The Complexity of HyperQPTL

Gaëtan Regaud (ENS Rennes, Rennes, France)
Martin Zimmermann (Aalborg University, Aalborg, Denmark)

Abstract

HyperQPTL and HyperQPTL$^+$ are expressive specification languages for hyperproperties, i.e., properties that relate multiple executions of a system. Tight complexity bounds are known for HyperQPTL finite-state satisfiability and model-checking.

Here, we settle the complexity of satisfiability for HyperQPTL as well as satisfiability, finite-state satisfiability, and model-checking for HyperQPTL$^+$: the former is equivalent to truth in second-order arithmetic, the latter are all equivalent to truth in third-order arithmetic, i.e., they are all four very undecidable.

1 Introduction

Hyperproperties [3] are properties relating multiple executions of a system and have found applications in security and privacy, epistemic reasoning, and verification. Temporal logics have been introduced to express hyperproperties, e.g., HyperLTL and HyperCTL* [2] (which extend LTL and CTL* with trace quantification), Hyper^{2}LTL [1] (second-order HyperLTL, which extends HyperLTL with quantification over sets of traces), and many more. Here, we are interested in the most important verification problems:

- Satisfiability: Given a sentence φ, does it have a model, i.e., is there a set T of traces such that $T \models \varphi$?
- Finite-state satisfiability: Given a sentence φ, is it satisfied by a transition system, i.e., is there a finite transition system $\mathfrak{T}$ such that $\mathfrak{T} \models \varphi$?
- Model-checking: Given a sentence φ and a finite transition system $\mathfrak{T}$, do we have $\mathfrak{T} \models \varphi$?

This work is part of a research program [6, 11, 4, 7, 9, 12] settling the complexity of these problems for hyperlogics. Most importantly for applications in verification, the model-checking problems for HyperLTL and HyperCTL* are decidable, albeit Tower-complete [6, 11, 10]. However, satisfiability is typically much harder. In fact, the satisfiability problems are typically highly undecidable, i.e., we measure their complexity by placing them in the arithmetic or analytic hierarchy, or even beyond: Intuitively, first-order arithmetic is predicate logic over the signature $(+, \cdot, <)$ where quantification ranges over natural numbers. Similarly, second-order arithmetic adds quantification over sets of natural numbers to first-order arithmetic while third-order arithmetic adds quantification over sets of sets of natural numbers to second-order quantification. Figure 1 gives an overview of the arithmetic, analytic, and "third" hierarchy, each spanned by the classes of languages definable by restricting the number of alternations of the highest-order quantifiers, i.e., Σ_n^0 contains languages definable by formulas of first-order arithmetic with $n-1$ quantifier alternations, starting with an existential one.

Here, we study the logics HyperQPTL and HyperQPTL$^+$ which extend HyperLTL by quantification over propositions [11, 5], just as QPTL extends LTL with quantification over propositions [13]. QPTL is, unlike LTL, able to express all ω-regular properties while HyperQPTL allows to express, e.g., promptness and epistemic properties, which are not expressible in HyperLTL [5].

The difference between HyperQPTL and HyperQPTL$^+$ manifests itself in the semantics of propositional quantification. Recall that hyperlogics are evaluated over sets T of traces. In HyperQPTL, the quantification of a proposition $\mathtt{p}$ reassigns the truth value of $\mathtt{p}$ in T in a uniform way over all traces, i.e., after

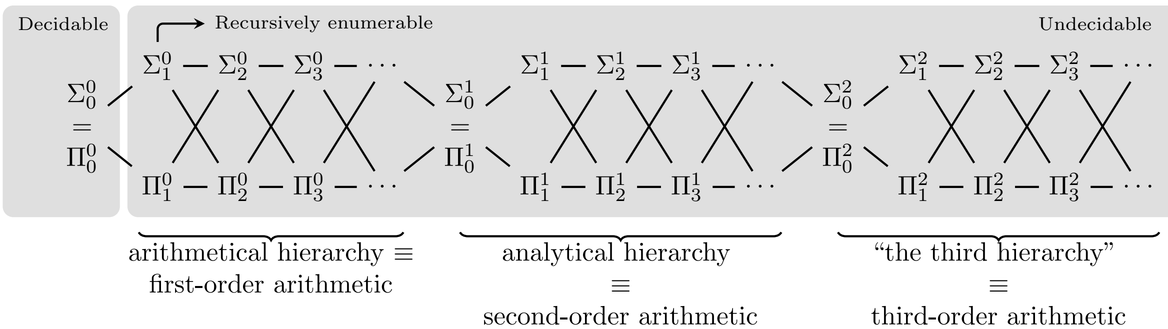

Figure 1: The arithmetical hierarchy, the analytical hierarchy, and beyond.

quantifying p all traces coincide on their truth values for p. In HyperQPTL$^+$ on the other hand, the quantification of a proposition p reassigns the truth value of p for each trace in T individually. Said differently, in HyperQPTL one quantifies over a single sequence of truth values (i.e., a sequence in $\{0,1\}^\omega$) while in HyperQPTL$^+$ one quantifies over a set of sequences of truth values (i.e., a subset of $\{0,1\}^\omega$). Hence, one expects HyperQPTL$^+$ to be more expressive than HyperQPTL. And indeed, Finkbeiner et al. showed that HyperQPTL$^+$ model-checking is undecidable [5], while it is decidable for HyperQPTL [11]. However, just how expressive HyperQPTL and HyperQPTL$^+$ are (and thus how undecidable their verification problems are) is an open problem.

It is known that HyperQPTL finite-state satisfiability is Σ^0_1-complete [11], i.e., complete for the recursive-enumerable languages, and HyperQPTL model-checking is Tower-complete [11], but the exact complexity of HyperQPTL satisfiability is open: it is only known to be undecidable, as satisfiability is already undecidable for its fragment HyperLTL [4]. For HyperQPTL$^+$, much less is known: As mentioned above, model-checking HyperQPTL$^+$ is undecidable, but its exact complexity is open, as is the complexity of satisfiability and finite-state satisfiability.

Here, we settle the complexity of all four open problems by showing that HyperQPTL satisfiability is equivalent to truth in second-order arithmetic while all three problems for HyperQPTL$^+$ are equivalent to truth in third-order arithmetic. These latter results are obtained by showing that HyperQPTL$^+$ and Hyper^{2}LTL have the same expressiveness. This confirms the expectation that HyperQPTL$^+$ is more expressive than HyperQPTL: the non-uniform quantification of propositions allows to simulate quantification over arbitrary sets of traces.

Table 1 presents our results (in bold) as well as results for the related logics mentioned above. There, Hyper^{2}LTL$_{\mathrm{mm}}$ and lfp-Hyper^{2}LTL$_{\mathrm{mm}}$ are the fragments of second-order HyperLTL obtained by restricting set quantification to minimal/maximal sets satisfying a guard formula respectively by restricting set quantification to least fixed points of LTL definable operators. Furthermore, Hyper^{2}LTL comes with two semantics, standard and closed-world (CW). In most cases, the choice of semantics does not influence the complexity of the verification problems. The only known exception is the satisfiability problem, which is Σ^2_1-complete for standard semantics, but only Σ^1_1-complete for closed-world semantics [9, 12].

2 Preliminaries

We denote the nonnegative integers by $\mathbb{N}$.

2.1 Words, Traces, and Transition Systems

An alphabet is a nonempty finite set. The set of infinite words over an alphabet Σ is denoted by Σ^ω. Let AP be a nonempty finite set of atomic propositions. A trace over AP is an infinite word over the alphabet 2^{AP}. Given a subset $\mathrm{AP}' \subseteq \mathrm{AP}$, the AP$'$-projection of a trace $t(0)t(1)t(2)\cdots$ over AP is the

Table 1: List of our results (in bold) and comparison to related logics. "T2A-equivalent" stands for "equivalent to truth in second-order arithmetic", "T3A-equivalent" for "equivalent to truth in third-order arithmetic".

Logic	Satisfiability	Finite-state satisfiability	Model-checking
LTL	PSPACE-complete	PSPACE-complete	PSPACE-complete
HyperLTL	Σ_1^1-complete	Σ_1^0-complete	TOWER-complete
Hyper^2LTL	T3A-equivalent	T3A-equivalent	T3A-equivalent
$\text{Hyper}^2\text{LTL}_{\text{mm}}$	T3A-equivalent	T3A-equivalent	T3A-equivalent
lfp-$\text{Hyper}^2\text{LTL}_{\text{mm}}$	Σ_1^2-complete/ Σ_1^1-complete (CW)	T2A-equivalent	T2A-equivalent
HyperQPTL	**T2A-equivalent**	Σ_1^0-complete	TOWER-complete
HyperQPTL^+	**T3A-equivalent**	**T3A-equivalent**	**T3A-equivalent**

trace $(t(0) \cap \text{AP}')(t(1) \cap \text{AP}')(t(2) \cap \text{AP}') \cdots \in (2^{\text{AP}'})^\omega$. The AP'-projection of $T \subseteq (2^{\text{AP}})^\omega$ is defined as the set of AP-projections of traces in T. We write $t =_{\text{AP}'} t'$ ($T =_{\text{AP}'} T'$) if the AP'-projections of t and t' (T and T') are equal. Now, let AP and AP' be two disjoint sets, let t be a trace over AP, and let t' be a trace over AP'. Then, we define $t^\frown t'$ as the pointwise union of t and t', i.e., $t^\frown t'$ is the trace over $\text{AP} \cup \text{AP}'$ defined as $(t(0) \cup t'(0))(t(1) \cup t'(1))(t(2) \cup t'(2)) \cdots$.

A transition system $\mathfrak{T} = (V, E, I, \lambda)$ consists of a finite nonempty set V of vertices, a set $E \subseteq V \times V$ of (directed) edges, a set $I \subseteq V$ of initial vertices, and a labeling $\lambda \colon V \to 2^{\text{AP}}$ of the vertices by sets of atomic propositions. We assume that every vertex has at least one outgoing edge. A path ρ through $\mathfrak{T}$ is an infinite sequence $\rho(0)\rho(1)\rho(2)\cdots$ of vertices with $\rho(0) \in I$ and $(\rho(n), \rho(n+1)) \in E$ for every $n \geq 0$. The trace of ρ is defined as $\lambda(\rho) = \lambda(\rho(0))\lambda(\rho(1))\lambda(\rho(2)) \cdots$. The set of traces of $\mathfrak{T}$ is $\text{Tr}(\mathfrak{T}) = \{\lambda(\rho) \mid \rho \text{ is a path through } \mathfrak{T}\}$.

2.2 Arithmetic

To capture the complexity of undecidable problems, we consider formulas of arithmetic, i.e., predicate logic with signature $(+, \cdot, <, \in)$, evaluated over the structure $(\mathbb{N}, +, \cdot, <, \in)$. A type 0 object is a natural number in $\mathbb{N}$, a type 1 object is a subset of $\mathbb{N}$, and a type 2 object is a set of subsets of $\mathbb{N}$.

First-order arithmetic allows to quantify over type 0 objects, second-order arithmetic allows to quantify over type 0 and type 1 objects, and third-order arithmetic allows to quantify over type 0, type 1, and type 2 objects. Note that every fixed natural number is definable in first-order arithmetic, so we freely use them as syntactic sugar. Similarly, equality can be eliminated if necessary, as it can be expressed using $<$.

Truth in second-order arithmetic is the following problem: given a sentence φ of second-order arithmetic, does $(\mathbb{N}, +, \cdot, <, \in)$ satisfy φ? Truth in third-order arithmetic is defined analogously.

3 HyperQPTL

Let $\mathcal{V}$ be a countable set of trace variables. The formulas of HyperQPTL are given by the grammar

$$\varphi ::= \exists \pi.\ \varphi \mid \forall \pi.\ \varphi \mid \exists \mathtt{q}.\ \varphi \mid \forall \mathtt{q}.\ \varphi \mid \psi$$
$$\psi ::= \mathtt{p}_\pi \mid \mathtt{q} \mid \neg\psi \mid \psi \vee \psi \mid \mathbf{X}\,\psi \mid \mathbf{F}\,\psi$$

where $\mathtt{p}$ and $\mathtt{q}$ range over AP and where π ranges over $\mathcal{V}$. Note that there are two types of atomic formulas, i.e., propositions labeled by traces on which they are evaluated ($\mathtt{p}_\pi$ with $\mathtt{p} \in \text{AP}$ and $\pi \in \mathcal{V}$) and unlabeled propositions ($\mathtt{q} \in \text{AP}$).[1] A formula is a sentence, if every occurrence of an atomic formula $\mathtt{p}_\pi$ is in the scope of a quantifier binding π and every occurrence of an atomic formula $\mathtt{q}$ is in the scope of a quantifier binding $\mathtt{q}$. Finally, note that the only temporal operators we have in the syntax are next ($\mathbf{X}$) and eventually ($\mathbf{F}$), as

[1] We use different letters for the propositions in these cases, but let us stress again that both $\mathtt{p}$ and $\mathtt{q}$ are propositions in AP.

the other temporal operators like always ($\mathbf{G}$) and until ($\mathbf{U}$) are syntactic sugar in HyperQPTL. Hence, we will use them freely in the following, just as we use conjunction, implication, and equivalence.

A (trace) variable assignment is a partial mapping $\Pi\colon \mathcal{V} \to (2^{\mathrm{AP}})^\omega$. Given a variable $\pi \in \mathcal{V}$, and a trace t, we denote by $\Pi[\pi \mapsto t]$ the assignment that coincides with Π on all variables but π, which is mapped to t. Furthermore, $\Pi[j,\infty)$ denotes the variable assignment mapping every $\pi \in \mathcal{V}$ in Π's domain to $\Pi(\pi)(j)\Pi(\pi)(j+1)\Pi(\pi)(j+2)\cdots$, the suffix of $\Pi(\pi)$ starting at position j. These are used to capture the semantics of quantification and temporal operators.

It remains to capture the semantics of quantification of propositions $\mathtt{q} \in \mathrm{AP}$. Let $t \in (2^{\mathrm{AP}})^\omega$ be a trace over AP and $t_{\mathtt{q}} \in (2^{\{\mathtt{q}\}})^\omega$ be a trace over $\{\mathtt{q}\}$. We define the trace $t[\mathtt{q} \mapsto t_{\mathtt{q}}] = t'^\frown t_{\mathtt{q}}$, where t' is the $(\mathrm{AP} \setminus \{\mathtt{q}\})$-projection of t: Intuitively, the occurrences of $\mathtt{q}$ in t are replaced according to $t_{\mathtt{q}}$. We lift this to sets T of traces by defining $T[\mathtt{q} \mapsto t_{\mathtt{q}}] = \{t[\mathtt{q} \mapsto t_{\mathtt{q}}] \mid t \in T\}$. Note that all traces in $T[\mathtt{q} \mapsto t_{\mathtt{q}}]$ have the same $\{\mathtt{q}\}$-projection, which is $t_{\mathtt{q}}$.

Now, for a trace assignment Π and a set T of traces, we define

- $\Pi \models_T \mathtt{p}_\pi$ if $\mathtt{p} \in \Pi(\pi)(0)$,
- $\Pi \models_T \mathtt{q}$ if for all $t \in T$ we have $\mathtt{q} \in t(0)$,
- $\Pi \models_T \neg\psi$ if $\Pi \not\models_T \psi$,
- $\Pi \models_T \psi_1 \vee \psi_2$ if $\Pi \models_T \psi_1$ or $\Pi \models_T \psi_2$,
- $\Pi \models_T \mathbf{X}\,\varphi$ if $\Pi[1,\infty) \models_T \varphi$,
- $\Pi \models_T \mathbf{F}\,\varphi$ if there is a $j \geq 0$ such that $\Pi[j,\infty) \models_T \varphi$,
- $\Pi \models_T \exists\pi.\,\varphi$ if there exists a trace $t \in T$ such that $\Pi[\pi \mapsto t] \models_T \varphi$,
- $\Pi \models_T \forall\pi.\ \varphi$ if for all traces $t \in T$ we have $\Pi[\pi \mapsto t] \models_T \varphi$,
- $\Pi \models_T \exists\mathtt{q}.\ \varphi$ if there exists a trace $t_{\mathtt{q}} \in (2^{\{\mathtt{q}\}})^\omega$ such that $\Pi \models_{T[\mathtt{q}\mapsto t_{\mathtt{q}}]} \varphi$, and
- $\Pi \models_T \forall\mathtt{q}.\ \varphi$ if for all traces $t_{\mathtt{q}} \in (2^{\{\mathtt{q}\}})^\omega$ we have $\Pi \models_{T[\mathtt{q}\mapsto t_{\mathtt{q}}]} \varphi$.

We say that a set T of traces satisfies a sentence φ, if $\Pi_\emptyset \models \varphi$ where $\Pi_\emptyset$ is the variable assignment with empty domain. We then also say that T is a model of φ. A transition system $\mathfrak{T}$ satisfies φ, written $\mathfrak{T} \models \varphi$, if $\mathrm{Tr}(\mathfrak{T}) \models \varphi$.

While it is known that HyperQPTL finite-state satisfiability is Σ^0_1-complete [11], i.e., complete for the recursive-enumerable languages, and HyperQPTL model-checking is Tower-complete [11], the exact complexity of HyperQPTL satisfiability is open: it is only known to be undecidable, as satisfiability is already undecidable for its fragment HyperLTL [4].

3.1 HyperQPTL Satisfiability

In this subsection, we settle the complexity of HyperQPTL satisfiability, showing that it is equivalent to truth in second-order arithmetic: intuitively, quantification over propositions (which, on the semantical level corresponds to quantification over an infinite string over $\{0,1\}$) is equivalent to set quantification: a trace $t \in \{0,1\}^\omega$ is encoded by the set $\{n \in \mathbb{N} \mid t(n) = 1\}$ and vice versa.

To this end, we need to be able to enforce that the model of a HyperQPTL sentence contains enough traces to encode all sets. Recall that $\mathfrak{c}$ denotes the cardinality of the continuum, or equivalently, the cardinality of $(2^{\mathrm{AP}})^\omega$ for each finite AP and the cardinality of $2^{\mathbb{N}}$. As models of HyperQPTL sentences are sets of traces, $\mathfrak{c}$ is a trivial upper bound on the size of models. It is straightforward to show that this upper bound is tight, which also implies that there are sentences whose models allow us to encode all subsets of $\mathbb{N}$.

Theorem 1. *There is a satisfiable* HyperQPTL *sentence that has only models of cardinality* $\mathfrak{c}$.

Proof. Fix $\mathrm{AP} = \{\mathtt{x}, \mathtt{q}\}$ and let $\varphi_{all} = \forall \mathtt{q}.\ \exists\pi.\ \mathbf{G}(\mathtt{q} \leftrightarrow \mathtt{x}_\pi)$, which requires that the $\{\mathtt{x}\}$-projection of any model of φ_{all} is equal to $(2^{\{\mathtt{x}\}})^\omega$. As $(2^{\{\mathtt{x}\}})^\omega$ is uncountable, every model of φ_{all} has cardinality $\mathfrak{c}$. □

This, and the fact that addition and multiplication can be "implemented" in HyperLTL [8] suffice to prove our lower bound on HyperQPTL satisfiability while the upper bound follows from a straightforward encoding of traces by sets of natural numbers [9].

Theorem 2. HyperQPTL *satisfiability is polynomial-time equivalent to truth in second-order arithmetic.*

Proof. We begin with the lower bound by reducing truth in second-order arithmetic to HyperQPTL satisfiability: we present a polynomial-time translation from sentences φ of second-order arithmetic to HyperQPTL sentences φ' such that $(\mathbb{N}, +, \cdot, <, \in) \models \varphi$ if and only if φ' is satisfiable.

We fix $\mathrm{AP} = \{\mathtt{x}, \mathtt{q}, \mathtt{arg1}, \mathtt{arg2}, \mathtt{res}, \mathtt{add}, \mathtt{mult}\}$. Recall that the sentence φ_{all} ensures that the $\{\mathtt{x}\}$-projection of each of its models is equal to $(2^{\{\mathtt{x}\}})^\omega$. Hence, by ignoring the other propositions, we can use traces over AP to encode sets of natural numbers and natural numbers (as singleton sets) and each model contains the encoding of each set of natural number. In our encoding, a trace bound to π encodes a singleton set if and only if the formula $(\neg\mathtt{x}_\pi)\,\mathbf{U}(\mathtt{x}_\pi \land \mathbf{X}\,\mathbf{G}\,\neg\mathtt{x}_\pi)$ is satisfied.

Fortin et al. showed that addition and multiplication can be "implemented" in HyperLTL [8]: Let $T_{(+,\cdot)}$ be the set of all traces $t \in (2^{\mathrm{AP}})^\omega$ such that

- there are unique $n_1, n_2, n_3 \in \mathbb{N}$ with $\mathtt{arg1} \in t(n_1)$, $\mathtt{arg2} \in t(n_2)$, and $\mathtt{res} \in t(n_3)$, and
- either
  - $\mathtt{add} \in t(n)$ and $\mathtt{mult} \notin t(n)$ for all n, and $n_1 + n_2 = n_3$, or
  - $\mathtt{mult} \in t(n)$ and $\mathtt{add} \notin t(n)$ for all n, and $n_1 \cdot n_2 = n_3$.

There is a satisfiable HyperLTL sentence $\varphi_{(+,\cdot)}$ such that the $\{\mathtt{arg1}, \mathtt{arg2}, \mathtt{res}, \mathtt{add}, \mathtt{mult}\}$-projection of every model of $\varphi_{(+,\cdot)}$ is $T_{(+,\cdot)}$ [8, Theorem 5.5]. As HyperLTL is a fragment of HyperQPTL, we can use $\varphi_{(+,\cdot)}$ to construct our desired formula.

Now, given a sentence φ of second-order arithmetic, we define

$$\varphi' = \varphi_{all} \land \varphi_{(+,\cdot)} \land \mathit{hyp}(\varphi)$$

where $\mathit{hyp}(\varphi)$ is defined inductively as follows:

- For second-order variables Y, $\mathit{hyp}(\exists Y.\ \psi) = \exists\pi_Y.\ \mathit{hyp}(\psi)$.
- For second-order variables Y, $\mathit{hyp}(\forall Y.\ \psi) = \forall\pi_Y.\ \mathit{hyp}(\psi)$.
- For first-order variables y,
  $\mathit{hyp}(\exists y.\ \psi) = \exists\pi_y.\ ((\neg\mathtt{x}_{\pi_y})\,\mathbf{U}(\mathtt{x}_{\pi_y} \land \mathbf{X}\,\mathbf{G}\,\neg\mathtt{x}_{\pi_y})) \land \mathit{hyp}(\psi)$,
- For first-order variables y,
  $\mathit{hyp}(\forall y.\ \psi) = \forall\pi_y.\ ((\neg\mathtt{x}_{\pi_y})\,\mathbf{U}(\mathtt{x}_{\pi_y} \land \mathbf{X}\,\mathbf{G}\,\neg\mathtt{x}_{\pi_y})) \rightarrow \mathit{hyp}(\psi)$,
- $\mathit{hyp}(\psi_1 \lor \varphi_2) = \mathit{hyp}(\psi_1) \lor \mathit{hyp}(\psi_2)$,
- $\mathit{hyp}(\neg\psi) = \neg\mathit{hyp}(\psi)$,
- For second-order variables Y and first-order variables y,
  $\mathit{hyp}(y \in Y) = \mathbf{F}(\mathtt{x}_y \land \mathtt{x}_{\pi_Y})$,
- For first-order variables y, y', $\mathit{hyp}(y < y') = \mathbf{F}(\mathtt{x}_y \land \mathbf{X}\,\mathbf{F}\,\mathtt{x}_{y'})$,
- For first-order variables y_1, y_2, y, $\mathit{hyp}(y_1 + y_2 = y) = \exists\pi.\ \mathtt{add}_\pi \land$
  $\mathbf{F}(\mathtt{x}_{\pi_{y_1}} \land \mathtt{arg1}_\pi) \land \mathbf{F}(\mathtt{x}_{\pi_{y_2}} \land \mathtt{arg2}_\pi) \land \mathbf{F}(\mathtt{x}_{\pi_y} \land \mathtt{res}_\pi)$.

- For first-order variables y_1, y_2, y, $hyp(y_1 \cdot y_2 = y) = \exists \pi.\ \mathtt{mult}_\pi \wedge$ $\mathbf{F}(\mathtt{x}_{\pi_{y_1}} \wedge \mathtt{arg1}_\pi) \wedge \mathbf{F}(\mathtt{x}_{\pi_{y_2}} \wedge \mathtt{arg2}_\pi) \wedge \mathbf{F}(\mathtt{x}_{\pi_y} \wedge \mathtt{res}_\pi)$.

While φ' is not in prenex normal form, it can easily be brought into prenex normal form, as there are no quantifiers under the scope of a temporal operator. An induction shows that we indeed have that $(\mathbb{N}, +, \cdot, <, \in) \models \varphi$ if and only if φ' is satisfiable.

For the upper bound, we conversely reduce HyperQPTL satisfiability to truth in second-order arithmetic: we present a polynomial-time translation from HyperQPTL sentences φ to sentences φ' of second-order arithmetic such that φ is satisfiable if and only if $(\mathbb{N}, +, \cdot, <, \in) \models \varphi'$. Here, we assume AP to be fixed, so that we can use $|\mathrm{AP}|$ as a constant in our formulas (which is definable in arithmetic).

Here, we encode traces as sets of natural numbers. To do to, we need to introduce some notation following Frenkel and Zimmermann [9]: Let $pair\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ denote Cantor's pairing function defined as $pair(i,j) = \frac{1}{2}(i+j)(i+j+1)+j$, which is a bijection and can be implemented in arithmetic. Furthermore, we fix a bijection $e\colon \mathrm{AP} \to \{0, 1, \ldots, |\mathrm{AP}|-1\}$. Then, we encode a trace $t \in (2^{\mathrm{AP}})^\omega$ by the set $S_t = \{pair(j, e(\mathtt{p})) \mid j \in \mathbb{N} \text{ and } \mathtt{p} \in t(j)\} \subseteq \mathbb{N}$. Now, one can write a formula $\varphi_{isTrace}(Y)$ which is satisfied in $(\mathbb{N}, +, \cdot, <, \in)$ if and only if the interpretation of Y encodes a trace over AP [9].

Now, given a HyperQPTL sentence over AP, let $\varphi' = ar(\varphi)(0)$, where $ar(\varphi)$ is defined inductively with a free variable i (capturing the position at which the current subformulas is evaluated) as follows:

- $ar(\exists \pi.\ \psi) = \exists Y_\pi.\ \varphi_{isTrace}(Y_\pi) \wedge ar(\psi)$. Here, the free variable of $ar(\exists \pi.\ \psi)$ is the free variable of $ar(\psi)$.
- $ar(\forall \pi.\ \psi) = \forall Y_\pi.\ \varphi_{isTrace}(Y_\pi) \to ar(\psi)$. Here, the free variable of $ar(\forall \pi.\ \psi)$ is the free variable of $ar(\psi)$.
- $ar(\exists \mathtt{q}.\ \psi) = \exists Y_{\mathtt{q}}.\ ar(\psi)$. Here, the free variable of $ar(\exists \mathtt{q}.\ \psi)$ is the free variable of $ar(\psi)$.

  Note that we do not quantify over sets encoding traces in $ar(\exists \mathtt{q}.\ \psi)$ to capture the trace $t_{\mathtt{q}}$ assigned to $\mathtt{q}$, but instead use a *plain* set. This is sufficient, as $t_{\mathtt{q}}$ is a trace over $\{\mathtt{q}\}$, which can be identified by a subset of $\mathbb{N}$.
- $ar(\forall \mathtt{q}.\ \psi) = \forall Y_{\mathtt{q}}.\ ar(\psi)$. Here, the free variable of $ar(\forall \mathtt{q}.\ \psi)$ is the free variable of $ar(\psi)$.
- $ar(\psi_1 \vee \varphi_2) = ar(\psi_1) \vee ar(\psi_2)$. Here, we assume that $ar(\psi_1)$ and $ar(\psi_2)$ have the same free variable (which is then the free variable of $ar(\psi_1 \vee \varphi_2)$), which can always be achieved by renaming variables if necessary.
- $ar(\neg\psi) = \neg ar(\psi)$. Here, the free variable of $ar(\neg\psi)$ is the free variable of $ar(\psi)$.
- $ar(\mathbf{X}\,\psi) = \exists i'(i' = i+1) \wedge ar(\psi)$. Here, i' is the free variable of $ar(\psi)$ and i is the free variable of $ar(\mathbf{X}\,\psi)$.
- $ar(\mathbf{F}\,\psi) = \exists i'(i' \geq i) \wedge ar(\psi)$. Here, i' is the free variable of $ar(\psi)$ and i is the free variable of $ar(\mathbf{F}\,\psi)$.
- $ar(\mathtt{p}_\pi) = pair(i, e(\mathtt{p})) \in Y_\pi$, i.e., i is the free variable of $ar(\mathtt{p}_\pi)$. Here, we use *pair* and e as syntactic sugar, as both are implementable in first-order arithmetic.
- $ar(\mathtt{q}) = i \in Y_{\mathtt{q}}$, i.e., i is the free variable of $ar(\mathtt{q})$.

An induction shows that φ is satisfiable if and only if $(\mathbb{N}, +, \cdot, <, \in) \models \varphi'$. □

4 HyperQPTL$^+$

In HyperQPTL quantification over an proposition $\mathtt{q}$ is interpreted as labeling each trace by the same sequence $t_{\mathtt{q}}$ of truth values for $\mathtt{q}$, i.e., the assignment of truth values is uniform. However, one can also consider a non-uniform labeling by truth values for $\mathtt{q}$. This results in the logic HyperQPTL$^+$.

The syntax of HyperQPTL$^+$ is very similar to that of HyperQPTL, one just drops the atomic formulas of the form $\mathtt{q}$, i.e., atomic propositions that are not labeled by trace variables:

$$\varphi ::= \exists\pi.\ \varphi \mid \forall\pi.\ \varphi \mid \exists\mathtt{q}.\ \varphi \mid \forall\mathtt{q}.\ \varphi \mid \psi$$
$$\psi ::= \mathtt{p}_\pi \mid \neg\psi \mid \psi \vee \psi \mid \mathbf{X}\,\psi \mid \mathbf{F}\,\psi$$

Here $\mathtt{p}$ and $\mathtt{q}$ range over AP and π ranges over $\mathcal{V}$. The semantics are also similar, we just change the definition of propositional quantification as follows:

- $\Pi \models_T \exists\mathtt{q}.\ \varphi$ if there exists a $T' \subseteq (2^{\{\mathrm{AP}\}})^\omega$ such that $T =_{\mathrm{AP}\setminus\{\mathtt{q}\}} T'$ and $\Pi \models_{T'} \varphi$, and
- $\Pi \models_T \forall\mathtt{q}.\ \varphi$ if for all $T' \subseteq (2^{\{\mathrm{AP}\}})^\omega$ such that $T =_{\mathrm{AP}\setminus\{\mathtt{q}\}} T'$ we have $\Pi \models_{T'} \varphi$.

It is known that model-checking HyperQPTL$^+$ is undecidable [5], but its exact complexity is open, as is the complexity of satisfiability and finite-state satisfiability.

In the following, we show that HyperQPTL$^+$ is equally expressive as Hyper^{2}LTL, which allows us to transfer the complexity results for Hyper^{2}LTL to HyperQPTL$^+$.

4.1 Second-order HyperLTL

We begin by introducing the syntax and semantics of Hyper^{2}LTL. Let $\mathcal{V}_1$ be a set of first-order trace variables (i.e., ranging over traces) and $\mathcal{V}_2$ be a set of second-order trace variables (i.e., ranging over sets of traces) such that $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$. We typically use π (possibly with decorations) to denote first-order variables and X, Y, Z (possibly with decorations) to denote second-order variables. Also, we assume the existence of two distinguished second-order variables $X_a, X_d \in \mathcal{V}_2$ such that X_a refers to the set $(2^{\mathrm{AP}})^\omega$ of all traces, and X_d refers to the universe of discourse (the set of traces the formula is evaluated over).

The formulas of Hyper^{2}LTL are given by the grammar

$$\varphi ::= \exists X.\ \varphi \mid \forall X.\ \varphi \mid \exists\pi \in X.\ \varphi \mid \forall\pi \in X.\ \varphi \mid \psi$$
$$\psi ::= \mathtt{p}_\pi \mid \neg\psi \mid \psi \vee \psi \mid \mathbf{X}\,\psi \mid \psi\,\mathbf{U}\,\psi$$

where $\mathtt{p}$ ranges over AP, π ranges over $\mathcal{V}_1$, X ranges over $\mathcal{V}_2$, and $\mathbf{X}$ (next) and $\mathbf{U}$ (until) are temporal operators. A sentence is a formula in which only the variables X_a, X_d can be free. Conjunction ($\wedge$), exclusive disjunction ($\oplus$), implication ($\rightarrow$), and equivalence ($\leftrightarrow$) are defined as usual, and the temporal operators eventually ($\mathbf{F}$) and always ($\mathbf{G}$) are derived as $\mathbf{F}\,\psi = \neg\psi\,\mathbf{U}\,\psi$ and $\mathbf{G}\,\psi = \neg\,\mathbf{F}\,\neg\psi$.

The semantics of Hyper^{2}LTL is defined with respect to a variable assignment, i.e., a partial mapping $\Pi \colon \mathcal{V}_1 \cup \mathcal{V}_2 \rightarrow (2^{\mathrm{AP}})^\omega \cup 2^{(2^{\mathrm{AP}})^\omega}$ such that

- if $\Pi(\pi)$ for $\pi \in \mathcal{V}_1$ is defined, then $\Pi(\pi) \in (2^{\mathrm{AP}})^\omega$ and
- if $\Pi(X)$ for $X \in \mathcal{V}_2$ is defined, then $\Pi(X) \in 2^{(2^{\mathrm{AP}})^\omega}$.

Given a variable assignment Π, a variable $\pi \in \mathcal{V}_1$, and a trace t, we denote by $\Pi[\pi \mapsto t]$ the assignment that coincides with Π on all variables but π, which is mapped to t. Similarly, for a variable $X \in \mathcal{V}_2$, and a set T of traces, $\Pi[X \mapsto T]$ is the assignment that coincides with Π everywhere but X, which is mapped to T. Furthermore, $\Pi[j, \infty)$ denotes the variable assignment mapping every $\pi \in \mathcal{V}_1$ in Π's domain to $\Pi(\pi)(j)\Pi(\pi)(j+1)\Pi(\pi)(j+2)\cdots$, the suffix of $\Pi(\pi)$ starting at position j (the assignment of variables $X \in \mathcal{V}_2$ is not updated).

For a variable assignment Π we define[2]

- $\Pi \models \mathtt{p}_\pi$ if $\mathtt{p} \in \Pi(\pi)(0)$,
- $\Pi \models \neg\psi$ if $\Pi \not\models \psi$,

[2]These are the standard semantics where second-order quantification ranges over arbitrary sets [1], not the closed-world semantics where second-order quantification only ranges over subsets of the model [9].

- $\Pi \models \psi_1 \vee \psi_2$ if $\Pi \models \psi_1$ or $\Pi \models \psi_2$,
- $\Pi \models \mathbf{X}\,\psi$ if $\Pi[1,\infty) \models \psi$,
- $\Pi \models \psi_1 \mathbf{U}\,\psi_2$ if there is a $j \geq 0$ such that $\Pi[j,\infty) \models \psi_2$ and for all $0 \leq j' < j$ we have $\Pi[j',\infty) \models \psi_1$,
- $\Pi \models \exists \pi \in X.\ \varphi$ if there exists a trace $t \in \Pi(X)$ such that $\Pi[\pi \mapsto t] \models \varphi$,
- $\Pi \models \forall \pi \in X.\ \varphi$ if for all traces $t \in \Pi(X)$ we have $\Pi[\pi \mapsto t] \models \varphi$,
- $\Pi \models \exists X.\ \varphi$ if there exists a set $T \subseteq (2^{\mathrm{AP}})^\omega$ such that $\Pi[X \mapsto T] \models \varphi$, and
- $\Pi \models \forall X.\ \varphi$ if for all sets $T \subseteq (2^{\mathrm{AP}})^\omega$ we have $\Pi[X \mapsto T] \models \varphi$.

The variable assignment with empty domain is denoted by $\Pi_\emptyset$. We say that a set T of traces satisfies a Hyper^{2}LTL sentence φ, written $T \models \varphi$, if $\Pi_\emptyset[X_a \mapsto (2^{\mathrm{AP}})^\omega, X_d \mapsto T] \models \varphi$, i.e., if we assign the set of all traces to X_a and the set T to the universe of discourse X_d. In this case, we say that T is a model of φ. A transition system $\mathfrak{T}$ satisfies φ, written $\mathfrak{T} \models \varphi$, if $\mathrm{Tr}(\mathfrak{T}) \models \varphi$.

4.2 HyperQPTL$^+$ "is" Second-order HyperLTL

In this subsection, we show that HyperQPTL$^+$ and Hyper^{2}LTL are equally expressive by translating Hyper^{2}LTL into HyperQPTL$^+$ and vice versa.

Lemma 1. *There is a polynomial-time computable function f mapping* Hyper^{2}LTL *sentences φ to* HyperQPTL$^+$ *sentences $f(\varphi)$ such that we have $T \models \varphi$ if and only if $T \models f(\varphi)$ for all nonempty $T \subseteq (2^{\mathrm{AP}})^\omega$.*

Proof. Let φ be a Hyper^{2}LTL sentence, let $X_1, \ldots, X_k$ be the second-order variables quantified in φ, and let $\{\mathtt{p}_1, \ldots, \mathtt{p}_n\}$ be the propositions appearing in φ. We assume without loss of generality that each (trace and set) variable in φ is quantified exactly once in φ. Further, we require that each X_j is different from X_d and X_a. These properties can always be achieved by renaming variables.

To construct $f(\varphi)$, we use additional propositions that will be quantified in $f(\varphi)$ to simulate set quantification, i.e., the propositions $\mathtt{p}^{all}$ and $\mathtt{p}^{temp}$ for each $\mathtt{p}$ in φ as well as the propositions $\mathtt{m}_j$ for each $j \in \{1, \ldots, k\}$. For the sake of readability, we define $\mathrm{AP}^{allSets} = \{\mathtt{p}_1^{all}, \ldots, \mathtt{p}_n^{all}\}$.

Now, consider the formula

$$f(\varphi) = \exists \mathtt{p}_1^{all}.\ \ldots \exists \mathtt{p}_n^{all}.\ (\psi_{complete} \wedge f'(\varphi))$$

with

$$\psi_{complete} = \forall \mathtt{p}_1^{temp}.\ \ldots \forall \mathtt{p}_n^{temp}.\ \forall \pi.\ \exists \pi'.\ \bigwedge_{i=1}^{n} \mathbf{G}((\mathtt{p}_i^{temp})_\pi \leftrightarrow (\mathtt{p}_i^{all})_{\pi'})$$

and $f'(\varphi)$ defined later. First note that we have $T \models f(\varphi)$ if and only if there exists a T' with $T =_{\mathrm{AP} \setminus \mathrm{AP}^{allSets}} T'$ and $T' \models \psi_{complete} \wedge f'(\varphi)$. Then, $\psi_{complete}$ requires that the $\mathrm{AP}^{allSets}$-projection of T' contains all traces over the fresh propositions in $\mathrm{AP}^{allSets}$ not used in φ. Note that a trace in T' therefore has the form $t^\frown t'$ where t is a trace from T and t' is a trace over $\mathrm{AP}^{allSets}$. Thus quantification of traces over T' mimics both quantification of traces from T and quantification of traces over $\mathrm{AP}^{allSets}$.

Now, we can mimic quantification of subsets of $\{\mathtt{p}_1, \ldots, \mathtt{p}_n\}^\omega$ by instead labeling traces over $\mathrm{AP}^{allSets}$ by a marker that denotes which traces are in the set. However, note that the traces over $\mathrm{AP}^{allSets}$ are obtained as projections of traces over $\mathrm{AP} \supsetneq \mathrm{AP}^{allSets}$. Hence, we need to require that the marking is consistent. This is captured by the formula

$$\psi_{cons}^j = \forall \pi.\ (\mathbf{X}\,\mathbf{G}\,\neg \mathtt{m}_j) \wedge \forall \pi'.\ \bigwedge_{i=1}^{n} (\mathbf{G}((\mathtt{p}_i^{all})_\pi \leftrightarrow (\mathtt{p}_i^{all})_{\pi'})) \rightarrow ((\mathtt{m}_j)_\pi \leftrightarrow (\mathtt{m}_j)_{\pi'})$$

which expresses that the marker $\mathtt{m}_j$ only holds at initial positions and if two traces in T' have the same $\mathrm{AP}^{allSets}$-projection, then they are either both marked or both unmarked.

As mentioned above, when quantifying a set X_j, we mark the traces over $\mathrm{AP}^{allSets}$. Thus, when we quantify a trace π from X_j, we need to use those traces, which requires us to replace each atomic formula of the form $\mathtt{p}_\pi$ by $(\mathtt{p}^{all})_\pi$. Similarly, when we quantify a trace π from X_a, we also need to replace $\mathtt{p}_\pi$ by $(\mathtt{p}^{all})_\pi$ so that π does indeed range over all traces. On the other hand, when we quantify a trace π from X_d, then we do not apply the replacement, as this quantification ranges over the *original* traces (those over $\{\mathtt{p}_1, \ldots, \mathtt{p}_n\}$).

Now, we define f' as

- $f'(\exists X_j.\ \psi) = \exists \mathtt{m}_j.\ \psi^j_{cons} \wedge f'(\psi)$,
- $f'(\forall X_j.\ \psi) = \forall \mathtt{m}_j.\ \psi^j_{cons} \rightarrow f'(\psi)$,
- $f'(\exists \pi \in X_j.\ \psi) = \exists \pi.\ (\mathtt{m}_j)_\pi \wedge f'(\mathit{repl}_\pi(\psi))$, where $\mathit{repl}_\pi(\psi)$ is the formula obtained from ψ by replacing each subformula $\mathtt{p}_\pi$ by $(\mathtt{p}^{all})_\pi$ (note that we only replace propositions labeled by π, the variable quantified here),
- $f'(\forall \pi \in X_j.\ \psi) = \forall \pi.\ (\mathtt{m}_j)_\pi \rightarrow f'(\mathit{repl}_\pi(\psi))$,
- $f'(\exists \pi \in X_a.\ \psi) = \exists \pi.\ f'(\mathit{repl}_\pi(\psi))$,
- $f'(\forall \pi \in X_a.\ \psi) = \forall \pi.\ f'(\mathit{repl}_\pi(\psi))$,
- $f'(\exists \pi \in X_d.\ \psi) = \exists \pi.\ f'(\psi)$,
- $f'(\forall \pi \in X_d.\ \psi) = \forall \pi.\ f'(\psi)$,
- $f'(\neg\psi) = \neg f'(\psi)$,
- $f'(\psi_1 \vee \psi_2) = f'(\psi_1) \vee f'(\psi_2)$,
- $f'(\mathbf{X}\,\psi) = \mathbf{X}\, f'(\psi)$,
- $f'(\mathbf{F}\,\psi) = \mathbf{F}\, f'(\psi)$,
- $f'(\mathtt{p}_\pi) = \mathtt{p}_\pi$, and
- $f'((\mathtt{p}^{all})_\pi) = (\mathtt{p}^{all})_\pi$.

An induction shows that we indeed have $T \models \varphi$ if and only if $T \models f(\varphi)$. □

Now, we show the other direction.

Lemma 2. *There is a polynomial-time computable function f mapping* HyperQPTL$^+$ *sentences φ to* Hyper^{2}LTL *sentences $f(\varphi)$ such that we have $T \models \varphi$ if and only if $T \models f(\varphi)$ for all $T \subseteq (2^{\mathrm{AP}})^\omega$.*

Proof. The semantics of HyperQPTL$^+$ can directly be expressed in Hyper^{2}LTL. To this end, we have a dedicated second-order variable that stores the set of traces that the trace quantifiers in the HyperQPTL$^+$ formula range over (i.e., the T in $\Pi \models_T \psi$). This set is initially equal to the set of traces the formula is evaluated over (i.e., the set $\Pi(X_d)$ in the setting of Hyper^{2}LTL), and is updated with each quantification over a proposition $\mathtt{q}$. As this update from T to T' has to satisfy $T =_{\mathrm{AP}\setminus\{\mathtt{q}\}} T'$, we need two set variables, one for the old value and one for the new value, to be able to compare these two. As the old value is no longer needed after the update, we can then reuse the variable.

Formally, we define $X_0 = X_d$ and let X_1 be some second-order variable other than X_d. Now, let

$$\begin{aligned}\psi_{\mathtt{q}}(X_b, X_{1-b}) = \forall \pi \in X_b.\ \exists \pi' \in X_{1-b}.\ &\bigwedge_{\mathtt{p}\in\mathrm{AP}\setminus\{\mathtt{q}\}} \mathbf{G}(\mathtt{p}_\pi \leftrightarrow \mathtt{p}_{\pi'}) \wedge \\ \forall \pi \in X_{1-b}.\ \exists \pi' \in X_b.\ &\bigwedge_{\mathtt{p}\in\mathrm{AP}\setminus\{\mathtt{q}\}} \mathbf{G}(\mathtt{p}_\pi \leftrightarrow \mathtt{p}_{\pi'}),\end{aligned}$$

which is satisfied by a variable assignment Π if and only if $\Pi(X_b) =_{\mathrm{AP}\setminus\{\mathtt{q}\}} \Pi(X_{1-b})$.

Now, we express the semantics of HyperQPTL$^+$ in Hyper^{2}LTL. Here, $b \in \{0, 1\}$ is a flag that keeps track of which of the two variables X_0 and X_1 is currently used to store the set of traces we are evaluating trace quantifiers over. We define

- $f'(\exists\pi.\ \psi, b) = \exists\pi \in X_b.\ f'(\psi, b)$,
- $f'(\forall\pi.\ \psi, b) = \forall\pi \in X_b.\ f'(\psi, b)$,
- $f'(\exists\mathtt{q}.\ \psi, b) = \exists X_{1-b}.\ (\psi_{\mathtt{q}}(X_{1-b}, X_b)) \wedge f'(\psi, 1-b)$,
- $f'(\forall\mathtt{q}.\ \psi, b) = \forall X_{1-b}.\ (\psi_{\mathtt{q}}(X_{1-b}, X_b)) \rightarrow f'(\psi, 1-b)$,
- $f'(\neg\psi, b) = \neg f'(\psi, b)$,
- $f'(\psi_1 \vee \psi_2, b) = f'(\psi_1, b) \vee f'(\psi_2, b)$,
- $f'(\mathbf{X}\,\psi, b) = \mathbf{X}\, f'(\psi, b)$,
- $f'(\mathbf{F}\,\psi, b) = \mathbf{F}\, f'(\psi, b)$, and
- $f'(\mathtt{p}_\pi, b) = \mathtt{p}_\pi$.

Now, define $f(\varphi) = f(\varphi, 0)$. An induction shows that we indeed have $T \models \varphi$ if and only if $T \models f(\varphi)$. $\square$

As Hyper^{2}LTL satisfiability, finite-state satisfiability, and model-checking are equivalent to truth in third-order arithmetic, the translations presented in Lemma 1 and Lemma 2 imply that the same is true for HyperQPTL$^+$.

Theorem 3. HyperQPTL$^+$ *satisfiability, finite-state satisfiability, and model-checking are equivalent to truth in third-order arithmetic.*

Let us remark that two second-order variables suffice to translate HyperQPTL$^+$ into Hyper^{2}LTL. Together with the converse translation presented in Lemma 1, we conclude that every Hyper^{2}LTL sentence is equivalent to one with only two second-order variables.

5 Conclusion

We have settled the exact complexity of the most important verification problems for HyperQPTL and HyperQPTL$^+$. For HyperQPTL, we proved that satisfiability is equivalent to truth in second-order arithmetic while for HyperQPTL$^+$, we proved that satisfiability, finite-state satisfiability, and model-checking are equivalent to truth in third-order arithmetic. The latter results were obtained by showing that HyperQPTL$^+$ and second-order HyperLTL have the same expressiveness.

Acknowledgments Supported by DIREC – Digital Research Centre Denmark.

References

[1] Raven Beutner, Bernd Finkbeiner, Hadar Frenkel, and Niklas Metzger. Second-order hyperproperties. In Constantin Enea and Akash Lal, editors, *CAV 2023, Part II*, volume 13965 of *LNCS*, pages 309–332. Springer, 2023.

[2] Michael R. Clarkson, Bernd Finkbeiner, Masoud Koleini, Kristopher K. Micinski, Markus N. Rabe, and César Sánchez. Temporal logics for hyperproperties. In Martín Abadi and Steve Kremer, editors, *POST 2014*, volume 8414 of *LNCS*, pages 265–284. Springer, 2014.

[3] Michael R. Clarkson and Fred B. Schneider. Hyperproperties. *J. Comput. Secur.*, 18(6):1157–1210, 2010.

[4] Bernd Finkbeiner and Christopher Hahn. Deciding hyperproperties. In Josée Desharnais and Radha Jagadeesan, editors, *CONCUR 2016*, volume 59 of *LIPIcs*, pages 13:1–13:14. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2016.

[5] Bernd Finkbeiner, Christopher Hahn, Jana Hofmann, and Leander Tentrup. Realizing omega-regular hyperproperties. In Shuvendu K. Lahiri and Chao Wang, editors, *CAV 2020, Part II*, volume 12225 of *LNCS*, pages 40–63. Springer, 2020.

[6] Bernd Finkbeiner, Markus N. Rabe, and César Sánchez. Algorithms for Model Checking HyperLTL and HyperCTL*. In Daniel Kroening and Corina S. Pasareanu, editors, *CAV 2015, Part I*, volume 9206 of *LNCS*, pages 30–48. Springer, 2015.

[7] Marie Fortin, Louwe B. Kuijer, Patrick Totzke, and Martin Zimmermann. HyperLTL satisfiability is Σ_1^1-complete, HyperCTL* satisfiability is Σ_1^2-complete. In Filippo Bonchi and Simon J. Puglisi, editors, *MFCS 2021*, volume 202 of *LIPIcs*, pages 47:1–47:19. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2021.

[8] Marie Fortin, Louwe B. Kuijer, Patrick Totzke, and Martin Zimmermann. HyperLTL satisfiability is highly undecidable, HyperCTL* is even harder. *arXiv*, 2303.16699, 2023. Journal version of [7]. Under submission.

[9] Hadar Frenkel and Martin Zimmermann. The complexity of second-order HyperLTL. In Jörg Endrullis and Sylvain Schmitz, editors, *CSL 2025*, volume 326 of *LIPIcs*. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2025. To appear. Available online as arXiv:2311.15675.

[10] Corto Mascle and Martin Zimmermann. The keys to decidable HyperLTL satisfiability: Small models or very simple formulas. In Maribel Fernández and Anca Muscholl, editors, *CSL 2020*, volume 152 of *LIPIcs*, pages 29:1–29:16. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2020.

[11] Markus N. Rabe. *A temporal logic approach to information-flow control.* PhD thesis, Saarland University, 2016.

[12] Gaëtan Regaud and Martin Zimmermann. The complexity of fragments of second-order HyperLTL, 2025. Under preparation.

[13] A.P̃rasad Sistla. *Theoretical Issues in the Design and Verification of Distributed Systems.* PhD thesis, Harvard University, 1983.