to π when picking π'. On the other hand, in the alternating game, Verifier only has access to the first vertex of the trace assigned to π when picking the first vertex of the trace assigned to π', which puts her at a disadvantage. That the alternating game can be incomplete is witnessed by the formula

$$\varphi_{\mathrm{inc}} = \forall \pi.\ \exists \pi'.\ \mathtt{p}_{\pi'} \leftrightarrow \mathbf{F}\, \mathtt{p}_{\pi}$$

expressing that for every trace π in $\mathfrak{T}$ there is a trace π' in $\mathfrak{T}$ such that $\mathtt{p}$ holds at the first position of π' if and only if there is a $\mathtt{p}$ somewhere in π, i.e., the first letter of π' depends (possibly) on every letter of π. So, by not picking an $\mathtt{p}$ in the first round, Falsifier forces Verifier to make a prediction about whether Falsifier will every pick a $\mathtt{p}$ or not in the future. However, Falsifier can easily contradict that prediction and thereby win, e.g., for a transition system $\mathfrak{T}_{\mathrm{all}}$ having all traces over $\mathtt{p}$. Thus, we indeed have $\mathfrak{T}_{\mathrm{all}} \models \varphi$, but Verifier does not win the alternating game.

However, a single bit of lookahead allows Verifier to win the alternating game induced by $\mathfrak{T}_{\mathrm{all}}$ and φ_{inc}, i.e., the answer to the query "will the trace picked by Falsifier contain a $\mathtt{p}$". If Falsifier has to provide this information with his first move, then Verifier can make her first move accordingly. For correctness, we additionally have to adapt the rules of the alternating game so that Verifier loses when contradicting his answer made during the first round, e.g., if he commits to there being no $\mathtt{p}$'s in his trace, but then picking one.

In general, it is not sufficient to ask a single query at the beginning of a play, but needs to ask queries at every move of Falsifier. To see this, let us consider another example, this time with the formula

$$\varphi'_{\mathrm{inc}} = \forall \pi.\ \exists \pi'.\ \mathbf{G}(\mathtt{p}_{\pi'} \leftrightarrow \mathbf{X}\, \mathtt{p}_{\pi}).$$

Here, Verifier has to always pick a $\mathtt{p}$ in her move if and only if Falsifier will pick a $\mathtt{p}$ in his next move (which Verifier does not have yet access to). Thus, Verifier wins if she gets the (truthful) answer to the query "does the next move by Falsifier contain a $\mathtt{p}$", but loses without the ability to query Falsifier. Note that in both cases, the queries are used to obtain a (binding) commitment about the future moves that Falsifier will make.

Coenen et al. [8] and Beutner and Finkbeiner [5] showed how to formalize this intuition using so-called prophecies [1]. A single prophecy is a language P of traces and is associated with a (Boolean) prophecy variable. Now, in addition to picking, vertex by vertex, a trace of $\mathfrak{T}$, Falsifier also picks a truth value for the prophecy variable with the interpretation that a value of 1 corresponds to the suffix of the trace picked by him from now on being in P and that a value of of 0 corresponds to the suffix of the trace picked by him from now on not being in P. If the language P is ω-regular, then one can employ an ω-automaton to check whether the predictions made by Falsifier are actually truthful (and make him lose if they are not truthful). We call the resulting game the alternating game with prophecies.

The main result of Beutner and Finkbeiner shows that for every $\forall^*\exists^*$ HyperLTL formula there is a finite set of ω-regular prophecies such that Verifier wins the alternating game with these prophecies if and only if $\mathfrak{T} \models \varphi$, i.e., the right prophecies ensure that the alternating game-based approach to model-checking is not only sound, but also complete. The resulting game with prophecies is finite-state and has a ω-regular winning condition, and can therefore be solved effectively. Note that this construction is still automata-based, as the prophecies are derived from a (deterministic) ω-automaton for the quantifier-free part of φ. But, here one "only" needs to translate temporal logic formulas into deterministic ω-automata, but no complementation is required. Beutner and Finkbeiner implemented their prophecy-based model-checking algorithm and presented encouraging results on small instances, e.g., their prototype implementation is in some cases the first tool that can prove $\mathfrak{T} \not\models \varphi$, a result that was out of reach for existing model-checking tools [5]. These results shows the potential of the game-based approach to HyperLTL model-checking.

However, one question remains open: can their alternating game-based approach be extended to full HyperLTL, i.e., beyond a single quantifier alternation. There is precedent for such a game-theoretic analysis of full HyperLTL: Recently, Winter and Zimmermann showed that the existence of (Turing) computable Skolem functions for existentially quantified variables can be characterized by a game [24]. For example, $\mathfrak{T}_{\mathrm{all}} \models \varphi'_{\mathrm{inc}}$ is witnessed by computable Skolem functions, as reading a prefix of length n

of π allows to compute the prefix of length $n-1$ of π' so that for every π, π and the resulting π' satisfy $\mathbf{G}(\mathsf{p}_{\pi'} \leftrightarrow \mathbf{X}\,\mathsf{p}_{\pi})$. On the other hand, $\mathfrak{T}_{\text{all}} \models \varphi_{\text{inc}}$ is not witnessed by computable Skolem functions, as the choice of the first letter of π' depends, as explained above, on all letters of π'. Such a Skolem function is not computable by a Turing machine, as it is not continuous.

The game characterizing the existence of computable Skolem functions, say for a formula

$$\varphi = \forall \pi_0.\ \exists \pi_1.\ \ldots \forall \pi_{k-2}.\ \exists \pi_{k-1}.\ \psi$$

with quantifier-free ψ, is a multi-player game played between a player in charge of selecting, vertex by vertex, a trace for each universally quantified variable (i.e., he has the role that Falsifier has in the previous games), and a coalition of players, one for each existentially quantifier variable (i.e., the coalition has the role that Verifier has in the previous games). Furthermore, the game must be of imperfect information in order to capture the semantics of HyperLTL, where the choice of π_i may only depend on the choice of the π_j with $j < i$. Thus, in the game, the player in charge of an existentially quantified π_i only has access to the choices made so far for the π_j for $j < i$. Furthermore, the game needs to incorporate a delay [16, 17, 12] between the moves of the different players to capture the fact that a choice by one of the existential players may depend on future moves by the universal player (see, e.g., the formula φ'_{inc} above). The main insight then is that a bounded delay is always sufficient, if there are computable Skolem functions at all (see, again, the difference between φ_{inc} (which has no computable Skolem functions over $\mathfrak{T}_{\text{all}}$) and φ'_{inc} (which has computable Skolem functions over $\mathfrak{T}_{\text{all}}$)).

Our Contribution. We present the first effective game-based characterization of model-checking for full HyperLTL (and even HyperQPTL, which allows to express all ω-regular hyperproperties [22, 13]), yielding a sound and complete imperfect information finite-state game with ω-regular winning condition. This result both generalizes the alternating game with prophecies from $\forall^*\exists^*$ formulas to formulas with arbitrary quantifier prefixes and the game of Winter and Zimmermann from characterizing the existence of computable Skolem functions witnessing $\mathfrak{T} \models \varphi$ to characterizing $\mathfrak{T} \models \varphi$.

However, $\mathfrak{T} \models \varphi_{\text{inc}}$ holds, but does not have computable Skolem functions, the games of Winter et al. are *not* a special case of the game we construct here: Our games here are still multi-player games of imperfect information (to capture the semantics of quantification) and use prophecies (for completeness), but do not require delayed moves, as prophecies can be seen as a form of infinite lookahead. And while the existence of computable Skolem functions is concerned with bounded lookahead, here we do indeed need infinite lookahead as witnessed by the formula φ_{inc}.

Our main result shows that there is again a finite, effectively computable, set of ω-regular prophecies so that the coalition of players in charge of the existentially quantified variables has a winning strategy in the game with these prophecies if and only if $\mathfrak{T} \models \varphi$. One challenge to overcome is a careful definition of the prophecies, so that they are not leaking any information about choices for variables π_j that a player in charge of π_i with $i < j$ may not have access to.

Our result can also be framed in terms of Skolem function implementable by letter-to-letter transducers with (ω-regular) lookahead: such a transducer computes a Skolem function for an existentially quantified variable π while reading values for the variables universally quantified before π while also being able to get a regular lookahead on the trace for those universally quantified variable (much like prophecies). The use of regular lookahead is well-studied in automata theory, see e.g., [9, 11, 3].

Note that our construction is sound and complete like the alternating games with prophecies for the $\forall^*\exists^*$-fragment, but we need to complement ω-automata to prevent the information-leak described above. It is open whether this can be avoided while staying sound and complete. On the other hand, we conjecture that our construction opens the door to proving decidability results for even more expressive logics like HyperRecHML, recursive Hennesey-Milner logic with trace quantification [2].

2 Preliminaries

For convenience, technical terms and notations in the electronic version of this manuscript are hyperlinked to their definitions (cf. `https://ctan.org/pkg/knowledge`).

Hereafter, we denote the set of nonnegative integers by $\mathbb{N}$.

Traces, Transition Systems and Automata. An alphabet is a nonempty finite set. The sets of finite and infinite words over an alphabet Σ are denoted by Σ^* and Σ^ω, respectively. The length of finite or infinite word w is denoted by $|w| \in \mathbb{N} \cup \{\infty\}$. For a word w of length at least n, we write $w[0, n)$ for the prefix of w of length n. Given n infinite words $w_0, \ldots, w_{n-1}$, let their *merge* (also known as zip), which is an infinite word over Σ^n, be defined as

$$\mathrm{mrg}(w_0, \ldots, w_{n-1}) = (w_0(0), \ldots, w_{n-1}(0))(w_0(1), \ldots, w_{n-1}(1))(w_0(2), \ldots, w_{n-1}(2)) \cdots .$$

We define $\mathrm{mrg}(w_0, \ldots, w_{n-1})$ for finite words $w_0, \ldots, w_{n-1}$ of the same length analogously.

Let AP be a nonempty finite set of atomic propositions. A *trace* over AP is an infinite word over the alphabet 2^{AP}. Given a subset $\mathrm{AP}' \subseteq \mathrm{AP}$, the AP'-projection of a trace $t(0)t(1)t(2)\cdots$ over AP is the trace $(t(0) \cap \mathrm{AP}')(t(1) \cap \mathrm{AP}')(t(2) \cap \mathrm{AP}') \cdots \in (2^{\mathrm{AP}'})^\omega$. Now, let AP and AP' be two disjoint sets, let t be a trace over AP, and let t' be a trace over AP'. Then, we define $t^\frown t'$ as the pointwise union of t and t', i.e., $t^\frown t'$ is the trace over $\mathrm{AP} \cup \mathrm{AP}'$ defined as $(t(0) \cup t'(0))(t(1) \cup t'(1))(t(2) \cup t'(2)) \cdots$.

A *transition system* $\mathfrak{T} = (V, E, V_I, \lambda)$ consists of a finite set V of vertices, a set $E \subseteq V \times V$ of (directed) edges, a nonempty set $V_I \subseteq V$ of initial vertices, and a labelling $\lambda \colon V \to 2^{\mathrm{AP}}$ of the vertices by sets of atomic propositions. We assume that every vertex has at least one outgoing edge. For $v \in V$, we denote by $\mathrm{S}(v)$ the set of its successors. A *path* ρ through $\mathfrak{T}$ is an infinite sequence $\rho = v_0 v_1 v_2 \cdots$ of vertices with $v_0 \in V_I$ and $(v_n, v_{n+1}) \in E$ for every $n \geq 0$. The *trace of* ρ is defined as $\lambda(\rho) = \lambda(v_0)\lambda(v_1)\lambda(v_2) \cdots \in (2^{\mathrm{AP}})^\omega$. The *set of traces* of $\mathfrak{T}$ is $\mathrm{Tr}(\mathfrak{T}) = \{\lambda(\rho) \mid \rho \text{ is a path of } \mathfrak{T}\}$. For $V' \subseteq V$, we write $\mathfrak{T}_{V'}$ to denote the transition system (V, E, V', λ) obtained from $\mathfrak{T}$ by making V' the set of initial states, and use $\mathfrak{T}_v$ as shorthand for $\mathfrak{T}_{\{v\}}$ for $v \in V$.

A (deterministic) *parity automaton*[1] $\mathcal{A} = (Q, \Sigma, q_I, \delta, \Omega)$ consists of a finite set Q of states containing the initial state $q_I \in Q$, an alphabet Σ, a transition function $\delta \colon Q \times \Sigma \to Q$, and a coloring $\Omega \colon Q \to \mathbb{N}$ of its states by natural numbers. Let $w = w(0)w(1)w(2) \cdots \in \Sigma^\omega$. The run of $\mathcal{A}$ on w is the sequence $q_0 q_1 q_2 \cdots$ with $q_0 = q_I$ and $q_{n+1} = \delta(q_n, w(n))$ for all $n \geq 0$. A run $q_0 q_1 q_2 \cdots$ is (parity) accepting if the maximal color appearing infinitely often in the sequence $\Omega(q_0)\Omega(q_1)\Omega(q_2) \cdots$ is even. The language (parity) recognized by $\mathcal{A}$, denoted by $L(\mathcal{A})$, is the set of infinite words over Σ such that the run of $\mathcal{A}$ on w is accepting.

Remark 1. *Deterministic parity automata accept exactly the ω-regular languages (see, e.g., [15] for definitions).*

HyperQPTL, HyperLTL and QPTL. Let $\mathcal{V}$ be a countable set of *trace variables*. The formulas of *HyperQPTL* are given by the grammar

$$\varphi ::= \exists \pi.\ \varphi \mid \forall \pi.\ \varphi \mid \psi \qquad \psi ::= \tilde{\exists} \mathtt{q}.\ \psi \mid \tilde{\forall} \mathtt{q}.\ \psi \mid \psi \mid \mathtt{p}_\pi \mid \mathtt{q} \mid \neg\psi \mid \psi \vee \psi \mid \mathbf{X}\,\psi \mid \mathbf{F}\,\psi$$

where $\mathtt{p}$ and $\mathtt{q}$ range over AP and where π ranges over $\mathcal{V}$. Here, we use a tilde to decorate propositional quantifiers to distinguish them from trace quantifiers.

Note that there are two types of atomic formulas, i.e., propositions labeled by trace variables on which they are evaluated ($\mathtt{p}_\pi$ with $\mathtt{p} \in \mathrm{AP}$ and $\pi \in \mathcal{V}$) and unlabeled propositions ($\mathtt{q} \in \mathrm{AP}$).[2] A formula is a sentence, if every occurrence of an atomic formula $\mathtt{p}_\pi$ is in the scope of a quantifier binding π and every occurrence of an atomic formula $\mathtt{q}$ is in the scope of a quantifier binding $\mathtt{q}$. Finally, note that the only temporal operators we have in the syntax are next ($\mathbf{X}$) and eventually ($\mathbf{F}$), as the other temporal operators like always ($\mathbf{G}$) and until ($\mathbf{U}$) are syntactic sugar in HyperQPTL. Hence, we will use them freely in the following, just as we use conjunction, implication, and equivalence.

A (trace) *variable assignment* is a partial mapping $\Pi \colon \mathcal{V} \to (2^{\mathrm{AP}})^\omega$. Given a variable $\pi \in \mathcal{V}$, and a trace t, we denote by $\Pi[\pi \mapsto t]$ the assignment that coincides with Π on all variables but π, which is mapped to t. It remains to capture the semantics of quantification of propositions $\mathtt{q} \in \mathrm{AP}$. Let $t \in (2^{\mathrm{AP}})^\omega$ be a trace over AP and $t_\mathtt{q} \in (2^{\{\mathtt{q}\}})^\omega$ be a trace over $\{\mathtt{q}\}$. We define the trace $t[\mathtt{q} \mapsto t_\mathtt{q}] = t'^\frown t_\mathtt{q}$, where t' is the $(\mathrm{AP} \setminus \{\mathtt{q}\})$-projection of t: Intuitively, the occurrences of $\mathtt{q}$ in t are replaced according to $t_\mathtt{q}$. We lift this to sets T of traces by defining $T[\mathtt{q} \mapsto t_\mathtt{q}] = \{t[\mathtt{q} \mapsto t_\mathtt{q}] \mid t \in T\}$. Note that all traces in $T[\mathtt{q} \mapsto t_\mathtt{q}]$ have the same $\{\mathtt{q}\}$-projection, which is $t_\mathtt{q}$.

Now, for a trace assignment Π, a position $i \in \mathbb{N}$, and a nonempty set T of traces, i.e., we disregard the empty set of traces as model, we define

[1] Note that we use parity acceptance, as we need deterministic automata for our proofs.

[2] We use different letters in these cases, but let us stress again that both $\mathtt{p}$ and $\mathtt{q}$ are propositions in AP.

- $T, \Pi, i \models \mathtt{p}_\pi$ if $\mathtt{p} \in \Pi(\pi)(i)$,
- $T, \Pi, i \models \mathtt{q}$ if for all $t \in T$ we have $\mathtt{q} \in t(i)$,
- $T, \Pi, i \models \neg\psi$ if $T, \Pi, i \not\models \psi$,
- $T, \Pi, i \models \psi_1 \vee \psi_2$ if $T, \Pi, i \models \psi_1$ or $T, \Pi, i \models \psi_2$,
- $T, \Pi, i \models \mathbf{X}\,\varphi$ if $T, \Pi, i+1 \models \varphi$,
- $T, \Pi, i \models \mathbf{F}\,\varphi$ if there is a $j \geq i$ such that $T, \Pi, j \models \varphi$,
- $T, \Pi, i \models \tilde{\exists}\mathtt{q}.\ \varphi$ if there exists a trace $t_\mathtt{q} \in (2^{\{\mathtt{q}\}})^\omega$ such that $T[\mathtt{q} \mapsto t_\mathtt{q}], \Pi, i \models \varphi$,
- $T, \Pi, i \models \tilde{\forall}\mathtt{q}.\ \varphi$ if for all traces $t_\mathtt{q} \in (2^{\{\mathtt{q}\}})^\omega$ we have $T[\mathtt{q} \mapsto t_\mathtt{q}], \Pi, i \models \varphi$,
- $T, \Pi, i \models \exists\pi.\ \varphi$ if there exists a trace $t \in T$ such that $T, \Pi[\pi \mapsto t], i \models \varphi$, and
- $T, \Pi, i \models \forall\pi.\ \varphi$ if for all traces $t \in T$ we have $T, \Pi[\pi \mapsto t], i \models \varphi$.

We say that an (again nonempty) set T of traces satisfies a sentence φ, written $T \models \varphi$, if $T, \Pi_\emptyset, 0 \models \varphi$ where $\Pi_\emptyset$ is the variable assignment with empty domain. We then also say that T is a model of φ. A transition system $\mathfrak{T}$ satisfies φ, written $\mathfrak{T} \models \varphi$, if $\mathrm{Tr}(\mathfrak{T}) \models \varphi$. Let ψ be a trace quantifier-free formula that has no free (unlabeled) propositions. A labeled proposition of the form $\mathtt{p}_\pi$ in ψ is evaluated on the traces assigned to π and an unlabeled proposition of the form $\mathtt{q}$ is evaluated on a trace "selected" by the quantifier binding $\mathtt{q}$. Hence, $T, \Pi, i \models \psi$ is independent of T and we often write $\Pi \models \psi$ instead of $T, \Pi, 0 \models \psi$.

Some comments on the definition of HyperQPTL are due.

Remark 2. *We have, for technical reasons, defined HyperQPTL so that every formula has a prefix of trace quantifiers followed by a formula (possibly) containing propositional quantifiers, but no more trace quantifiers. On the other hand, Finkbeiner et al. [13] require formulas to be in prenex normal form, but allow to mix trace and propositional quantification in the quantifier prefix. We refrained from doing so, as one can modify the transition system one is interested in, so that it has enough paths to simulate propositional quantification by trace quantification. We leave the straightforward details to the reader.*

Remark 3. HyperLTL *is the fragment of HyperQPTL sentences that do not use the propositional quantifiers $\tilde{\exists}$ and $\tilde{\forall}$ (and thus also not use unlabeled propositions).*

We say that a HyperQPTL formula φ is a *QPTL* sentence if it has no trace quantifiers and no free propositional variables, i.e., all unlabeled propositional variables are in the scope of a propositional quantifier. But a QPTL sentence may have free trace variables, say $\pi_0, \ldots, \pi_{k-1}$ for some $k \geq 0$. Typically, QPTL is defined without trace variables labelling propositions and QPTL formulas define languages over the alphabet 2^{AP} [23]. For technical necessity, we allow such labels, which implies that our formulas define languages over the alphabet $(2^{\mathrm{AP}})^k$, where k is the number of trace variables occurring in the formula. More formally, a QPTL sentence with trace variables $\pi_0, \pi_1, \ldots, \pi_{k-1}$ defines the language

$$L(\varphi) = \{\mathrm{mrg}(t_0, \ldots, t_{k-1}) \mid t_0, \ldots, t_{k-1} \in (2^{\mathrm{AP}})^\omega : \Pi_\emptyset[\pi_0 \mapsto t_0, \ldots, \pi_{k-1} \mapsto t_{k-1}] \models \varphi\}.$$

Now, let $\varphi = Q_0\pi_0 Q_1\pi_1 \cdots Q_{k-1}\pi_{k-1}.\ \psi$ with $Q_i \in \{\exists, \forall\}$ and trace quantifier-free ψ be a HyperQPTL sentence and define $\varphi_i = Q_{i+1}\pi_{i+1} Q_{i+2}\pi_{i+2} \cdots Q_{k-1}\pi_{k-1}.\ \psi$ for $i \in \{0, 1, \ldots, k-1\}$ and $\varphi_{-1} = \varphi$. Note that $\varphi_{k-1} = \psi$ and that the free trace variables of each φ_i with $i \in \{0, 1, \ldots, k-1\}$ are exactly $\pi_0, \ldots, \pi_i$. The following results follows by combining and adapting automata constructions for classical QPTL and HyperLTL [6, 14, 23].

Proposition 1. *1. Let $\mathfrak{T}$ be a transition system. For every $i \in \{-1, 0, \ldots, k-1\}$ there is an (effectively constructible) parity automaton $\mathcal{A}_i^{\mathfrak{T}}$ such that*

$$L(\mathcal{A}_i^{\mathfrak{T}}) = \{\mathrm{mrg}(\Pi(\pi_0), \ldots, \Pi(\pi_i)) \mid \Pi(\pi_j) \in \mathrm{Tr}(\mathfrak{T}) \text{ for } 0 \leq j \leq i \text{ and } \mathrm{Tr}(\mathfrak{T}), \Pi, 0 \models \varphi_i\}.$$

2. A language over $(2^{\mathrm{AP}})^k$ is ω-regular if and only if it is of the form $L(\varphi)$ for a QPTL-sentence φ over some $\mathrm{AP}' \supseteq \mathrm{AP}$ with k free trace variables.[3]

[3]For the sake of readability, in the following, we do not distinguish between AP and AP′, i.e., we assume that AP always contains enough propositions not used in our languages to quantify over.

3 Skolem Functions for HyperQPTL

Let $\varphi = Q_0\pi_0 \cdots Q_{k-1}\pi_{k-1}.\ \psi$ be a HyperQPTL sentence such that ψ is trace quantifier-free and let T be a nonempty set of traces. Moreover, let $i \in \{0, 1, \ldots, k-1\}$ be such that $Q_i = \exists$ and let $U_i = \{j < i \mid Q_j = \forall\}$ be the indices of the universal quantifiers preceding Q_i. Furthermore, let $f_i \colon T^{|U_i|} \to T$ for each such i (note that f_i is a constant, if U_i is empty). We say that a trace assignment Π with $\mathrm{dom}(\Pi) \supseteq \{\pi_0, \pi_1, \ldots, \pi_{k-1}\}$ is consistent with the f_i if $\Pi(\pi_i) \in T$ for all i with $Q_i = \forall$ and $\Pi(\pi_i) = f_i(\Pi(\pi_{i_0}), \Pi(\pi_{i_1}), \ldots, \Pi(\pi_{i_{|U_i|-1}}))$ for all i with $Q_i = \exists$, where $U_i = \{i_0 < i_1 < \cdots < i_{|U_i|-1}\}$. If $\Pi \models \psi$ for each Π that is consistent with the f_i, then we say that the f_i are *Skolem functions* witnessing $T \models \varphi$.

Remark 4. *$T \models \varphi$ iff there are Skolem functions for the existentially quantified variables of φ that witness $T \models \varphi$.*

Note that only traces for universal variables are inputs for Skolem functions, but not those for existentially quantified variables. As usual, this is not a restriction, as the inputs of a Skolem function for an existentially quantified variable π_i are a superset of the inputs of a Skolem function for another existentially quantified variable π_j with $j < i$.

Example 1 ([25])**.** *Let $\varphi = \forall\pi\exists\pi_1\exists\pi_2.\ \mathbf{G}(a_\pi \leftrightarrow (a_{\pi_1} \oplus a_{\pi_2}))$. We have $(2^{\{a\}})^\omega \models \varphi$. Now, for every function $f_1 \colon (2^{\{a\}})^\omega \to (2^{\{a\}})^\omega$, there is a function $f_2 \colon (2^{\{a\}})^\omega \to (2^{\{a\}})^\omega$ such that f_1, f_2 are Skolem functions witnessing $(2^{\{a\}})^\omega \models \varphi$, i.e., we need to define f_2 such that $(f_2(t))(n) = (f_1(t))(n)$ for all $n \in \mathbb{N}$ such that $t(n) = \emptyset$ and $(f_2(t))(n) = \overline{(f_1(t))(n)}$ for all $n \in \mathbb{N}$ such that $t(n) = \{a\}$, where $\overline{\{a\}} = \emptyset$ and $\overline{\emptyset} = \{a\}$. Hence, f_2 depends on f_1, but the value of $f_1(t)$ (for the existentially quantified π_1) does not need to be an input to f_2, it can be determined from the input t for the universally quantified π. This is not surprising, but needs to be taken into account in our constructions.*

4 Gamed-based Model-Checking for HyperQPTL

In this section, we present our game-theoretic characterization of $\mathfrak{T} \models \varphi$ for finite transition systems $\mathfrak{T}$ and HyperQPTL sentences φ. To this end, we introduce multi-player games with hierarchical information in Section 4.1 and then present the construction of our a game in Section 4.2, while we introduce prophecies and prove their soundness in Section 4.3. Then, in Section 5, we show how to construct complete prophecies for the special case of safety formulas (to be defined formally there). Finally, in Section 6, we show how to construct complete prophecies for arbitrary formulas. This allows us to first explain how to generalize the prophecy definition from $\forall^*\exists^*$ to arbitrary quantifier prefixes and then, in a second step, move from safety to ω-regular languages.

4.1 Multi-player Graph Games with Hierarchical Information

We develop a game-based characterization of model-checking for HyperQPTL via (multi-player) graph games with hierarchical information, using the notations of Berwanger et al. [4]. First, we introduce the necessary definitions and then present our game in Section 4.2. The games considered by Berwanger et al. are concurrent games (i.e., the players make their moves simultaneously), while for our purpose turn-based games (i.e., the players make their moves one after the other) are sufficient. Turn-based games are simpler versions of concurrent games. To avoid cumbersome notation, we introduce a turn-based variant of these games.

Fix some finite set C of players forming a coalition playing against a distinguished agent called Nature (which is *not* in C). For each player $i \in C$ we fix a finite set B^i of observations. A game graph $G = (V, E, v_I, (\beta^i)_{i\in C})$ consists of a finite set $V = \biguplus_{i\in C} V_i \uplus V_{\mathrm{Nat}}$ of positions partitioned into sets controlled by some player resp. Nature, an edge relation $E \subseteq V \times V$ representing moves, an initial position $v_I \in V$, and a collection $(\beta^i)_{i\in C}$ of observation functions $\beta^i \colon V \to B^i$ that label, for each player, the positions with observations. We require that E has no dead-ends, i.e., for every $v \in V$ there is a $v' \in V$ with $(v, v') \in E$.

A game graph $(V, E, v_I, (\beta^i)_{i\in C})$ yields hierarchical information if there exists a total order $\preceq$ over C such that if $i \preceq j$ then for all $v, v' \in V$, $\beta^i(v) = \beta^i(v')$ implies $\beta^j(v) = \beta^j(v')$, i.e., if Player i cannot distinguish v and v', then neither can Player j for $i \preceq j$.

Intuitively, a play starts at position $v_I \in V$. At position v, the player that controls this position chooses a successor position v' such that $(v, v') \in E$. Now, each player $i \in C$ receives the observation $\beta^i(v')$ and the play continues from position v'. Thus, a play of G is an infinite sequence $v_0 v_1 v_2 \cdots$ of vertices such that $v_0 = v_I$ and for all $r \geq 0$ we have $(v_r, v_{r+1}) \in E$.

A history is a prefix $v_0 v_1 \cdots v_r$ of a play. We denote the set of all histories by $\mathrm{Hist}(G)$ and extend $\beta^i \colon V \to B^i$ to plays and histories by defining $\beta^i(v_0 v_1 v_2 \cdots) = \beta^i(v_1)\beta^i(v_2)\beta^i(v_3)\cdots$. Note that the observation of the initial position is discarded for technical reasons [4]. We say two histories h and h' are indistinguishable to Player $i \in C$, denoted by $h \sim_i h'$, if $\beta^i(h) = \beta^i(h')$.

A strategy for Player $i \in C$ is a mapping $s^i \colon V^* \to V$ that satisfies $s^i(h) = s^i(h')$ for all histories h, h' with $h \sim_i h'$ (i.e., the move selected by the strategy only depends on the observations of the history). A play $v_0 v_1 v_2 \cdots$ is consistent with s^i if for every $r \geq 0$, we have $v_{r+1} = s^i(v_0 v_1 \cdots v_r)$. A play is consistent with a collection of strategies $(s^i)_{i \in C}$ if it is consistent with each s^i. The set of possible outcomes of a collection of strategies is the set of all plays that are consistent with it. As usual, a strategy is finite-state, if it is implemented by some Moore machine.

Lastly, a game $\mathcal{G}$ consists of a game graph G and a winning condition $W \subseteq V^\omega$, where V is the set of positions of G. A play is winning if it is in W and a collection of strategies is winning if all its outcomes are winning.

Proposition 2 ([21, 4])**.** *1. It is decidable, given a game with hierarchical information with ω-regular winning condition, whether it has a winning collection of strategies?*

2. A game with hierarchical information with ω-regular winning condition has a winning collection of strategies if and only if it has a winning collection of finite-state strategies.

4.2 The Model-checking Game

For the remainder of this paper, we fix a HyperQPTL sentence φ and a transition system $\mathfrak{T}$. We assume (w.l.o.g.)[4]

$$\varphi = \forall \pi_0 \exists \pi_1 \cdots \forall \pi_{k-2} \exists \pi_{k-1}.\ \psi$$

such that ψ is trace quantifier-free, define

$$\varphi_i = Q_{i+1}\pi_{i+1} Q_{i+2}\pi_{i+2} \cdots \forall \pi_{k-2} \exists \pi_{k-1}.\ \psi$$

for $i \in \{0, 1, \ldots, k-1\}$ and use the automata $\mathcal{A}_i^{\mathfrak{T}}$ constructed in Proposition 1 satisfying

$$L(\mathcal{A}_i^{\mathfrak{T}}) = \{\mathrm{mrg}(\Pi(\pi_0), \ldots, \Pi(\pi_i)) \mid \Pi(\pi_j) \in \mathrm{Tr}(\mathfrak{T}) \text{ for all } 0 \leq j \leq i \text{ and } \mathrm{Tr}(\mathfrak{T}), \Pi, 0 \models \varphi_i\}.$$

We use the notation $(\mathcal{A}_i^{\mathfrak{T}})_q$ to denote the parity automaton obtained from $\mathcal{A}_i^{\mathfrak{T}}$ by making its state q the initial state. Finally, let $\mathcal{A}_{k-1}^{\mathfrak{T}} = (Q, \Sigma, q_I, \delta, \Omega)$. Note that $\mathcal{A}_{k-1}^{\mathfrak{T}}$ accepts the trace assignments coming from paths trough $\mathfrak{T}$ that satisfy ψ.

We define a multi-player game $\mathcal{G}(\mathfrak{T}, \varphi)$ with hierarchical information induced by the transition system $\mathfrak{T}$ and the HyperQPTL sentence φ. This game is played between *Falsifier* (who takes on the role of Nature, cf. Section 4.1), who is in charge of providing traces (from paths trough the transition system) for the universally quantified variables, and a coalition of *Verifier-players* $\{1, 3, \ldots, k-1\}$ (*Verifier i* for short), who are in charge of providing traces (also from paths trough the transition system) for the existentially quantified variables. The goal of Falsifier is to prove $\mathfrak{T} \not\models \varphi$ and the goal of the coalition of Verifier-players is to prove $\mathfrak{T} \models \varphi$. Therefore, the k traces built during a play are read synchronously by the parity automaton $\mathcal{A}_{k-1}^{\mathfrak{T}}$ accepting the trace assignments that satisfy ψ (recall that ψ is the trace quantifier-free part of φ). We say that $\mathcal{A}_k^{\mathfrak{T}}$ checks ψ.

The game has two phases. An initialization phase where initial vertices for all paths through the transition system are picked, and a second phase (of infinite duration) where the paths (which induce the trace assignment) are build. Formally, in the initialization phase, a position of the game is of the form $((v_0, \ldots, v_{i-1}, \underbrace{\bullet, \ldots, \bullet}_{k-i \text{ times}}), q_I, i)$ where $v_0, \ldots, v_{i-1} \in V_I$ are initial vertices in the transition system $\mathfrak{T}$, $\bullet$ is

[4] The following reasoning can easily be extended to general sentences with arbitrary quantifier prefixes, albeit at the cost of more complex notation.

a fresh (placeholder) symbol, $q_I \in Q$ is the initial state of $\mathcal{A}^{\mathfrak{T}}_{k-1}$, and $i \in \{0, 1, \cdots, k-1\}$. In the second phase, a position of the game is of the form $((v_0, \ldots, v_{k-1}), q, i)$ where $v_0, v_1, \ldots, v_{k-1} \in V$, $q \in Q$, and $i \in \{0, 1, \ldots, k-1\}$. Vertices of the i-th Verifier-player are those with $i \in \{1, 3, \ldots, k-1\}$ and vertices of Falsifier are those with $i \in \{0, 2, \ldots, k-2\}$.

The edges (also called moves) of the game graph are defined as follows, where the first two items are the moves in the initialization phase:

- $\big(((v_0, \ldots, v_{i-1}, \bullet, \ldots, \bullet), q_I, i), ((v_0, \ldots, v_{i-1}, v_i, \bullet, \ldots, \bullet), q_I, i+1)\big)$ is a move for all $v_i \in V_I$ and all $i \in \{0, 1, \ldots, k-2\}$: An initial vertex in $\mathfrak{T}$ for the i-th path is picked.
- $\big(((v_0, \ldots, v_{k-2}, \bullet), q_I, k-1), ((v_0, \ldots, v_{k-2}, v_{k-1}), q, 0)\big)$ is a move for all $v_{k-1} \in V_I$, where $q = \delta(q_I, \mathrm{mrg}(\lambda(v_0), \ldots, \lambda(v_{k-1})))$: An initial vertex in $\mathfrak{T}$ for the last path is picked. With this move, the initialization phase is over and the state of $\mathcal{A}^{\mathfrak{T}}_{k-1}$ checking ψ is updated for the first time.
- $\big(((v'_0, \ldots, v'_{i-1}, v_i, \ldots, v_{k-1}), q, i), ((v'_0, \ldots, v'_{i-1}, v'_i, v_{i+1} \ldots, v_{k-1}), q, i+1)\big)$ is a move for all $(v_i, v'_i) \in E$ and all $i \in \{0, 1, \ldots, k-2\}$: The i-th path is updated by moving to a successor vertex in $\mathfrak{T}$.
- $\big(((v'_0, \ldots, v'_{k-2}, v_{k-1}), q, k-1), ((v'_0, \ldots, v'_{k-2}, v'_{k-1}), q', 0)\big)$ is a move for all $(v_{k-1}, v'_{k-1}) \in E$, where $q' = \delta(q, \mathrm{mrg}(\lambda(v'_0), \lambda(v'_1), \ldots, \lambda(v'_{k-1})))$: The last path is updated by moving to a successor vertex in $\mathfrak{T}$. Simultaneously, the state of $\mathcal{A}^{\mathfrak{T}}_{k-1}$ checking ψ is updated.

The initial position is $((\bullet, \ldots, \bullet), q_I, 0)$. We use a parity winning condition. The color of all positions of the initialization phase is 0 (the color is of no consequence as these vertices are seen only once during the course of a play). The color of positions of the form $((v_0, \ldots, v_{k-1}), q, i)$ is the color that q has in $\mathcal{A}^{\mathfrak{T}}_{k-1}$, i.e., a play is winning for the verifier players if and only if the trace assignments picket by them and Falsifier satisfies ψ.

The game described must be a game of hierarchical information to capture the fact that the Skolem function for an existentially quantified π_i depends only on the universally quantified variables π_j with $j \in \{0, 2, \ldots, i-1\}$. To capture that, we define an equivalence relation $\equiv_i$ between positions of the game for $i \in \{1, 3, \ldots, k-1\}$ which ensures that for Verifier i, two positions are indistinguishable if they coincide on their first $i+1$ components and additionally belong to the same player. Formally, regarding positions from the initialization phase, let $((v_0, \ldots, v_{m-1}, \bullet, \ldots, \bullet), q_I, m)$ and $((v'_0, \ldots, v'_{n-1}, \bullet, \ldots, \bullet), q_I, n)$ be $\equiv_i$-equivalent if $v_j = v'_j$ for all $j \leq i$ and $m = n$. Regarding all other positions, let $((v_0, \ldots, v_{k-1}), p, m)$ and $((v'_0, \ldots, v'_{k-1}), q, n)$ be $\equiv_i$-equivalent if $v_j = v'_j$ for all $j \leq i$ and $m = n$. Now, we define the observation functions: For Verifier i, it maps positions to their $\equiv_i$-equivalence classes.

4.3 The Model-Checking Game with Prophecies

The game $\mathcal{G}(\mathfrak{T}, \varphi)$ as introduced in Section 4.2 does not capture $\mathfrak{T} \models \varphi$. While it is sound (see Lemma 1 for the empty set of prophecies), i.e., the coalition of Verifier-players has a winning collection of strategies for $\mathcal{G}(\mathfrak{T}, \varphi)$ implies that $\mathfrak{T} \models \varphi$, the converse is not necessarily true. This is witnessed, e.g., by a transition system $\mathfrak{T}$ with $\mathrm{Tr}(\mathfrak{T}) = (2^{\{\mathtt{p}\}})^\omega$ and the sentence $\forall \pi.\ \exists \pi.\ \mathtt{p}_{\pi'} \leftrightarrow \mathbf{F}\, \mathtt{p}_{\pi}$. As explained in the introduction, the Verifier-player does not have a strategy to select, step-by-step, a trace t' for π' while given, again step-by-step, a trace t for π, as the choice of the first letter of t' depends on all positions of t. Hence, the coalition does not win $\mathcal{G}(\mathfrak{T}, \varphi)$, even though $\mathfrak{T} \models \varphi$.

In the following, we show how *prophecies*, binding commitments about future moves by Falsifier, make the game-based approach to model-checking complete. In our example, Falsifier has to make, with his first move, a commitment about whether he will ever play a p or not. This allows the Verifier-player to pick the "right" first letter of t' and thereby win the game, if Falsifier honors the commitment. If not, the rules of the game make him loose. To add prophecies to $\mathcal{G}(\mathfrak{T}, \varphi)$, we do not change the rules of the game, but instead modify the transition system the game is played on (to allow Falsifier to select truth values for the *prophecy variables*[5], which is the mechanism he uses to make the commitments) and modify the formula (to ensure that Falsifier loses if he breaks a commitment). Hence, given $\mathfrak{T}$ and φ, we construct $\mathfrak{T}^{\mathcal{P}}$ and $\varphi^{\mathcal{P},\Xi}$ such that the following two properties are satisfied:

[5]Note that prophecy variables are Boolean variables that are set by Falsifier during each move of a play and should not be confused with trace variables.

- Soundness: If the coalition of Verifier-players has a winning collection of strategies for $\mathcal{G}(\mathfrak{T}^{\mathcal{P}}, \varphi^{\mathcal{P},\Xi})$, then $\mathfrak{T} \models \varphi$.
- Completeness: If $\mathfrak{T} \models \varphi$, then the coalition of Verifier-players has a winning collection of strategies for $\mathcal{G}(\mathfrak{T}^{\mathcal{P}}, \varphi^{\mathcal{P},\Xi})$.

The challenge here is to construct the "right" prophecies that allow us to prove completeness, as soundness is independent of the prophecies chosen.

Recall that $\pi_0, \pi_2, \dots, \pi_{k-2}$ resp. $\pi_1, \pi_3, \dots, \pi_{k-1}$ are the universally resp. existentially quantified variables in our fixed formula φ. We frequently need to refer to their indices. Hence, we define $I_\forall = \{0, 2, \dots, k-2\}$ and $I_\exists = \{1, 3, \dots, k-1\}$.

We begin by defining the transition system $\mathfrak{T}^{\mathcal{P}}$ from the fixed transition system $\mathfrak{T}$, in which paths additionally determine truth values for prophecy variables.

Definition 1 (*System manipulation*)**.** *Let $\mathcal{P} = (\mathrm{P}_i)_{i \in I_\forall}$ be a collection of sets of atomic propositions such that* AP*, the P_i, and $\{\mathtt{m}_i \mid i \in I_\forall\}$ are all pairwise disjoint, where the $\mathtt{m}_i$ are propositions used to mark copies of the transition system $\mathfrak{T} = (V, E, V_I, \lambda)$.*

For $i \in I_\forall$, we define $\mathfrak{T}^{\mathrm{P}_i} = (V^{\mathrm{P}_i}, E^{\mathrm{P}_i}, V_I^{\mathrm{P}_i}, \lambda^{\mathrm{P}_i})$ over $\mathrm{AP} \uplus \mathrm{P}_i \uplus \{\mathtt{m}_i\}$ where $V^{\mathrm{P}_i} = V \times 2^{\mathrm{P}_i} \times \{i\}$, $E^{\mathrm{P}_i} = \{((s, A, i), (s', A', i)) \mid (s, s') \in E$ and $A, A' \in 2^{\mathrm{P}_i}\}$, $V_I^{\mathrm{P}_i} = V_I \times 2^{\mathrm{P}_i} \times \{i\}$ and $\lambda^{\mathrm{P}_i}(s, A, i) = \lambda(s) \cup A \cup \{\mathtt{m}_i\}$.

Furthermore, we define $\mathfrak{T}^{\mathcal{P}} = (V^{\mathcal{P}}, E^{\mathcal{P}}, V_I^{\mathcal{P}}, \lambda^{\mathcal{P}})$ as the disjoint union of the $\mathfrak{T}^{\mathrm{P}_i}$, where a vertex of $\mathfrak{T}^{\mathcal{P}}$ is in $V_I^{\mathcal{P}}$ if and only if it is in some $V_I^{\mathrm{P}_i}$. Consequently, we have $\mathrm{Tr}(\mathfrak{T}^{\mathcal{P}}) = \bigcup_{i \in I_\forall} \{t^\frown t'^\frown \{\mathtt{m}_i\}^\omega \mid t \in \mathrm{Tr}(\mathfrak{T})$ and $t' \in (2^{\mathrm{P}_i})^\omega\}$.

An effect of this definition is that all paths trough the manipulated system select truth values for the prophecy variables with each move. For the Verifier-players, these are simply ignored, as can be seen from the manipulated formula (cf. Definition 2) we introduce now.

We define the sentence $\varphi^{\mathcal{P},\Xi}$ which ensures that Falsifier loses, if he breaks his commitments made via prophecy variables: for every prophecy variable, we have an associated *prophecy*, a language $\mathfrak{P} \subseteq ((2^{\mathrm{AP}})^i)^\omega$ for some i. Note that Falsifier can only make commitments about his own moves, as the Verifier-players could otherwise falsify the commitments (made by Falsifier) about their moves: Prophecies can only refer to traces for universally quantified variables. Also, we will have prophecies for each universally quantified variable.

Definition 2 (*Property manipulation*)**.** *Let $\Xi = (\Xi_i)_{i \in I_\forall}$ be a family of sets $\Xi_i = \{\xi_{i,1}, \dots, \xi_{i,n_i}\}$ of QPTL sentences (over* AP*) such that each $\xi \in \Xi_i$ uses only trace variables π_j with even $j \leq i$. Let $\mathcal{P} = (\mathrm{P}_i)_{i \in I_\forall}$ satisfy the disjointness requirements of Definition 1 and $\mathrm{P}_i = \{\mathtt{p}_{i,1}, \dots, \mathtt{p}_{i,n_i}\}$, i.e., we have $|\Xi_i| = |\mathrm{P}_i|$. We define the HyperQPTL sentence $\varphi^{\mathcal{P},\Xi}$ as*

$$\forall \pi_0 \exists \pi_1 \cdots \forall \pi_{k-2} \exists \pi_{k-1}.\ \left[\bigwedge\nolimits_{i \in I_\forall} (\mathtt{m}_i)_{\pi_i} \wedge \mathbf{G}\left(\bigwedge\nolimits_{\ell=1}^{n_i} ((\mathtt{p}_{i,\ell})_{\pi_i} \leftrightarrow \xi_{i,\ell})\right)\right] \rightarrow \psi.$$

We denote the trace quantifier-free part of $\varphi^{\mathcal{P},\Xi}$ by $\psi^{\mathcal{P},\Xi}$.

Note that $\varphi^{\mathcal{P},\Xi}$ has the same quantifier prefix as φ and that restricting each $\xi_{i,\ell}$ to trace variables π_j with even $j \leq i$ ensures that the prophecies only refer to trace variables under the control of Falsifier. Also note that the truth values of prophecy variables on trace variables under the control of the Verifier-players are not used in the formula. Finally, Falsifier has to pick the i-th path in $\mathfrak{T}^{\mathrm{P}_i}$ (and thus select valuations for the prophecy variables in P_i), otherwise he loses immediately.

We show that our construction is sound, independently of the choice of prophecies.

Lemma 1. *Let Ξ and $\mathcal{P}$ satisfy the requirements of Definition 2. If the coalition of Verifier-players has a winning collection of strategies in $\mathcal{G}(\mathfrak{T}^{\mathcal{P}}, \varphi^{\mathcal{P},\Xi})$, then $\mathfrak{T} \models \varphi$.*

Proof. Let the coalition of Verifier-players have a winning collection of strategies in $\mathcal{G}(\mathfrak{T}^{\mathcal{P}}, \varphi^{\mathcal{P},\Xi})$. We construct Skolem functions f_π for the variables π existentially quantified in φ witnessing $\mathfrak{T} \models \varphi$.

Let $t_0, t_2, \dots t_{k-2}$ be traces of $\mathfrak{T}$ for the universally quantified variables. For each such t_i, we fix a path ρ_i through $\mathfrak{T}^{\mathcal{P}}$ such that the (point-wise) AP-projection of the trace of ρ_i is t_i, such that $\mathtt{m}_i$ holds on every position of ρ_i, and such that the prophecy variables $\mathtt{p}_{i,\ell}$ are picked correctly, i.e., such that $\mathtt{p}_{i,\ell}$ is satisfied at position n if and only if the suffixes of $t_0, t_2, \dots, t_i$ starting at position n satisfy $\xi_{i,\ell}$.

Consider the play of $\mathcal{G}(\mathfrak{T}^{\mathcal{P}}, \varphi^{\mathcal{P},\Xi})$ where Falsifier plays such that he constructs the ρ_i and the Verifier-players use their winning collection of strategies to construct paths $\rho_1, \rho_3, \dots, \rho_{k-1}$.

Recall that $\lambda^{\mathcal{P}}$ is the labelling function of $\mathfrak{T}^{\mathcal{P}}$. As the collection of strategies is winning,

$$\mathrm{mrg}(\lambda^{\mathcal{P}}(\rho_0), \lambda^{\mathcal{P}}(\rho_1), \dots, \lambda^{\mathcal{P}}(\rho_{k-1}))$$

is accepted by the automaton constructed from $\varphi^{\mathcal{P},\Xi}$, i.e., the trace assignment Π mapping each π_i to $\lambda^{\mathcal{P}}(\rho_i)$ satisfies

$$\left[\bigwedge\nolimits_{i \in I_\forall} (\mathtt{m}_i)_{\pi_i} \wedge \mathbf{G}\left(\bigwedge\nolimits_{\ell=1}^{n_i}((\mathtt{p}_{i,\ell})_{\pi_i} \leftrightarrow \xi_{i,\ell})\right)\right] \rightarrow \psi.$$

By construction of the ρ_i for $i \in I_\forall$, Π satisfies the premise

$$\bigwedge\nolimits_{i \in I_\forall} (\mathtt{m}_i)_{\pi_i} \wedge \mathbf{G}\left(\bigwedge\nolimits_{\ell=1}^{n_i}((\mathtt{p}_{\ell,i})_{\pi_i} \leftrightarrow \xi_{\ell,i})\right).$$

Hence, Π must satisfy ψ.

Finally, by the definition of the hierarchical information in the game $\mathcal{G}(\mathfrak{T}^{\mathcal{P}}, \varphi^{\mathcal{P},\Xi})$, ρ_i for even i only depends on the paths $\rho_0, \rho_1, \dots, \rho_{i-1}$, but not on the paths $\rho_{i+1}, \rho_{i+2}, \dots, \rho_{k-1}$ (as they are hidden to the i-th Verifier-player). Thus, we can inductively, for $i \in I_\exists$, define $f_{\pi_i}(t_0, t_2, \dots, t_{i-1})$ as the AP-projection of $\lambda^{\mathcal{P}}(\rho_i)$.

Then, the functions f_π just defined are indeed Skolem functions witnessing $\mathfrak{T} \models \varphi$. ◻

5 Complete Prophecies for Safety Properties

We show that there are sets Ξ and $\mathcal{P}$ such that if $\mathfrak{T} \models \varphi$, then the coalition of Verifier-players wins $\mathcal{G}(\mathfrak{T}^{\mathcal{P}}, \varphi^{\mathcal{P},\Xi})$. We first consider the case where ψ is a safety property, i.e., the automaton $\mathcal{A}^{\mathfrak{T}}_{k-1}$, which checks ψ, is a deterministic safety automaton. A *safety automaton* is a parity automaton using only two colors, one even and one odd one, and moreover, all its states are even-colored except for an odd-colored sink state. In other words, all runs avoiding the single unsafe state are accepting. We start with safety as this is allows us to work with "simpler" prophecies while presenting all underlying concepts needed for arbitrary quantifier prefixes. The idea for safety is that a prophecy should indicate which successor vertices are safe for the Verifier-players to move to, i.e., from which successor vertices it is possible to successfully continue the play without immediately losing. In the general case, we have to additionally handle a more complex acceptance condition. This is shown in Section 6.

Definition 3 (Prophecy construction for safety)**.** *For each $i \in I_\forall$, each vector $\bar{v} = (v_0, v_1, \dots, v_{i+1})$ of vertices of $\mathfrak{T}$, and each state q of $\mathcal{A}^{\mathfrak{T}}_{i+1}$ we define*

$$(\mathfrak{S}_i)^{\bar{v}}_q = \{\mathrm{mrg}(t_0, t_2, \dots, t_i) \mid t_0 \in \mathrm{Tr}(\mathfrak{T}_{v_0}), t_2 \in \mathrm{Tr}(\mathfrak{T}_{v_2}), \dots, t_i \in \mathrm{Tr}(\mathfrak{T}_{v_i}), \text{ and}$$
$$\text{there are } t_1 \in \mathrm{Tr}(\mathfrak{T}_{v_1}), t_3 \in \mathrm{Tr}(\mathfrak{T}_{v_3}), \dots, t_{i+1} \in \mathrm{Tr}(\mathfrak{T}_{v_{i+1}}) \text{ s.t. } \mathrm{mrg}(t_0, \dots, t_{i+1}) \in L((\mathcal{A}^{\mathfrak{T}}_{i+1})_q)\}.$$

Note that the t_i for $i \in I_\exists$ may depend on all t_j with $j \in I_\forall$. Our game definition ensures that the choice for an existential variable π_i only depends on the choices for π_j with $j < i$.

Also note that each prophecy is an ω-regular language, as ω-regular languages are closed under projection, the analogue of existential quantification. So, due to Proposition 1, there are QPTL-sentences expressing the prophecies allowing us to verify them in the formula $\varphi^{\mathcal{P},\Xi}$.

Definition 4. *We define $\Xi = (\Xi_i)_{i \in I_\forall}$ and $\mathcal{P} = (\mathrm{P}_i)_{i \in I_\forall}$. Let Ξ_i be the set of QPTL formulas that contains sentences $(\xi_i)^{\bar{v}}_q$ for each state q of $\mathcal{A}^{\mathfrak{T}}_{i+1}$ and each vector $\bar{v}$ of vertices expressing the prophecy $(\mathfrak{S}_i)^{\bar{v}}_q$ using only trace variables π_j with even $j \le i$. Let P_i be the set that contains the prophecy variable $(\mathfrak{p}_i)^{\bar{v}}_q$ for each state q of $\mathcal{A}^{\mathfrak{T}}_{i+1}$ and each vector $\bar{v}$ of vertices. The prophecy variable $(\mathfrak{p}_i)^{\bar{v}}_q$ corresponds to $(\xi_i)^{\bar{v}}_q$.*

Our main technical lemma shows that the prophecies defined above are indeed sound.

Lemma 2. *Let $\mathfrak{T} \models \varphi$ and ψ expresses a safety property. Then the coalition of Verifier-players has a winning collection of strategies in $\mathcal{G}(\mathfrak{T}^{\mathcal{P}}, \varphi^{\mathcal{P},\Xi})$ with Ξ and $\mathcal{P}$ as in Definition 4.*

Before we turn to the proof of Lemma 2, we introduce some notation about the game $\mathcal{G}(\mathfrak{T}^{\mathcal{P}}, \varphi^{\mathcal{P},\Xi})$ and define the strategies we will prove winning.

Firstly, recall that $\mathfrak{T}^{\mathcal{P}}$ is the union $\biguplus_{P_i \in \mathcal{P}} \mathfrak{T}^{P_i}$ where $\mathcal{P} = (\mathrm{P}_i)_{i \in I_\forall}$. A vertex v of $\mathfrak{T}^{\mathcal{P}}$ is of the form $(s, A, i) \in V \times 2^{\mathrm{P}_i} \times \{i\}$ for some P_i where V is the set of vertices of $\mathfrak{T}$ and its label $\lambda^{\mathrm{P}_i}(v)$ is $\lambda(s) \cup A \cup \{\mathtt{m}_i\}$, i.e., the union of its original (meaning AP-based) label in $\mathfrak{T}$, its associated prophecy variables, and its marker $\mathtt{m}_i$ indicating that v belongs to $\mathfrak{T}^{\mathrm{P}_i}$. We need to access these bits of information. Hence, we define $\lambda_{\mathrm{AP}}(v) = \lambda(s)$, $\mathrm{Prophecies}(v) = A$, and $\mathrm{mark}(v) = \mathtt{m}_i$. Given a path $\rho = v_0 v_1 \cdots$ through $\mathfrak{T}^{\mathcal{P}}$, let $\lambda_{\mathrm{AP}}(\rho) = \lambda_{\mathrm{AP}}(v_0)\lambda_{\mathrm{AP}}(v_1)\cdots$. Moreover, for $v = (s, A, i)$ in $\mathfrak{T}^{\mathrm{P}_i}$, we are often interested in reasoning about s in $\mathfrak{T}$. To simplify our notation, we will also write v for the (unique) vertex s in $\mathfrak{T}$ induced by v.

Secondly, recall that $\psi^{\mathcal{P},\Xi}$ is the trace quantifier-free part of $\varphi^{\mathcal{P},\Xi}$. Let $\mathcal{B}$ denote the parity automaton $(Q_{\mathcal{B}}, \Sigma, q_I^{\mathcal{B}}, \delta_{\mathcal{B}}, \Omega_{\mathcal{B}})$ that accepts the trace assignments that satisfy $\psi^{\mathcal{P},\Xi}$.

Lastly, recall that a position of $\mathcal{G}(\mathfrak{T}^{\mathcal{P}}, \varphi^{\mathcal{P},\Xi})$ is of the form $((v_0', \ldots, v_{i-1}', v_i, \ldots, v_{k-1}), q, i)$ where $v_0', \ldots, v_{i-1}', v_i, \ldots, v_{k-1}$ are vertices in the transition system $\mathfrak{T}^{\mathcal{P}}$, $q \in Q_{\mathcal{B}}$ is a state of the parity automaton $\mathcal{B}$ checking $\psi^{\mathcal{P},\Xi}$, and $i \in \{0, 1, \ldots, k-1\}$. Technically, in the initialization phase, a position is of the form $((v_0, \ldots, v_{i-1}, \underbrace{\bullet, \ldots, \bullet}_{k-i \text{ times}}), q_I, i)$ where $v_0, \ldots, v_{i-1}$ are vertices and $\bullet$ is a placeholder. For simplicity, we always refer to a position with $((v_0', \ldots, v_{i-1}', v_i, \ldots, v_{k-1}), q, i)$, even though $\bullet$ might be present. Also, for convenience, we define the successors $\mathrm{S}(\bullet)$ of the placeholder symbol $\bullet$ to be V_I, the set of initial vertices of $\mathfrak{T}$.

We say a play in $\mathcal{G}(\mathfrak{T}^{\mathcal{P}}, \varphi^{\mathcal{P},\Xi})$ is in *Round* (r, i) for $r \geq 0$ and $i \in \{0, 1, \ldots, k-1\}$ if the play is in a position of the form $((v_0', \ldots, v_{i-1}', v_i, \ldots, v_{k-1}), q, i)$ for the $(r+1)$-th time. Also, we say we are in *Round* r if the play is in Round (r, i) for some i.

Given a play α in $\mathcal{G}(\mathfrak{T}^{\mathcal{P}}, \varphi^{\mathcal{P},\Xi})$, for $i \in \{0, 1, \ldots, k-1\}$, let ρ_i^α denote the path through $\mathfrak{T}^{\mathcal{P}}$ that is induced by α when considering the moves made in Round (r, i) for $r \geq 0$: Formally, if in Round (r, i) the move from position $p = ((v_0', \ldots, v_{i-1}', v_i, \ldots, v_{k-1}), q, i)$ to $p' = ((v_0', \ldots, v_{i-1}', v_i', \ldots, v_{k-1}), q', (i+1) \bmod k)$ is made, then $\rho_i^\alpha(r) = v_i'$. Note that $q' = q$, unless $i = k-1$, then $q' = \delta_{\mathcal{B}}(q, \mathrm{mrg}(\lambda^{\mathcal{P}}(v_0'), \ldots, \lambda^{\mathcal{P}}(v_{k-1}')))$. Furthermore, note that the move (p, p') is uniquely identified by p and v_i'. So in the future, we simply write the move from v_i to v_i' when p is clear from the context. We drop the index α from ρ_i^α and write ρ_i when α is clear from the context.

We continue by defining a collection of strategies for the Verifier-players and then show that $\mathfrak{T} \models \varphi$ implies that this collection is winning in $\mathcal{G}(\mathfrak{T}^{\mathcal{P}}, \varphi^{\mathcal{P},\Xi})$. Recall that $\varphi^{\mathcal{P},\Xi}$ is $\forall \pi_0 \exists \pi_1 \cdots \forall \pi_{k-2} \exists \pi_{k-1}.\ \psi^{\mathcal{P},\Xi}$. Assume the play is in Round (r, i) for $r \geq 0$ and $i \in I_\exists$. Thus, it is in a position of the form

$$((v_0', \ldots, v_{i-1}', v_i, \ldots, v_{k-1}), q, i)$$

and Verifier i has to move. We review some information Verifier i can use to base her choice on. In Round r, Verifier i has access to the first r letters of the traces $\lambda_{\mathrm{AP}}(\rho_0), \lambda_{\mathrm{AP}}(\rho_1), \ldots, \lambda_{\mathrm{AP}}(\rho_i)$ induced by the play.[6] Let q_i be the state that $\mathcal{A}_i^{\mathfrak{T}}$ has reached on $\mathrm{mrg}(\lambda_{\mathrm{AP}}(\rho_0)[0, r), \lambda_{\mathrm{AP}}(\rho_1)[0, r), \ldots, \lambda_{\mathrm{AP}}(\rho_i)[0, r))$ for $i \in I_\exists$.

Definition 5 (Strategy definition for safety)**.** *Let*

$$M_i(v_0', \ldots, v_{i-1}', v_i, q_i) = \{v_i' \in \mathrm{S}(v_i) \mid (\mathtt{p}_{i-1})_{q_i}^{\bar{v}} \in \mathrm{Prophecies}(v_{i-1}')\}$$

*where $\bar{v} = (v_0', \ldots, v_i')$. If $M_i(v_0', \ldots, v_{i-1}', v_i, q_i)$ is nonempty, then Verifier i can move to any v_i' in the set (the "*Nonempty-case*"). If it is empty, then Verifier i can move to any $v_i' \in \mathrm{S}(v_i)$ (the "*Empty-case*").*

Proof of Lemma 2. Let us prove that the strategies defined above form a winning collection. To this end, let α be a play in $\mathcal{G}(\mathfrak{T}^{\mathcal{P}}, \varphi^{\mathcal{P},\Xi})$ that is consistent with the strategies and satisfies the following assumption.

Assumption 1. *Let $\rho_0^\alpha, \ldots, \rho_{k-1}^\alpha$ be the paths induced by α and let Π be the trace assignment mapping π_i to $\lambda^{\mathcal{P}}(\rho_i)$ for $i \in I_\forall$. We assume that Π satisfies the premise of $\psi^{\mathcal{P},\Xi}$.*

[6] For $j < i$, she actually has access to $r+1$ letters, but our construction is independent of the last ones.

Intuitively, this assumption is satisfied by Falsifier, if he picks paths that start in the intended sub-parts of the manipulated transition system, and furthermore, he always truthfully indicates which prophecies hold. Plays in which the assumption is violated are trivially won by the Verifier-players. Thus, in the following we can focus on α's satisfying the assumption.

We show by induction over Round (r, i) for $r \geq 0$ and $i \in I_\exists$ that under this assumption, Verifier i used the NONEMPTY-case in this round to pick her move. To proceed by nested induction, i.e., with an outer induction going from Round $(r, 0)$ to Round $(r+1, 0)$ for each $r \geq 0$, and an inner induction going from Round (r, i) to Round $(r, i+2)$ for fixed $r \geq 0$ and $i \in I_\exists \setminus \{k-1\}$.

We begin with the outer induction. At the beginning of Round $(r, 0)$, the play α is in a position of the form $((v_0, \ldots, v_{k-1}), q, 0)$. Let q_{k-1} be the state that $\mathcal{A}^{\mathfrak{T}}_{k-1}$ has reached on

$$\mathrm{mrg}(\lambda_{\mathrm{AP}}(\rho_0)[0, r), \lambda_{\mathrm{AP}}(\rho_1)[0, r), \ldots, \lambda_{\mathrm{AP}}(\rho_{k-1})[0, r)).$$

We prove by induction on r the following auxiliary statement:

$$\forall t_0 \in \mathrm{Tr}(\mathfrak{T}_{\mathrm{S}(v_0)})\ \exists t_1 \in \mathrm{Tr}(\mathfrak{T}_{\mathrm{S}(v_1)}) \cdots \forall t_{k-2} \in \mathrm{Tr}(\mathfrak{T}_{\mathrm{S}(v_{k-2})})\ \exists t_{k-1} \in \mathrm{Tr}(\mathfrak{T}_{\mathrm{S}(v_{k-1})}) : \\ \mathrm{mrg}(t_0, \ldots, t_{k-1}) \in L((\mathcal{A}^{\mathfrak{T}}_{k-1})_{q_{k-1}}). \tag{1}$$

Outer Base Case, Round $(0, 0)$. The starting position of the game, which is the position in Round $(0, 0)$, is $((\bullet, \ldots, \bullet), q^{\mathcal{B}}_I, 0)$. As $\mathrm{S}(\bullet) = V_I$, $\mathfrak{T}_{V_I} = \mathfrak{T}$, and $\mathcal{A}^{\mathfrak{T}}_{k-1}$ has reached q_I on the empty prefix, Eq. (1) is equivalent to $\mathfrak{T} \models \varphi$, which is true by assumption of the statement of Lemma 2.

Outer Inductive Step, Round $(r, 0)$ to Round $(r+1, 0)$. To show Eq. (1) for Round $(r+1, 0)$, we inductively go over the rounds Round (r, i) for each Verifier i.

In Round (r, i), as mentioned in the strategy definition (cf. Definition 5), Verifier i has access to the first r letters of the traces $\lambda_{\mathrm{AP}}(\rho_0), \lambda_{\mathrm{AP}}(\rho_1), \ldots, \lambda_{\mathrm{AP}}(\rho_i)$ induced by the play prefix. Let q_i be the state that $\mathcal{A}^{\mathfrak{T}}_i$ has reached on

$$\mathrm{mrg}(\lambda_{\mathrm{AP}}(\rho_0)[0, r), \lambda_{\mathrm{AP}}(\rho_1)[0, r), \ldots, \lambda_{\mathrm{AP}}(\rho_j)[0, r))$$

for $i \in I_\exists$. In Round (r, i), the play is in a position $((v'_0, \ldots, v'_{i-1}, v_i, \ldots, v_{k-1}), q, i)$. We prove by induction on $i \in I_\exists$ another auxiliary statement and our original claim: there exists some $v'_i \in \mathrm{S}(v_i)$ such that

$$\forall t_0 \in \mathrm{Tr}(\mathfrak{T}_{v'_0})\ \exists t_1 \in \mathrm{Tr}(\mathfrak{T}_{v'_1}) \cdots \forall t_{i-1} \in \mathrm{Tr}(\mathfrak{T}_{v'_{i-1}})\ \exists t_i \in \mathrm{Tr}(\mathfrak{T}_{v'_i}) \\ \forall t_{i+1} \in \mathrm{Tr}(\mathfrak{T}_{\mathrm{S}(v_{i+1})})\ \exists t_{i+2} \in \mathrm{Tr}(\mathfrak{T}_{\mathrm{S}(v_{i+2})}) \cdots \forall t_{k-2} \in \mathrm{Tr}(\mathfrak{T}_{\mathrm{S}(v_{k-2})})\ \exists t_{k-1} \in \mathrm{Tr}(\mathfrak{T}_{\mathrm{S}(v_{k-1})}) : \\ \mathrm{mrg}(t_0, \ldots, t_{k-1}) \in L((\mathcal{A}^{\mathfrak{T}}_{k-1})_{q_{k-1}}), \tag{2}$$

and

$$\mathrm{Prophecies}(v'_{i-1}) \text{ contains } (\mathtt{p}_{i-1})^{\bar{v}}_{q_{i+1}} \text{ with } \bar{v} = (v'_0, \ldots, v'_i). \tag{3}$$

Note that Eq. (3) for Round (r, i) witnesses that Verifier i used the NONEMPTY-case to pick her move in Round (r, i), which was our original claim we prove by induction.

Inner Base Case, Round $(r, 1)$. In Round $(r, 0)$, Eq. (1) is true. We now restrict the considered traces for the first two quantifiers. Clearly, the statement remains true when restricting the first quantifier which is universal to traces from paths starting with $v'_0 \in \mathrm{S}(v_0)$. Furthermore, there exists some $v'_1 \in \mathrm{S}(v_1)$ such that the second quantifier which is existential can be restricted to all traces from paths starting with v'_1 and the statement remains true. This is the statement of Eq. (2) for Round $(r, 1)$. Recall that

$$(\mathfrak{S}_0)^{(v'_0, v'_1)}_{(q_1)} = \{t_0 \mid t_0 \in \mathrm{Tr}(\mathfrak{T}_{v'_0}) \text{ and there is } t_1 \in \mathrm{Tr}(\mathfrak{T}_{v'_1}) \text{ s.t. } \mathrm{mrg}(t_0, t_1) \in L((\mathcal{A}^{\mathfrak{T}}_1)_{q_1})\}.$$

Eq. (2) for Round $(r, 1)$ yields that

$$(\mathfrak{S}_0)^{(v'_0, v'_1)}_{(q_2)} = \mathrm{Tr}(\mathfrak{T}_{v'_0}).$$

Since Falsifier has moved to v'_0, he is committed to play one of those paths, i.e.,

$$\lambda^{\mathcal{P}}(\rho_0)[r,\infty) \in (\mathfrak{S}_{i-1})^{(v'_0,v'_1)}_{(q_1)}.$$

Assumption 1 implies that Falsifier has truthfully indicated this, i.e.,

$$(\mathrm{p}_0)^{(v'_0,v'_1)}_{(q_1)} \in \mathrm{Prophecies}(v'_0).$$

This is the statement of Eq. (3). Hence, Verifier 1 can move to v'_1 in Round $(r,1)$, i.e., we are in the NONEMPTY-case. This completes the inner base case.

Inner Inductive step, Round (r,i) to Round $(r,i+2)$. We use similar arguments as for the inner base case. By induction hypothesis, Eq. (2) for Round (r,i) is true. We now restrict the considered traces for the $(i+1)$-th and $(i+2)$-th quantifier. The $(i+1)$-th quantifier is universal, so clearly the statement remains true if we restrict the traces for this quantifier to traces coming from paths starting with $v'_{i+1} \in \mathrm{S}(v_{i+1})$. The $i+2$-th quantifier is existential, hence, there exists some $v'_{i+2} \in \mathrm{S}(v_{i+2})$ such that this quantifier can be restricted to traces from paths starting with v'_{i+2} and the statement remains true. This is exactly the statement of Eq. (2) for Round $(r,i+2)$. Recall that the prophecy $(\mathfrak{S}_{i+1})^{(v'_0,\ldots,v'_{i+2})}_{q_{i+2}}$ is defined as

$$\begin{aligned}\{\mathrm{mrg}(t_0,t_2,\ldots,t_{i+1}) \mid t_0 \in \mathrm{Tr}(\mathfrak{T}_{v'_0}), t_2 \in \mathrm{Tr}(\mathfrak{T}_{v'_2}),\ldots,t_{i+1} \in \mathrm{Tr}(\mathfrak{T}_{v'_{i+1}})\\ \text{and there exist } t_1 \in \mathrm{Tr}(\mathfrak{T}_{v'_1}), t_3 \in \mathrm{Tr}(\mathfrak{T}_{v'_3}),\ldots,t_{i+2} \in \mathrm{Tr}(\mathfrak{T}_{v'_{i+2}})\\ \text{such that } \mathrm{mrg}(t_0,\ldots,t_{i+2}) \in L((\mathcal{A}^{\mathfrak{T}}_{i+2})_{q_{i+2}})\}.\end{aligned}$$

We claim (shown below) that Eq. (2) for Round $(r,i+2)$ implies that

$$(\mathfrak{S}_{i+1})^{(v'_0,\ldots,v'_{i+2})}_{q_{i+2}} = \mathrm{Tr}(\mathfrak{T}_{v'_0}) \times \mathrm{Tr}(\mathfrak{T}_{v'_2}) \times \cdots \times \mathrm{Tr}(\mathfrak{T}_{v'_{i+1}}).$$

Consequently,

$$\mathrm{mrg}(\lambda^{\mathcal{P}}(\rho_0)[r,\infty), \lambda^{\mathcal{P}}(\rho_2)[r,\infty),\ldots,\lambda^{\mathcal{P}}(\rho_{i+1})[r,\infty)) \in (\mathfrak{S}_{i+1})^{(v'_0,\ldots,v'_{i+2})}_{q_{i+2}}.$$

In words, Falsifier has committed to play such that the remainders of his induced traces satisfy the above prophecy (which includes information about possible plays for Verifier-players). Due to Assumption 1, Falsifier truthfully indicates this, i.e.,

$$(\mathrm{p}_{i+1})^{(v'_0,\ldots,v'_{i+2})}_{(q_1,q_3,\ldots,q_{i+2})} \in \mathrm{Prophecies}(v'_{i-1}).$$

This is the statement of Eq. (3). Hence, Verifier $i+2$ can move to v'_{i+2} in Round $(r,i+2)$ which meets the requirements imposed by the strategy construction.

It is left to prove that

$$(\mathfrak{S}_{i+1})^{(v'_0,\ldots,v'_{i+2})}_{q_{i+2}} = \mathrm{Tr}(\mathfrak{T}_{v'_0}) \times \mathrm{Tr}(\mathfrak{T}_{v'_2}) \times \cdots \times \mathrm{Tr}(\mathfrak{T}_{v'_{i+1}}).$$

Towards a contradiction, assume there are $t_0 \in \mathrm{Tr}(\mathfrak{T}_{v'_0}), t_2 \in \mathrm{Tr}(\mathfrak{T}_{v'_2}),\ldots,t_{i+1} \in \mathrm{Tr}(\mathfrak{T}_{v'_{i+1}})$ such that

$$\mathrm{mrg}(t_0,t_2,\ldots,t_{i+1}) \notin (\mathfrak{S}_{i+1})^{(v'_0,\ldots,v'_{i+2})}_{q_{i+2}}.$$

Eq. (2) for Round $(r,i+2)$, by successively restricting the quantifiers, implies that there are $t_1 \in \mathrm{Tr}(\mathfrak{T}_{v'_1}), t_3 \in \mathrm{Tr}(\mathfrak{T}_{v'_1}),\ldots,t_{i+2} \in \mathrm{Tr}(\mathfrak{T}_{v'_{i+2}})$ such that $\mathrm{mrg}(t_0,\ldots,t_{i+2}) \in L((\mathcal{A}^{\mathfrak{T}}_{i+2})_{q_{i+2}})$. This implies that

$$\mathrm{mrg}(t_0,t_2,\ldots,t_{i+1}) \in (\mathfrak{S}_{i+1})^{(v'_0,\ldots,v'_{i+2})}_{q_{i+2}}$$

which contradicts the assumption.

Now that we have completed the inner induction, we return to the proof of Eq. (1) for Round $r+1$. We proved that Eq. (2) is true in Round $(r, k-1)$, meaning that

$$\forall t_0 \in \mathrm{Tr}(\mathfrak{T}_{v'_0})\ \exists t_1 \in \mathrm{Tr}(\mathfrak{T}_{v'_1}) \cdots \forall t_{k-2} \in \mathrm{Tr}(\mathfrak{T}_{v'_{k-2}})\ \exists t_{k-1} \in \mathrm{Tr}(\mathfrak{T}_{v'_{k-1}}) : \\ \mathrm{mrg}(t_0, \ldots, t_{k-1}) \in L((\mathcal{A}^{\mathfrak{T}}_{k-1})_{q_{k-1}})$$

Let $q'_{k-1} = \delta(q_{k-1}, (\lambda_{\mathrm{AP}}(v'_0), \ldots, \lambda_{\mathrm{AP}}(v'_{k-1})))$. We reformulate the previous equation as

$$\forall t_0 \in \mathrm{Tr}(\mathfrak{T}_{v'_0})\ \exists t_1 \in \mathrm{Tr}(\mathfrak{T}_{v'_1}) \cdots \forall t_{k-2} \in \mathrm{Tr}(\mathfrak{T}_{v'_{k-2}})\ \exists t_{k-1} \in \mathrm{Tr}(\mathfrak{T}_{v'_{k-1}}) : \\ \mathrm{mrg}(t_0[1,\infty), \ldots, t_{k-1}[1,\infty)) \in L((\mathcal{A}^{\mathfrak{T}}_{k-1})_{q'_{k-1}}) \tag{4}$$

In Round $(r, k-1)$, the play is in a position $((v'_0, \ldots, v'_{k-2}, v_{k-1}), q, k-1)$. As described, Verifier $k-1$ picks a suitable vertex $v'_{k-1} \in \mathrm{S}(v_{k-1})$ and the play moves to position $((v'_0, \ldots, v'_{k-1}), q', 0)$ where $q' = \delta_{\mathcal{B}}(q, (\lambda^{\mathcal{P}}(v'_0), \ldots, \lambda^{\mathcal{P}}(v'_{k-1})))$ and is now in Round $(r+1, 0)$. So Eq. (4) clearly implies that Eq. (1) is true in Round $r+1$. We completed the proof of the outer induction, i.e., in every play that is consistent with the strategies constructed above and that satisfies Assumption 1, the Verifier-players always use the Nonempty-case.

Next, we show that the strategies are consistent with the available observations. We note that the strategy we constructed for Verifier i is solely based on the information visible to her, i.e., it is based only on the paths $\rho_0, \ldots, \rho_i$ and derived information such as information about $\mathcal{A}^{\mathfrak{T}}_i$ (processing traces $\lambda_{\mathrm{AP}}(\rho_0), \ldots, \lambda_{\mathrm{AP}}(\rho_i)$) and related prophecy information.

It is left to argue that the resulting collection of strategies for the Verifier-players is winning. We review the winning condition: Recall that the parity automaton $\mathcal{B}$ accepts trace assignments (stemming from $\mathfrak{T}^{\mathcal{P}}$) satisfying $\psi^{\mathcal{P},\Xi}$ which is the trace quantifier-free part of $\varphi^{\mathcal{P},\Xi}$. The Verifier-players win if $\mathcal{B}$ accepts $\mathrm{mrg}(\lambda^{\mathcal{P}}(\rho_0), \ldots, \lambda^{\mathcal{P}}(\rho_{k-1}))$.

As mentioned at the beginning of the proof, Assumption 1 holds. Recall that $\pi_0, \ldots, \pi_{k-1}$ are the quantified trace variables of $\varphi^{\mathcal{P},\Xi}$. Under Assumption 1, considering the trace assignment which maps each π_j to $\lambda^{\mathcal{P}}(\rho_j)$, the premise of $\psi^{\mathcal{P},\Xi}$ is true. Note that traces induced by Verifier-players do not influence the truth-value of the premise, because the prophecies only make predictions about traces induced by Falsifier. Hence, we have to show that the conclusion is true, i.e., that the above trace assignment satisfies ψ. Recall that $\mathcal{A}^{\mathfrak{T}}_{k-1}$ checks this. We have to show that $\mathcal{A}^{\mathfrak{T}}_{k-1}$ accepts $\mathrm{mrg}(\lambda_{\mathrm{AP}}(\rho_0), \ldots, \lambda_{\mathrm{AP}}(\rho_{k-1}))$. Playing accordingly to the strategy we constructed, in each Round $(r, k-1)$, the play is in a position $((v'_0, \ldots, v'_{k-2}, v_{k-1}), q, k-1)$ and Verifier $k-1$ has picked some $v'_{k-1} \in \mathrm{S}(v_{k-1})$ such that

$$(\mathtt{p}_{k-2})^{(v'_0, \ldots, v'_{k-1})}_{q_{k-1}} \in \mathrm{Prophecies}(v'_{k-2}),$$

where q_{k-1} is the state of $\mathcal{A}^{\mathfrak{T}}_{k-1}$ reached on

$$\mathrm{mrg}(\lambda_{\mathrm{AP}}(\rho_0)[0,r), \lambda_{\mathrm{AP}}(\rho_1)[0,r), \ldots, \lambda_{\mathrm{AP}}(\rho_{k-1})[0,r)).$$

If $L((\mathcal{A}^{\mathfrak{T}}_{k-1})_{q_{k-1}}) = \emptyset$, it would not have been possible to indicate that $(\mathfrak{S}_{k-2})^{(v'_0, \ldots, v'_{k-1})}_{q_k}$ holds. Consequently, the state q_{k-1} must be safe, because $\mathcal{A}^{\mathfrak{T}}_{k-1}$ is a safety automaton since ψ is a safety condition (safe simply means able to still accept something). Thus, the run of $\mathcal{A}^{\mathfrak{T}}_{k-1}$ on $\mathrm{mrg}(\lambda_{\mathrm{AP}}(\rho_0), \ldots, \lambda_{\mathrm{AP}}(\rho_{k-1}))$ is accepting which implies that the run of $\mathcal{B}$ on $\mathrm{mrg}(\lambda^{\mathcal{P}}(\rho_0), \ldots, \lambda^{\mathcal{P}}(\rho_{k-1}))$ is accepting. □

Combining Lemma 1 and Lemma 2, we obtain our main result of this section.

Theorem 2. *Let Ξ, $\mathcal{P}$ be as in Definition 4, let ψ be a safety property. The coalition of Verifier-players has a winning collection of strategies for $\mathcal{G}(\mathfrak{T}^{\mathcal{P}}, \varphi^{\mathcal{P},\Xi})$ if and only if $\mathfrak{T} \models \varphi$.*

6 Complete Prophecies for ω-Regular Properties

In this section, we show how to construct complete prophecies for quantifier-free ψ beyond the safety fragment. Let us recall the safety case. The prophecies used indicate moves which prevent the Verifier-players from losing. In the case of safety, not losing equals winning. However for general ω-regular objectives, this is not enough. It is also necessary to (again and again) make progress towards satisfying the acceptance condition.

In this section, we assume that the automaton $\mathcal{A}_{k-1}^{\mathfrak{T}}$, which checks ψ, is a *Rabin automaton*. The class of languages recognized by Rabin automata is the class of ω-regular languages. A (deterministic) Rabin automaton is a deterministic ω-automaton whose acceptance condition is given as a set of pairs $(B_1, F_1), \ldots, (B_m, F_m)$. A run is accepting if for some i the run visits states in B_i only finitely many times and states in F_i infinitely many times.

6.1 One-Pair Rabin Automata

First, we present our construction for one-pair Rabin automata, then show how to generalize to Rabin automata in Section 6.2. Here, we assume that for each $i \in I_\exists$, the automaton $\mathcal{A}_i^{\mathfrak{T}}$ is a one-pair Rabin automaton. We always denote the single pair as (B, F), it will be clear from the context to which automaton it belongs. Intuitively, making progress towards winning in a one-pair Rabin automaton means avoiding states in B and visiting states in F.

Let κ be a run of such an automaton. We define $\mathrm{last}_B(\kappa)$ as $\bot$ if κ never visits B, as i if $\kappa(i)$ is the last visit of κ in B, and as ∞ if κ visits B infinity many times (meaning there is no last visit). Furthermore, we define $\mathrm{first}_F(\kappa)$ as i if $\kappa(i)$ is the first visit of κ in F and as ∞ if κ never visits F. Finally, we define the value $\mathrm{val}(\kappa)$ of a run κ as $\mathrm{first}_F(\kappa)$ if $\mathrm{last}_B(\kappa) = \bot$ and as $\max\{\mathrm{last}_B(\kappa), \mathrm{first}_F(\kappa)\}$ otherwise.

Given two runs κ, κ', we say that κ "makes progress faster" than κ' if $\mathrm{val}(\kappa) < \mathrm{val}(\kappa')$.

Before we can define our new prophecies, we need additional notation that gives us the value of a best run (over several traces) if we fix all but the last trace. For each $i \in I_\exists$, each vertex v of $\mathfrak{T}$, each state q of $\mathcal{A}_i^{\mathfrak{T}}$, and traces $t_0, t_1, \ldots, t_{i-1} \in \mathrm{Tr}(\mathfrak{T})$, we define

$$\mathrm{opt}_i(q, v, t_0, \ldots, t_{i-1}) = \min_{t_i \in \mathrm{Tr}(\mathfrak{T}_{\mathrm{S}(v)})} \mathrm{val}(\kappa_i),$$

where κ_i is the run of $(\mathcal{A}_i^{\mathfrak{T}})_q$ on $\mathrm{mrg}(t_0, \ldots, t_{i-1}, t_i)$.

We are ready to introduce our prophecies.

Definition 6 (Prophecy construction)**.** *For each $i \in I_\forall$, vectors $\bar{u} = (v_0, v_1, \ldots, v_{i+1})$ and $\bar{v} = (v'_0, v'_1, \ldots, v'_{i+1})$ of vertices such that $v'_j \in \mathrm{S}(v_j)$ for $j \leq i+1$, and each vector $\bar{q} = (q_1, q_3, \ldots, q_{i+1})$ of states q_j of $\mathcal{A}_j^{\mathfrak{T}}$, we define*

$$\begin{aligned}(\mathfrak{P}_i)_{\bar{q}}^{\bar{u},\bar{v}} = \{\mathrm{mrg}(t_0, t_2, \ldots, t_i) \mid t_0 \in \mathrm{Tr}(\mathfrak{T}_{v'_0}), t_2 \in \mathrm{Tr}(\mathfrak{T}_{v'_2}), \ldots, t_i \in \mathrm{Tr}(\mathfrak{T}_{v'_i}),\ \textit{and}\\ \textit{there exist } t_1 \in \mathrm{Tr}(\mathfrak{T}_{v'_1}), t_3 \in \mathrm{Tr}(\mathfrak{T}_{v'_3}), \ldots, t_{i+1} \in \mathrm{Tr}(\mathfrak{T}_{v'_{i+1}}) \textit{ such that}\\ \textit{the run } \kappa_j \textit{ of } (\mathcal{A}_j^{\mathfrak{T}})_{q_j} \textit{ on } \mathrm{mrg}(t_0, \ldots, t_j) \textit{ is accepting and satisfies}\\ \mathrm{val}(\kappa_j) = \mathrm{opt}_j(q_j, v_j, t_0, \ldots, t_{j-1}) \textit{ for all odd } j \leq i+1\}.\end{aligned}$$

Intuitively, the prophecy $(\mathfrak{P}_i)_{\bar{q}}^{\bar{u},\bar{v}}$ indicates that it is safe for Verifier $i+1$ to move from v_{i+1} to v'_{i+1} as before, and additionally, among all successors of v_{i+1}, picking v'_{i+1} allows Verifier $i+1$ to make the most progress. This prophecy is intended to be used, if for odd $j < i+1$, Verifier j already picked the move v_j to v'_j which was optimal. In particular, the prophecy repeats the prophecies made (in the same round) for Verifier j with odd $j < i+1$.

We show that the prophecies are ω-regular.

Lemma 3. *For each $i \in I_\forall$, vector $\bar{q}$ of states and vectors $\bar{u}, \bar{v}$ such that the ℓ-th vertex of $\bar{v}$ is a successor of the ℓ-th vertex of $\bar{u}$, the set $(\mathfrak{P}_i)_{\bar{q}}^{\bar{u},\bar{v}}$ is ω-regular.*

Proof. For ease of presentation, we speak of value of a trace t when we actually mean the value of the unique run κ on t. We also sometimes write $\mathrm{val}(t)$ instead of $\mathrm{val}(\kappa)$. In both cases, the automaton we consider will be clear from context.

Let $\mathcal{R}$ be a one-pair Rabin automaton. One can construct an ω-automaton that recognizes

$$\{\mathrm{mrg}(t, t') \mid \mathrm{val}(\kappa) > \mathrm{val}(\kappa'),\ \text{where } \kappa \text{ resp. } \kappa' \text{ is the run of } \mathcal{R} \text{ on } t \text{ resp. } t'\}.$$

This automaton checks whether one of the following cases holds:

1. κ visits B infinitely many times (hence, $\mathrm{val}(\kappa) = \infty$), κ' visits B finitely many times and F at least once (hence, $\mathrm{val}(\kappa') < \infty$).
2. κ and κ' visit B finitely many times and F at least once, and
   (a) $\mathrm{last}_B(\kappa), \mathrm{last}_B(\kappa'), \mathrm{first}_F(\kappa') < \mathrm{first}_F(\kappa)$, or
   (b) $\mathrm{first}_F(\kappa), \mathrm{first}_F(\kappa'), \mathrm{last}_B(\kappa') < \mathrm{last}_B(\kappa)$
3. κ and κ' do not visit B, and $\mathrm{first}_F(\kappa') < \mathrm{first}_F(\kappa)$.

By projection to the first component of the set, one obtains an automaton that recognizes traces t such that there is some trace t' with $\mathrm{val}(t) > \mathrm{val}(t')$. The complement of this set (in intersection with valid traces) yields traces that have an optimal value for $\mathcal{R}$.

Using these techniques, one can construct an automaton $\mathcal{B}_1$ that accepts those $\mathrm{mrg}(t_0, t_1)$ where $\mathrm{mrg}(t_0, t_1) \in L(\mathcal{A}_1^{\mathfrak{T}})$ such that t_1 is optimal in combination with t_0 for $\mathcal{A}_1^{\mathfrak{T}}$. Again, using these techniques, one can construct an automaton $\mathcal{B}_3$ that accepts those $\mathrm{mrg}(t_0, t_1, t_2, t_3)$ where $\mathrm{mrg}(t_0, t_1) \in L(\mathcal{B}_1)$, $\mathrm{mrg}(t_0, t_1, t_2, t_3) \in L(\mathcal{A}_3^{\mathfrak{T}})$ and t_3 is optimal in combination with t_0, t_1, t_2 for $\mathcal{A}_3^{\mathfrak{T}}$.

Iterating this, one arrives at an automaton $\mathcal{B}_i$ that accepts those $\mathrm{mrg}(t_0, t_1, \ldots, t_i)$. Projecting away the traces t_j for odd $j \leq i$, one obtains an automaton that accepts those $\mathrm{mrg}(t_0, t_2, \ldots, t_{i-1})$ required by the prophecy $(\mathfrak{P}_{i-1})_{\bar{q}}^{\bar{u},\bar{v}}$ (for some $\bar{q}, \bar{u}, \bar{v}$). □

We set up the manipulated system and formula, then state the completeness result.

Definition 7. *We define $\Xi = (\Xi_i)_{i \in I_\forall}$ and $\mathcal{P} = (\mathrm{P}_i)_{i \in I_\forall}$. Let Ξ_i be the set of QPTL formulas that contains sentences $(\xi_i)_{\bar{q}}^{\bar{u},\bar{v}}$ for each vector $\bar{q}$ of states and vectors $\bar{u}, \bar{v}$ such that the ℓ-th vertex of $\bar{v}$ is a successor of the ℓ-th vertex of $\bar{u}$. The sentence $(\xi_i)_{\bar{q}}^{\bar{u},\bar{v}}$ expresses the prophecy $(\mathfrak{P}_i)_{\bar{q}}^{\bar{u},\bar{v}}$ using only even trace variables π_j with $j \leq i$.*

Moreover, let P_i such that it contains the prophecy variable $(\mathfrak{p}_i)_{\bar{q}}^{\bar{u},\bar{v}}$ for each vector $\bar{q}$ of states and vectors $\bar{u}, \bar{v}$ such that the ℓ-th vertex of $\bar{v}$ is a successor of the ℓ-th vertex of $\bar{u}$. The prophecy variable $(\mathfrak{p}_i)_{\bar{q}}^{\bar{u},\bar{v}}$ corresponds to $(\xi_i)_{\bar{q}}^{\bar{u},\bar{v}}$.

Lemma 4. *Assume $\mathfrak{T} \models \varphi$. Then the coalition of Verifier-players has a winning collection of strategies in $\mathcal{G}(\mathfrak{T}^{\mathcal{P}}, \varphi^{\mathcal{P},\Xi})$ with Ξ and $\mathcal{P}$ as in Definition 7.*

We use the notations we already introduced for the proof of Lemma 2. We define a collection of strategies for the Verifier-players and then show, that $\mathfrak{T} \models \varphi$ implies that this collection is winning in $\mathcal{G}(\mathfrak{T}^{\mathcal{P}}, \varphi^{\mathcal{P},\Xi})$.

Assume the play is in Round (r, i) for $r \geq 0$ and $i \in I_\exists$. Thus, it is in a position

$$((v_0', \ldots, v_{i-1}', v_i, \ldots, v_{k-1}), q, i)$$

and Verifier i has to move. We review some information Verifier i can use to base her choice on. In Round r, Verifier i has access to the first r letters of the traces $\lambda_{\mathrm{AP}}(\rho_0), \lambda_{\mathrm{AP}}(\rho_1), \ldots, \lambda_{\mathrm{AP}}(\rho_i)$ induced by the play[6]. Let q_j be the state that $\mathcal{A}_j^{\mathfrak{T}}$ has reached on $\mathrm{mrg}(\lambda_{\mathrm{AP}}(\rho_0)[0,r), \lambda_{\mathrm{AP}}(\rho_1)[0,r), \ldots, \lambda_{\mathrm{AP}}(\rho_j)[0,r))$ for odd $j \leq i$. Also, let $((v_0, \ldots, v_{i-1}, v_i, \ldots, v_{k-1}), q, 0)$ be the position reached in Round $(r, 0)$.

Definition 8 (Strategy construction)**.** *Let*

$$M_i(\bar{u}, \bar{v}, \bar{q}) = \{v_i' \in \mathrm{S}(v_i) \mid (\mathfrak{p}_{i-1})_{\bar{q}}^{\bar{u},\bar{w}} \in \mathrm{Prophecies}(v_{i-1}')\}$$

where $\bar{u} = (v_0, \ldots, v_i)$, $\bar{v} = (v_0', \ldots, v_{i-1}', v_i)$, $\bar{w} = (v_0', \ldots, v_{i-1}', v_i')$ and $\bar{q} = (q_1, q_3, \ldots, q_i)$.

If $M_i(\bar{u}, \bar{v}, \bar{q})$ is nonempty, then Verifier i can move to any v_i' in the set (the "Nonempty-case"). If it is empty, then Verifier i can move to any $v_i' \in \mathrm{S}(v_i)$ (the "Empty-case").

We begin by showing that if the Verifier-players use the strategies defined in Definition 8, then the Verifier-players have always used the Nonempty-case. This is under the assumption that Falsifier adheres to Assumption 1, that is, he picks paths that start in the intended sub-parts of the manipulated transition system, and furthermore, he always truthfully indicates which prophecies hold.

Lemma 5. *Let α be a play in $\mathcal{G}(\mathfrak{T}^{\mathcal{P}}, \varphi^{\mathcal{P},\Xi})$ that is consistent with the strategies constructed in Definition 8 and that satisfies Assumption 1, then the Verifier-players have always used the* NONEMPTY*-case to pick their moves.*

Proof. Let $\rho_0, \dots, \rho_{k-1}$ be the paths induced by α. For each $r \geq 0$, and each $i \in I_\exists$, we show inductively that Verifier i can use the NONEMPTY-case in Round (r, i) to pick her move. Our induction hypothesis is that in each previous round of a Verifier-player the NONEMPTY-case was used.

We assume that in Round $(r, 0)$ the play was in a position of the form

$$((v_0, \dots, v_{i-1}, v_i, \dots, v_{k-1}), q, 0),$$

and in Round (r, i) the play is in a position of the form

$$((v_0', \dots, v_{i-1}', v_i, \dots, v_{k-1}), q, i).$$

Furthermore, let q_j be the state that $\mathcal{A}_j^{\mathfrak{T}}$ has reached on

$$\mathrm{mrg}(\lambda_{\mathrm{AP}}(\rho_0)[0, r), \lambda_{\mathrm{AP}}(\rho_1)[0, r), \dots, \lambda_{\mathrm{AP}}(\rho_j)[0, r))$$

for odd $j \leq i$.

Let $\bar{u} = (v_0, \dots, v_i)$, $\bar{v} = (v_0', \dots, v_{i-1}', v_i)$ and $\bar{q} = (q_1, q_3, \dots, q_i)$. Our goal is to show that there is some $v_i' \in \mathrm{S}(v_i)$ such that for $\bar{w} = (v_0', \dots, v_{i-1}', v_i')$ holds that

$$\mathrm{mrg}(\lambda_{\mathrm{AP}}(\rho_0)[0, r), \lambda_{\mathrm{AP}}(\rho_2)[0, r), \dots, \lambda_{\mathrm{AP}}(\rho_{i-1})[0, r)) \in (\mathfrak{P}_{i-1})_{\bar{q}}^{\bar{u}, \bar{w}} \tag{5}$$

which implies that $(\mathfrak{p}_{i-1})_{\bar{q}}^{\bar{u}, \bar{w}} \in \mathrm{Prophecies}(v_{i-1}')$ (because Falsifier adheres to Assumption 1). By definition, then $v_i' \in M_i(\bar{u}, \bar{v}, \bar{q})$, and Verifier i is in the NONEMPTY-case.

We first make an observation about the connection between prophecies used in the safety case and our new prophecies:

For each $j \in I_\exists$, vertex vectors $\bar{x}, \bar{y}$ such that the ℓ-th vertex of $\bar{y}$ is a successor vertex of the ℓ-th vertex of $\bar{x}$, each state vector $\bar{s}$, if

$$\mathrm{mrg}(t_0, t_2, \dots, t_{j-1}) \in (\mathfrak{P}_{j-1})_{\bar{s}}^{\bar{x}, \bar{y}},$$

then

$$\mathrm{mrg}(t_0, t_2, \dots, t_{j-1}) \in (\mathfrak{S}_{j-1})_s^y,$$

where s is the last state of $\bar{s}$ and y is the last vertex of $\bar{y}$. Note that the definition of the safety prophecies $(\mathfrak{S}_{j-1})_s^y$ does *not* depend on the automaton inducing the prophecies being a safety automaton: They just ensure that the remaining suffixes induced by the play can be accepted, if the Verifier-players play accordingly. Similarly, the induction at the beginning of the proof of Lemma 2, showing that the Verifier-players can always use the NONEMPTY-case, is also independent of the acceptance condition.

As a consequence, this allows us to reuse (without reproving) Eq. (1) and Eq. (2). To recap, we want to prove Eq. (5) for Round (r, i). Therefore, we first prove that Eq. (1) for Round $(r, 0)$ implies Eq. (5) for Round $(r, 1)$. Then, we prove that Eq. (2) for Round (r, i) together with Eq. (5) for Round $(r, i-2)$ implies Eq. (5) for Round (r, i) with $i > 1$.

We show Eq. (5) for Round $(r, 1)$. We have to show that $\lambda_{\mathrm{AP}}(\rho_0)[0, r) \in (\mathfrak{P}_0)_{(q_1)}^{(v_0, v_1), (v_0', v_1')}$ for some $v_1' \in \mathrm{S}(v_1)$. Let t_0 denote $\lambda_{\mathrm{AP}}(\rho_0)[r, \infty) \in \mathrm{Tr}(\mathfrak{T}_{v_0'})$. Eq. (1) implies that there is some $t_1 \in \mathrm{Tr}(\mathfrak{T}_{\mathrm{S}(v_1)})$ such that $(\mathcal{A}_1^{\mathfrak{T}})_{q_1}$ accepts $\mathrm{mrg}(t_0, t_1)$. Hence, there is a witness $t_1' \in \mathrm{Tr}(\mathfrak{T}_{\mathrm{S}(v_1)})$ such that the run κ_1 of $(\mathcal{A}_1^{\mathfrak{T}})_{q_1}$ on $\mathrm{mrg}(t_0, t_1')$ is accepting and $\mathrm{val}(\kappa_1) = \mathrm{opt}_1(q_1, v_1, t_0)$. Clearly, t_1' in $\mathrm{Tr}(\mathfrak{T}_{v_1'})$ for some $v_1' \in \mathrm{S}(v_1)$. Thus, $t_0 \in (\mathfrak{P}_0)_{(q_1)}^{(v_0, v_1), (v_0', v_1')}$. We completed the proof for Round $(r, 1)$.

We show Eq. (5) for Round (r, i) with $i > 1$. Let t_j denote $\lambda_{\mathrm{AP}}(\rho_j)[r, \infty) \in \mathrm{Tr}(\mathfrak{T}_{v_j'})$ for all even $j < i$. By assumption, Eq. (5) holds in Round $(r, i-2)$. Consequently,

$$\mathrm{mrg}(t_0, t_2, \dots, t_{i-3}) \in (\mathfrak{P}_{i-3})_{(q_1, q_3, \dots, q_{i-2})}^{(v_0, \dots, v_{i-2}), (v_0', \dots, v_{i-2}')}.$$

By definition of the above prophecy, we obtain that there is a witness $t_j' \in \mathrm{Tr}(\mathfrak{T}_{v_j'})$ such that

$$\mathrm{val}(\kappa_j) = \mathrm{opt}_j(q_{j+1}, v_j, t_0, t_1', t_2, t_3', \dots, t_{j-1}), \text{ and } \kappa_j \text{ is accepting,}$$

where κ_j is the run of $(\mathcal{A}_j^{\mathfrak{T}})_{q_j}$ on $\mathrm{mrg}(t_0, t'_1, t_2, t'_3, \ldots, t_{j-1}, t'_j)$ for all odd $j < i$. Note that the traces with odd indices are the witnesses of optimality and the traces with even indices are the traces of Falsifier.

Eq. (2) for Round (r, i) implies that there is some $t \in \mathrm{Tr}(\mathfrak{T}_{\mathrm{S}(v_i)})$ such that $(\mathcal{A}_i^{\mathfrak{T}})_{q_i}$ accepts $\mathrm{mrg}(t_0, t'_1, t_2, t'_3, \ldots, t_{i-1}, t)$. Hence, there is a witness $t'_i \in \mathrm{Tr}(\mathfrak{T}_{\mathrm{S}(v_i)})$ such that

$$\mathrm{val}(\kappa_i) = \mathrm{opt}_i(q_{i+1}, v_i, t_0, t'_1, t_2, t'_3, \ldots, t_{i-1}), \text{ and } \kappa_i \text{ is accepting,}$$

where κ_i is the run of $(\mathcal{A}_i^{\mathfrak{T}})_{q_i}$ on $\mathrm{mrg}(t_0, t'_1, t_2, t'_3, \ldots, t_{i-1}, t'_i)$.

Clearly, $t'_i \in \mathrm{Tr}(\mathfrak{T}_{v'_i})$ for some $v'_i \in \mathrm{S}(v_i)$. Thus, $\mathrm{mrg}(t_0, t_2, \ldots, t_{i-1}) \in (\mathfrak{P}_{i-1})_{\bar{q}}^{\bar{u},\bar{w}}$ where the last vertex of $\bar{w}$ is v'_i. We have completed the proof for Round (r, i). ☐

We show an auxiliary lemma (Lemma 6) that allows us to prove that playing according to the proposed strategies ensures that the play induces traces that are accepted by the $\mathcal{A}_i^{\mathfrak{T}}$ (Lemma 7). Intuitively, Lemma 6 states that making one move according to the proposed strategy definition (Definition 8) yields progress towards accepting.

Lemma 6. *For each $i \in I_\exists$, vectors $\bar{u} = (v_0, v_1, \ldots, v_i)$ and $\bar{v} = (v'_0, v'_1, \ldots, v'_i)$ of vertices such that $v'_j \in \mathrm{S}(v_j)$ for $j \leq i$, each vector $\bar{q} = (q_1, q_3, \ldots, q_i)$ of states, and traces $t_j \in \mathrm{Tr}(\mathfrak{T}_{v'_j})$ for $j < i$ such that*

$$\mathrm{val}(\kappa_j) = \mathrm{opt}_j(q_j, v_j, t_0, \ldots, t_{j-1}), \textit{ and } \kappa_j \textit{ is accepting,}$$

where κ_j is the run of $\mathcal{A}_j^{\mathfrak{T}}$ on $\mathrm{mrg}(t_0, \ldots, t_j)$ for all odd $j < i$, and

$$\mathrm{mrg}(t_0, t_2, \ldots, t_{i-1}) \in (\mathfrak{P}_{i-1})_{\bar{q}}^{\bar{u},\bar{v}}$$

holds the following:

$$\mathrm{opt}_i(q_i, v_i, t_0, \ldots, t_{i-1}) < \infty,$$

and if $q_i \notin F$ then

$$\mathrm{opt}_i(q'_i, v'_i, t_0[1,\infty), \ldots, t_{i-1}[1,\infty)) < \mathrm{opt}_i(q_i, v_i, t_0, \ldots, t_{i-1}),$$

where $q'_i = \delta_i(q_i, \lambda(v'_0), \ldots, \lambda(v'_i))$ and δ_i is the transition function of $\mathcal{A}_i^{\mathfrak{T}}$.

Proof. Since $\mathrm{val}(\kappa_j) = \mathrm{opt}_j(q_j, v_j, t_0, \ldots, t_{j-1})$ and κ_j is accepting for all odd $j < i$, and $\mathrm{mrg}(t_0, t_2, \ldots, t_{i-1}) \in (\mathfrak{P}_{i-1})_{\bar{q}}^{\bar{u},\bar{v}}$, we obtain, using the definition of $(\mathfrak{P}_{i-1})_{\bar{q}}^{\bar{u},\bar{v}}$, that

$$\mathrm{opt}_i(q_i, v_i, t_0, \ldots, t_{i-1}) < \infty.$$

Furthermore, the above implies also that there is a trace $t_i \in \mathrm{Tr}(\mathfrak{T}_{v'_i})$ such that

$$\mathrm{opt}_i(q_i, v_i, t_0, \ldots, t_{i-1}) = \mathrm{val}(\kappa_i), \tag{6}$$

where κ_i is the run of $(\mathcal{A}_i^{\mathfrak{T}})_{q_i}$ on $\mathrm{mrg}(t_0, \ldots, t_{i-1}, t_i)$.

We show that

$$\mathrm{opt}_i(q'_i, v'_i, t_0[1,\infty), \ldots, t_{i-1}[1,\infty)) \leq \mathrm{val}(\kappa_i[1,\infty)). \tag{7}$$

For Eq. (7) to hold, it suffices that $t(0) = \lambda(v'_i)$ and that $\kappa_i[1,\infty)$ is a run of $(\mathcal{A}_i^{\mathfrak{T}})_{q'_i}$. Both conditions are true.

Furthermore, we show that

$$\mathrm{val}(\kappa_i[1,\infty)) + 1 = \mathrm{val}(\kappa_i). \tag{8}$$

Since $\mathrm{val}(\kappa_i) < \infty$, we have $\mathrm{last}_B(\kappa_i) \neq \infty$ and $\mathrm{first}_F(\kappa_i) < \infty$. If $\mathrm{last}_B(\kappa_i) = \bot$, then $\mathrm{last}_B(\kappa_i[1,\infty)) = \bot$, otherwise $\mathrm{last}_B(\kappa_i[1,\infty)) + 1 = \mathrm{last}_B(\kappa_i)$. Furthermore, $\mathrm{first}_F(\kappa_i[1,\infty)) + 1 = \mathrm{first}_F(\kappa_i)$, because $q_{i+1} \neq F$ and κ_i starts in q_{i+1}. Eq. (8) follows.

Finally, by combining Eqs. (6) to (8), we obtain

$$\mathrm{opt}_i(q_i, v_i, t_0, \ldots, t_{i-1}) = \mathrm{val}(\rho) > \mathrm{val}(\rho[1,\infty)) \geq \mathrm{opt}_i(q'_i, v'_i, t_0[1,\infty), \ldots, t_{i-1}[1,\infty)).$$

☐

We show that playing according to the proposed strategies yields traces that make optimal progress towards being accepted (for each of the $\mathcal{A}_i^{\mathfrak{T}}$) implying that in the limit they are actually accepted. Ultimately, to show that the Verifier-players win a play in the game $\mathcal{G}(\mathfrak{T}^{\mathcal{P}}, \psi^{\mathcal{P},\Xi})$, it suffices to show that $\mathcal{A}_{k-1}^{\mathfrak{T}}$ accepts the traces induced by the play. However, to prove this, we need that all $\mathcal{A}_i^{\mathfrak{T}}$ for $i \in I_\exists$ accept their input traces, to ensure that the traces up to $k-2$ yield a valid basis to be completed with the $(k-1)$-th trace.

Lemma 7. *Let α be a play in $\mathcal{G}(\mathfrak{T}^{\mathcal{P}}, \varphi^{\mathcal{P},\Xi})$ that is consistent with the strategies constructed in Definition 8 and that satisfies Assumption 1. Let $\rho_0^\alpha, \dots, \rho_{k-1}^\alpha$ be the paths induced by α, and let $t_i = \lambda_{\mathrm{AP}}(\rho_i^\alpha)$ for $i \in \{0, \dots, k-1\}$. For $r \geq 0$ and $i \in I_\exists$, we obtain*

$$\mathrm{val}(\kappa_i)[r, \infty) = \mathrm{opt}_i(\kappa_i(r), \rho_i^\alpha(r), t_0[r, \infty), \dots, t_{i-1}[r, \infty)), \tag{9}$$

where κ_i is the run of $\mathcal{A}_i^{\mathfrak{T}}$ on $\mathrm{mrg}(t_0, \dots, t_i)$. Furthermore, $\mathrm{mrg}(t_0, \dots, t_i) \in L(\mathcal{A}_i^{\mathfrak{T}})$.

Proof. We proceed by induction on $i \in I_\exists$. We prove the statement for i and assume the statement holds for odd $j < i$.

Recall that in Round (r, i) with $r \geq 0$, we are in a position of the form

$$((v_0', \dots, v_{i-1}', v_i, \dots, v_{k-1}), q, i),$$

where $\rho_i^\alpha(r) = v_i$. We define

$$x_r := \mathrm{opt}_i(\kappa_i(r), v_i, t_0[r, \infty), \dots, t_{i-1}[r, \infty)).$$

We claim that for any $r \geq 0$, we have $x_r < \infty$ and either $\kappa_i(r) \in F$ or $x_r > x_{r+1}$. This implies that κ_i is an accepting run: The run κ_i only visits finitely many times B, because $x_0 < \infty$. And κ_i visits F infinitely many times, because there is no infinitely long decreasing sequence of natural numbers.

Assumption 1 yields that

$$\mathrm{mrg}(t_0[r, \infty), t_2[r, \infty), \dots, t_{i-1}[r, \infty])) \in (\mathfrak{P}_{i-1})_{\bar{q}}^{\bar{u}, \bar{v}}, \tag{10}$$

where $\bar{u} = (v_0, \dots, v_i)$, $\bar{v} = (v_0', \dots, v_i')$, and $\bar{q} = (\kappa_1(r), \kappa_3(r), \dots, \kappa_i(r))$, and $v_\ell' \in \mathrm{S}(v_\ell)$ is the move chosen in Round (r, ℓ), i.e., $v_\ell = \rho_\ell^\alpha(r)$ and $v_\ell' = \rho_\ell^\alpha(r+1)$ for $\ell \leq i$.

By induction hypothesis, the statement of Lemma 7 is true for all odd $j < i$. Note that $\mathrm{mrg}(t_0, \dots, t_j) \in \mathcal{A}_j^{\mathfrak{T}}$ implies that $\mathrm{mrg}(t_0[r, \infty), \dots, t_j[r, \infty)) \in (\mathcal{A}_j^{\mathfrak{T}})_{\kappa_j(r)}$. Thus, using Eq. (10) and Eq. (9) for all odd $j < i$, the pre-conditions of Lemma 6 are satisfied. Consequently, we get $x_r < \infty$, and if $\kappa_i(r) \notin F$,

$$\begin{aligned} x_{r+1} &= \mathrm{opt}_i(\kappa_i(r+1), v_i', t_0[r+1, \infty), \dots, t_{i-1}[r+1, \infty)) \\ &< \mathrm{opt}_i(\kappa_i(r), v_i, t_0[r, \infty), \dots, t_{i-1}[r, \infty)) = x_r, \end{aligned}$$

which shows that κ_i is accepting, i.e., $\mathrm{mrg}(t_0, \dots, t_i) \in L(\mathcal{A}_i^{\mathfrak{T}})$.

It is left to prove that Eq. (9) holds, i.e., for all $r \geq 0$, we have

$$x_r = \mathrm{opt}_i(\kappa_i(r), \rho_i^\alpha(r), t_0[r, \infty), \dots, t_{i-1}[r, \infty)) = \mathrm{val}(\kappa_i)[r, \infty).$$

Recall that $\rho_i^\alpha(r) = v_i$ and that $\rho_i^\alpha(r+1) = v_i'$, and let $\kappa_i(r) = q$. By definition of $\mathrm{opt}_i(\cdot)$, there is a path ρ with $\rho(0) \in \mathrm{S}(v_i)$ such that $\mathrm{val}(\kappa) = x_r$, where κ is the run of $(\mathcal{A}_i^{\mathfrak{T}})_q$ on $\mathrm{mrg}(t_0[r, \infty), \dots, t_{i-1}[r, \infty), \lambda(\rho))$. By Eq. (10) and Eq. (9) for all odd $j < i$, we obtain that ρ can be chosen such that $\rho(0) = v_i' \in \mathrm{S}(v_i)$.

Hence, the path ρ_i^α ensures that $x_r = \mathrm{val}(\kappa_i)[r, \infty)$ is true for all $r \geq 0$. ☐

Finally, we are ready to prove completeness.

Proof of Lemma 4. We prove that the strategies defined above form a winning collection. To this end, let α be a play in $\mathcal{G}(\mathfrak{T}^{\mathcal{P}}, \varphi^{\mathcal{P},\Xi})$ that is consistent with the strategies and satisfies Assumption 1. Plays in which the assumption is violated are trivially won by the Verifier-players. Furthermore, let $\rho_0^\alpha, \dots, \rho_{k-1}^\alpha$ be the paths induced by α, and let $t_i = \lambda_{\mathrm{AP}}(\rho_i^\alpha)$ for $i \in \{0, \dots, k-1\}$. Since the premise of $\psi^{\mathcal{P},\Xi}$ holds, we must show that its conclusion ψ holds, i.e., that $\mathcal{A}_{k-1}^{\mathfrak{T}}$ accepts $\mathrm{mrg}(t_0, \dots, t_{k-1})$. This is implied by Lemma 7 for $i = k-1$. ☐

Combining Lemma 1 and Lemma 4, we directly obtain our main result.

Theorem 3. *Let Ξ and $\mathcal{P}$ as in Definition 7. The coalition of Verifier-players has a winning collection of strategies for $\mathcal{G}(\mathfrak{T}^{\mathcal{P}}, \varphi^{\mathcal{P},\Xi})$ if and only if $\mathfrak{T} \models \varphi$.*

6.2 Beyond One-Pair Rabin Automata

We have proven Theorem 3 for the case that all involved automata are one-pair Rabin automata. We sketch how to drop this constraint following Beutner and Finkbeiner [5].

A Rabin automaton $\mathcal{R}$ with pairs $(B_1, F_1), \ldots, (B_m, F_m)$ can be seen as a union of m one-pair Rabin automata. A run that is accepting in (B_i, F_i)-$\mathcal{R}$ is also accepting in $\mathcal{R}$.

The idea is to annotate our prophecies (which refer to $\mathcal{A}_1^{\mathfrak{T}}, \mathcal{A}_3^{\mathfrak{T}}, \ldots, \mathcal{A}_{k-1}^{\mathfrak{T}}$) with the indices of the pairs used for acceptance. More concretely, for a prophecy $(\mathfrak{P}_i)_{\bar{q}}^{\bar{u},\bar{v}}$, we introduce prophecies of the form $(\mathfrak{P}_i)_{\bar{q},\bar{a}}^{\bar{u},\bar{v}}$, where $\bar{a}$ is a vector of indices with the same length as $\bar{q}$. Say the j-th entry of $\bar{q}$ is q_ℓ, then the prophecy $(\mathfrak{P}_i)_{\bar{q}}^{\bar{u},\bar{v}}$ refers to $(\mathcal{A}_\ell^{\mathfrak{T}})_{q_\ell}$ (among other automata), and say the j-th entry of $\bar{a}$ is a_j, then $(\mathfrak{P}_i)_{\bar{q},\bar{a}}^{\bar{u},\bar{v}}$ refers to the one-pair Rabin automaton (B_{a_j}, F_{a_j})-$(\mathcal{A}_\ell^{\mathfrak{T}})_{q_\ell}$ (among other automata).

In a play of the game $\mathcal{G}(\mathfrak{T}^{\mathcal{P}}, \varphi^{\mathcal{P},\Xi})$, Falsifier indicates with his prophecies for which pairs the automata can accept. In Round $(0,0)$, it is indicated for $\mathcal{A}_1^{\mathfrak{T}}$, in Round $(0,1)$, Verifier 1 chooses one option indicated by Falsifier for $\mathcal{A}_1^{\mathfrak{T}}$ and then has to stick to it for the rest of the play. In Round $(0,2)$, it is indicated also for $\mathcal{A}_3^{\mathfrak{T}}$, in Round $(0,3)$, Verifier 3 chooses one option for $\mathcal{A}_3^{\mathfrak{T}}$ and then has to stick to it for the rest of the play. Additionally, she must pick the option chosen by Verifier 1 for $\mathcal{A}_1^{\mathfrak{T}}$ to ensure consistency. The same principle applies for the next rounds. After Round $(0, k-1)$, for each automaton a pair has been fixed to which the Verifier-players stick for the rest of the play.

7 Conclusion

We have presented the first effective game-theoretic characterization of full HyperLTL (even HyperQPTL) model-checking via multi-player games with hierarchical information, thereby extending the prophecy-based framework of Coenen et al. and Beutner and Finkbeiner from the $\forall^*\exists^*$-fragment to arbitrary quantifier prefixes. Similarly, we have extended Winter and Zimmermann's game-based characterization of the existence of computable Skolem functions witnessing $\mathfrak{T} \models \varphi$. One way of understanding our result is that we compute finite-state implementations of Skolem functions via transducers with ω-regular lookahead.

One of the main advantages of the prophecy-based game for the $\forall^*\exists^*$-fragment is that it avoids complementation of ω-automata, which is necessary for the classical automata-based model-checking algorithm. Our construction does require complementation of ω-automata to handle the "hiding" of information. Further research is necessary to determine whether a complementation-free prophecy construction for full HyperLTL is possible. Note that solving multi-player games of hierarchical information is already Tower-complete (in the number of players), so even prophecies of elementary-size would not contradict any complexity-lower bounds, as HyperLTL model-checking is Tower-complete. However, as it is, the number of prophecies we use is only bounded by an n-fold exponential, where n is linear in the number of quantifier alternations, as the size of the automata $\mathcal{A}_i^{\mathfrak{T}}$ grows like that.

We conjecture that the techniques developed here can be applied to even more expressive logics, e.g., HyperRecHML, recursive Hennesy-Milner logic with trace quantification [2].

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