

History-deterministic Parikh Automata

Enzo Erlich

ENS Rennes, France

`enzo.erlich@ens-rennes.fr`

Shibashis Guha

Tata Institute of Fundamental Research, Mumbai, India

`shibashis.guha@tifr.res.in`

Ismaël Jecker

University of Warsaw, Poland

`ismael.jecker@gmail.com`

Karoliina Lehtinen

CNRS, Aix-Marseille University and University of Toulon, LIS, Marseille, France

`lehtinen@lis-lab.fr`

Martin Zimmermann

University of Aalborg, Denmark

`mzi@cs.aau.dk`

Abstract

Parikh automata extend finite automata by counters that can be tested for membership in a semilinear set, but only at the end of a run. Thereby, they preserve many of the desirable properties of finite automata. Deterministic Parikh automata are strictly weaker than nondeterministic ones, but enjoy better closure and algorithmic properties.

This state of affairs motivates the study of intermediate forms of nondeterminism. Here, we investigate history-deterministic Parikh automata, i.e., automata whose nondeterminism can be resolved on the fly. This restricted form of nondeterminism is well-suited for applications which classically call for determinism, e.g., solving games and composition.

We show that history-deterministic Parikh automata are strictly more expressive than deterministic ones, incomparable to unambiguous ones, and enjoy almost all of the closure properties of deterministic automata.

1 Introduction

Some of the most profound (and challenging) questions of theoretical computer science are concerned with the different properties of deterministic and nondeterministic computation, the P vs. NP problem being arguably the most important and surely the most well-known one. However, even in the more modest setting of automata theory, there is a tradeoff between deterministic and nondeterministic automata with far-reaching consequences for, e.g., the automated verification of finite-state systems. In the automata-based approach to model checking, for example, one captures a finite-state system \mathcal{S} and a specification φ by automata $\mathcal{A}_{\mathcal{S}}$ and \mathcal{A}_{φ} and then checks whether $L(\mathcal{A}_{\mathcal{S}}) \subseteq L(\mathcal{A}_{\varphi})$ holds, i.e., whether every execution of \mathcal{S} satisfies the specification φ . To do so, one tests $L(\mathcal{A}_{\mathcal{S}}) \cap \overline{L(\mathcal{A}_{\varphi})}$ for emptiness. Hence, one is interested in expressive automata models that have good closure and algorithmic properties. Nondeterminism yields conciseness (think DFA's

vs. NFA's) or even more expressiveness (think pushdown automata) while deterministic automata often have better algorithmic properties and better closure properties (again, think, e.g., pushdown automata).

Limited forms of nondeterminism constitute an appealing middle ground as they often combine the best of both worlds, e.g., increased expressiveness in comparison to deterministic automata and better algorithmic and closure properties than nondeterministic ones. A classical, and well-studied, example are unambiguous automata, i.e., nondeterministic automata that have at most one accepting run for every input. For example, unambiguous finite automata can be exponentially smaller than deterministic ones while unambiguous pushdown automata are more expressive than deterministic ones [26].

Another restricted class of nondeterministic automata is that of residual automata [12], automata where every state accepts a residual language of the automaton's language. For every regular language there exists a residual automaton. While there exist residual automata that can be exponentially smaller than DFA, there also exist languages for which NFA can be exponentially smaller than residual automata [12].

More recently, another notion of limited nondeterminism has received considerable attention: history-deterministic automata [9, 25]¹ are nondeterministic automata whose nondeterminism can be resolved based on the run constructed thus far, but independently of the remainder of the input. This property makes history-deterministic automata suitable for the composition with games, trees, and other automata, applications which classically require deterministic automata. History-determinism has been studied in the context of regular [1, 25, 30], pushdown [23, 31], quantitative [3, 9], and timed automata [24]. For automata that can be determinized, history-determinism offers the potential for succinctness (e.g., co-Büchi automata [30]) while for automata that cannot be determinized, it even offers the potential for increased expressiveness (e.g., pushdown automata [23, 31]). In the quantitative setting, the exact power of history-determinism depends largely on the type of quantitative automata under consideration. So far, it has been studied for quantitative automata in which runs accumulate weights into a value using a value function such as **Sum**, **LimInf**, **Average**, and that assign to a word the supremum among the values of its runs. For these automata, history-determinism turns out to have interesting applications for quantitative synthesis [2]. Here, we continue this line of work by investigating history-deterministic Parikh automata, a mildly quantitative form of automata.

Parikh automata, introduced by Klaedtke and Rueß [29], consist of finite automata, augmented with counters that can only be incremented. A Parikh automaton only accepts a word if the final counter-configuration is within a semilinear set specified in the automaton. As the counters do not interfere with the control flow of the automaton, that is, counter values do not affect whether transitions are enabled, they allow for mildly quantitative computations without the full power of vector addition systems or other more powerful models.

For example the language of words over the alphabet $\{0, 1\}$ having a prefix with strictly more 1's than 0's is accepted by a Parikh automaton that starts by counting the number of 0's and 1's and after some prefix nondeterministically stops counting during the processing of the input. It accepts if the counter counting the 1's is, at the end of the run, indeed larger than the counter counting the 0's. Note that the nondeterministic choice can be made based on the word processed so far, i.e., as soon as a prefix with more 1's than 0's is encountered, the counting is stopped. Hence, the automaton described above is in fact history-deterministic.

Klaedtke and Rueß [29] showed Parikh automata to be expressively equivalent to a quantitative version of existential weak MSO that allows for reasoning about set cardinalities. Their expressiveness also coincides with that of reversal-bounded counter machines [29], in which counters can go from decrementing to incrementing only a bounded number of times, but in which counters affect control flow [27]. The weakly unambiguous restriction of Parikh automata, that is, those that have at most one accepting run, on the other hand, coincide with unambiguous reversal-bounded counter machines [4]. Parikh automata are also expressively equivalent to weighted finite automata over the groups $(\mathbb{Z}^k, +, 0)$ [11, 33] for $k \geq 1$. This shows that Parikh automata accept a natural class of quantitative specifications.

Despite their expressiveness, Parikh automata retain some decidability: nonemptiness, in particular, is NP-complete [14]. For weakly unambiguous Parikh automata, inclusion [7] and regular separability [8] are

¹There is a closely related notion, good-for-gameness, which is often, but not always equivalent [2] (despite frequently being used interchangeably in the past).

decidable as well. Figueira and Libkin [14] also argued that this model is well-suited for querying graph databases, while mitigating some of the complexity issues related with more expressive query languages. Further, they have been used in the model checking of transducer properties [16].

As Parikh automata have been established as a robust and useful model, many variants thereof exist: pushdown (visibly [10] and otherwise [35]), two-way with [10] and without stack [15], unambiguous [6], and weakly unambiguous [4] Parikh automata, to name a few.

Our contribution We introduce history-deterministic Parikh automata (HDPa) and study their expressiveness, their closure properties, and their algorithmic properties.

Our main result shows that history-deterministic Parikh automata are more expressive than deterministic ones (DPA), but less expressive than nondeterministic ones (PA). Furthermore, we show that they are of incomparable expressiveness to both classes of unambiguous Parikh automata found in the literature, but equivalent to history-deterministic reversal-bounded counter machines, another class of history-deterministic automata that is studied here for the first time. These results show that history-deterministic Parikh automata indeed constitute a novel class of languages capturing quantitative features.

Secondly, we show that history-deterministic Parikh automata satisfy almost the same closure properties as deterministic ones, the only difference being non-closure under complementation. This result has to be contrasted with unambiguous Parikh automata being closed under complement [6]. Thus, history-determinism is a too strong form of nondeterminism to preserve closure under complementation, a phenomenon that has already been observed in the case of pushdown automata [23, 31].

Finally, we study the algorithmic properties of history-deterministic Parikh automata. Most importantly, safety model checking for HDPa is decidable, as it is for PA. The problem asks, given a system and a set of *bad* prefixes specified by an automaton, whether the system has an execution that has a bad prefix. This allows, for example, to check properties of an arbiter of some shared resource like “the accumulated waiting time between requests and responses of client 1 is always at most twice the accumulated waiting time for client 2 and vice versa”. Note that this property is not ω -regular.

Non-emptiness and finiteness are also both decidable for HDPa (as they are for nondeterministic automata), but universality, inclusion, equivalence, and regularity are not. This is in stark contrast to unambiguous Parikh automata (and therefore also deterministic ones), for which *all* of these problems are decidable. Finally, we show that it is undecidable whether a Parikh automaton is history-deterministic and whether it is equivalent to a history-deterministic one.

Note that we consider only automata over finite words here, but many of our results can straightforwardly be transferred to Parikh automata over infinite words, introduced independently by Guha et al. [22] and Grobler et al. [19, 20].

All proofs omitted due to space restrictions can be found in the appendix.

2 Definitions

An alphabet is a finite nonempty set Σ of letters. As usual, ε denotes the empty word, Σ^* denotes the set of finite words over Σ , Σ^+ denotes the set of finite nonempty words over Σ , and Σ^ω denotes the set of infinite words over Σ . The length of a finite word w is denoted by $|w|$ and, for notational convenience, we define $|w| = \infty$ for all infinite words w . Finally, $|w|_a$ denotes the number of occurrences of the letter a in a finite word w .

Semilinear Sets We denote the set of nonnegative integers by \mathbb{N} . Given vectors $\vec{v} = (v_0, \dots, v_{d-1}) \in \mathbb{N}^d$ and $\vec{v}' = (v'_0, \dots, v'_{d'-1}) \in \mathbb{N}^{d'}$, we define their concatenation $\vec{v} \cdot \vec{v}' = (v_0, \dots, v_{d-1}, v'_0, \dots, v'_{d'-1}) \in \mathbb{N}^{d+d'}$. We lift the concatenation of vectors to sets $D \subseteq \mathbb{N}^d$ and $D' \subseteq \mathbb{N}^{d'}$ via $D \cdot D' = \{\vec{v} \cdot \vec{v}' \mid \vec{v} \in D \text{ and } \vec{v}' \in D'\}$.

Let $d \geq 1$. A set $C \subseteq \mathbb{N}^d$ is linear if there are vectors $\vec{v}_0, \dots, \vec{v}_k \in \mathbb{N}^d$ such that

$$C = \left\{ \vec{v}_0 + \sum_{i=1}^k c_i \vec{v}_i \mid c_i \in \mathbb{N} \text{ for } i = 1, \dots, k \right\}.$$

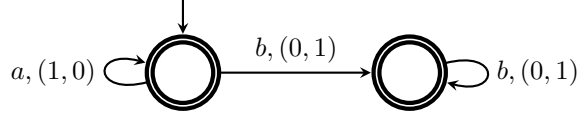


Figure 1: The automaton for Example 2.

Furthermore, a subset of \mathbb{N}^d is semilinear if it is a finite union of linear sets.

Example 1. *The sets $\{(n, n) \mid n \in \mathbb{N}\} = \{(0, 0) + c(1, 1) \mid c \in \mathbb{N}\}$ and $\{(n, 2n) \mid n \in \mathbb{N}\} = \{(0, 0) + c(1, 2) \mid c \in \mathbb{N}\}$ are linear, so their union is semilinear. Further, the set $\{(n, n') \mid n < n'\} = \{(0, 1) + c_1(1, 1) + c_2(0, 1) \mid c_1, c_2 \in \mathbb{N}\}$ is linear and thus also semilinear.*

Proposition 1 ([18]). *If $C, C' \subseteq \mathbb{N}^d$ are semilinear, then so are $C \cup C'$, $C \cap C'$, $\mathbb{N}^d \setminus C$, as well as $\mathbb{N}^d \cdot C$ and $C \cdot \mathbb{N}^d$ for every $d' \geq 1$.*

Finite Automata A (nondeterministic) finite automaton (NFA) $\mathcal{A} = (Q, \Sigma, q_I, \Delta, F)$ over Σ consists of the finite set Q of states containing the initial state q_I , the alphabet Σ , the transition relation $\Delta \subseteq Q \times \Sigma \times Q$, and the set $F \subseteq Q$ of accepting states. The NFA is deterministic (i.e., a DFA) if for every state $q \in Q$ and every letter $a \in \Sigma$, there is at most one $q' \in Q$ such that (q, a, q') is a transition of \mathcal{A} .

A run of \mathcal{A} is a (possibly empty) sequence $(q_0, a_0, q_1)(q_1, a_1, q_2) \cdots (q_{n-1}, a_{n-1}, q_n)$ of transitions with $q_0 = q_I$. It processes the word $a_0 a_1 \cdots a_{n-1} \in \Sigma^*$. The run is accepting if it is either empty and the initial state is accepting or if it is nonempty and q_n is accepting. The language $L(\mathcal{A})$ of \mathcal{A} contains all finite words $w \in \Sigma^*$ such that \mathcal{A} has an accepting run processing w .

Parikh Automata Let Σ be an alphabet, $d \geq 1$, and D a finite subset of \mathbb{N}^d . Furthermore, let $w = (a_0, \vec{v}_0) \cdots (a_{n-1}, \vec{v}_{n-1})$ be a word over $\Sigma \times D$. The Σ -projection of w is $p_\Sigma(w) = a_0 \cdots a_{n-1} \in \Sigma^*$ and its *extended Parikh image* is $\Phi_e(w) = \sum_{j=0}^{n-1} \vec{v}_j \in \mathbb{N}^d$ with the convention $\Phi_e(\varepsilon) = \vec{0}$, where $\vec{0}$ is the d -dimensional zero vector.

A Parikh automaton (PA) is a pair (\mathcal{A}, C) such that \mathcal{A} is an NFA over $\Sigma \times D$ for some input alphabet Σ and some finite $D \subseteq \mathbb{N}^d$ for some $d \geq 1$, and $C \subseteq \mathbb{N}^d$ is semilinear. The language of (\mathcal{A}, C) consists of the Σ -projections of words $w \in L(\mathcal{A})$ whose extended Parikh image is in C , i.e.,

$$L(\mathcal{A}, C) = \{p_\Sigma(w) \mid w \in L(\mathcal{A}) \text{ with } \Phi_e(w) \in C\}.$$

The Parikh automaton (\mathcal{A}, C) is deterministic if for every state q of \mathcal{A} and every $a \in \Sigma$, there is at most one pair $(q', \vec{v}) \in Q \times D$ such that $(q, (a, \vec{v}), q')$ is a transition of \mathcal{A} . Note that this definition does *not* coincide with \mathcal{A} being a DFA: As mentioned above, \mathcal{A} accepts words over $\Sigma \times D$ while (\mathcal{A}, C) accepts words over Σ . Therefore, determinism is defined with respect to Σ only.

Note that the above definition of $L(\mathcal{A}, C)$ coincides with the following alternative definition via accepting runs: A run ρ of (\mathcal{A}, C) is a run

$$\rho = (q_0, (a_0, \vec{v}_0), q_1)(q_1, (a_1, \vec{v}_1), q_2) \cdots (q_{n-1}, (a_{n-1}, \vec{v}_{n-1}), q_n)$$

of \mathcal{A} . We say that ρ *processes* the word $a_0 a_1 \cdots a_{n-1} \in \Sigma^*$, i.e., the \vec{v}_j are ignored, and that ρ 's extended Parikh image is $\sum_{j=0}^{n-1} \vec{v}_j$. The run is accepting if it is either empty and both the initial state of \mathcal{A} is accepting and the zero vector (the extended Parikh image of the empty run) is in C , or if it is nonempty, q_n is accepting, and ρ 's extended Parikh image is in C . Finally, (\mathcal{A}, C) accepts $w \in \Sigma^*$ if it has an accepting run processing w .

Example 2. *Consider the deterministic PA (\mathcal{A}, C) with \mathcal{A} in Figure 1 and $C = \{(n, n) \mid n \in \mathbb{N}\} \cup \{(n, 2n) \mid n \in \mathbb{N}\}$ (cf. Example 1). It accepts the language $\{a^n b^n \mid n \in \mathbb{N}\} \cup \{a^n b^{2n} \mid n \in \mathbb{N}\}$.*

3 History-deterministic Parikh Automata

In this section, we introduce history-deterministic Parikh automata and give examples.

Let (\mathcal{A}, C) be a PA with $\mathcal{A} = (Q, \Sigma \times D, q_I, \Delta, F)$. For a function $r: \Sigma^+ \rightarrow \Delta$ we define its iteration $r^*: \Sigma^* \rightarrow \Delta^*$ via $r^*(\varepsilon) = \varepsilon$ and $r^*(a_0 \cdots a_n) = r^*(a_0 \cdots a_{n-1}) \cdot r(a_0 \cdots a_n)$. We say that r is a resolver for (\mathcal{A}, C) if, for every $w \in L(\mathcal{A}, C)$, $r^*(w)$ is an accepting run of (\mathcal{A}, C) processing w . Further, we say that (\mathcal{A}, C) is history-deterministic (i.e., an HDPA) if it has a resolver.

Example 3. Fix $\Sigma = \{0, 1\}$ and say that a word $w \in \Sigma^*$ is non-Dyck if $|w|_0 < |w|_1$. We consider the language $N \subseteq \Sigma^+$ of words that have a non-Dyck prefix. It is accepted by the PA (\mathcal{A}, C) where \mathcal{A} is depicted in Figure 2 and $C = \{(n, n') \mid n < n'\}$ (cf. Example 1). Intuitively, in the initial state q_c , the automaton counts the number of 0's and 1's occurring in some prefix, nondeterministically decides to stop counting by moving to q_n (this is the only nondeterminism in \mathcal{A}), and accepts if there are more 1's than 0's in the prefix.

The nondeterministic choice can be made only based on the prefix processed so far, i.e., as soon as the first non-Dyck prefix is encountered, the resolver proceeds to state q_n , thereby ending the prefix. Formally, the function

$$wb \mapsto \begin{cases} (q_c, (b, (1-b, b)), q_c) & \text{if } wb \text{ has no non-Dyck prefix,} \\ (q_c, (b, (1-b, b)), q_n) & \text{if } wb \text{ is non-Dyck, but } w \text{ has no non-Dyck prefix,} \\ (q_n, (b, (0, 0)), q_n) & \text{if } w \text{ has a non-Dyck prefix,} \end{cases}$$

is a resolver for (\mathcal{A}, C) .

Remark 1. As a resolver resolves nondeterminism and a DPA has no nondeterminism to resolve, every DPA is history-deterministic.

4 Expressiveness

In this section, we study the expressiveness of HDPA by comparing them to related automata models, e.g., deterministic and nondeterministic Parikh automata, unambiguous Parikh automata (capturing another restricted notion of nondeterminism), and reversal-bounded counter machines (which are known to be related to Parikh automata). Overall, we obtain the relations shown in Figure 3, where the additional classes of languages and the separating languages will be introduced throughout this section.

We begin by stating and proving a pumping lemma for HDPA. The basic property used here, just as for the pumping lemmata for PA and DPA [5], is that shuffling around cycles of a run does not change whether it is accepting or not, as acceptance only depends on the last state of the run being accepting and the vectors (and their multiplicity) that appear on the run, but not the order of their appearance.

Lemma 1. Let (\mathcal{A}, C) be an HDPA with $L(\mathcal{A}, C) \subseteq \Sigma^*$. Then, there exist $p, \ell \in \mathbb{N}$ such that every $w \in \Sigma^*$ with $|w| > \ell$ can be written as $w = uvxvz$ such that

- $0 < |v| \leq p$, $|x| > p$, and $|uvxv| \leq \ell$, and
- for all $z' \in \Sigma^*$: if $uvxvz' \in L(\mathcal{A}, C)$, then also $uv^2xz' \in L(\mathcal{A}, C)$ and $uxv^2z' \in L(\mathcal{A}, C)$.

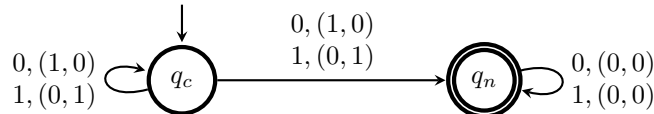


Figure 2: The automaton for Example 3

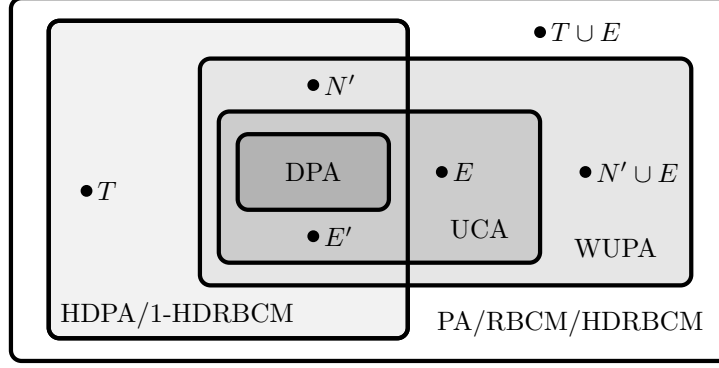


Figure 3: The classes of languages accepted by different models of Parikh automata.

Proof. Fix some resolver r for (\mathcal{A}, C) . Note that the definition of a resolver only requires $r^*(w)$ to be a run processing w for those $w \in L(\mathcal{A}, C)$. Here, we assume without loss of generality that $r^*(w)$ is a run processing w for each $w \in \Sigma^*$. This can be achieved by completing \mathcal{A} (by adding a nonaccepting sink state and transitions to the sink where necessary) and redefining r where necessary (which is only the case for inputs that cannot be extended to a word in $L(\mathcal{A}, C)$).

A cycle is a nonempty finite run infix

$$(q_0, a_0, q_1)(q_1, a_1, q_2) \cdots (q_{n-1}, a_{n-1}, q_n)(q_n, a_n, q_0)$$

starting and ending in the same state and such that the q_j are pairwise different. Now, let p be the number of states of \mathcal{A} and let m be the number of cycles of \mathcal{A} . Note that every run infix containing at least p transitions contains a cycle.

We define $\ell = (p + 1)(2m + 1)$, consider a word $w \in \Sigma^*$ with $|w| > \ell$, and let $\rho = r^*(w)$ be the run of \mathcal{A} induced by r which processes w . We split ρ into $\rho_0\rho_1 \cdots \rho_{2m}\rho'$ such that each ρ_j contains $p + 1$ transitions. Then, each ρ_j contains a cycle and there are j_0, j_1 with $j_1 > j_0 + 1$ such that ρ_{j_0} and ρ_{j_1} contain the same cycle. Now, let

- ρ_v be the cycle in ρ_{j_0} and ρ_{j_1} ,
- ρ_u be the prefix of ρ before the first occurrence of ρ_v in ρ_{j_0} , and
- ρ_x be the infix of ρ between the first occurrences of ρ_v in ρ_{j_0} and ρ_{j_1} .

Furthermore, let $u, v, x \in \Sigma^*$ be the inputs processed by ρ_u, ρ_v , and ρ_x respectively. Then, we indeed have $0 < |v| \leq p$ (as we consider simple cycles), $|x| > p$ (as $j_1 > j_0 + 1$), and $|uvxv| \leq \ell$.

Note that $\rho_u\rho_v\rho_x\rho_v$, $\rho_u\rho_v^2\rho_x$, and $\rho_u\rho_x\rho_v^2$ are all runs of \mathcal{A} which process $uvxv$, uv^2x , and uxv^2 respectively. Furthermore, all three runs end in the same state and their extended Parikh images are equal, as we only shuffled pieces around.

Now, consider some z' such that $uvxz' \in L(\mathcal{A}, C)$. Then $r^*(uvxz')$ is an accepting run, and of the form $\rho_u\rho_v\rho_x\rho_v\rho_{z'}$ for some $\rho_{z'}$ processing z' . Now, $\rho_u\rho_v^2\rho_x\rho_{z'}$, and $\rho_u\rho_x\rho_v^2\rho_{z'}$ are accepting runs of (\mathcal{A}, C) (although not necessarily induced by r) processing uv^2xz' and uxv^2z' , respectively. Thus, $uv^2xz' \in L(\mathcal{A}, C)$ and $uxv^2z' \in L(\mathcal{A}, C)$. \square

It is instructive to compare our pumping lemma for HDPa to those for PA and DPA [5]:

- The pumping lemma for PA states that every *long* word accepted by a PA can be decomposed into $uvxz$ as above such that both uv^2xz and uxv^2z are accepted as well. This statement is weaker than ours, as it only applies to the two words obtained by moving a v while our pumping lemma applies to any suffix z' . This is possible, as the runs of an HDPa on words of the form $uvxz'$ (for fixed uvx),

induced by a resolver, all coincide on their prefixes processing $uvxv$. This is not necessarily the case in PA.

- The pumping lemma for DPA states that every *long* word (not necessarily accepted by the automaton) can be decomposed into $uvxvz$ as above such that $uvxv$, uv^2x , and uxv^2 are all equivalent with respect to the Myhill-Nerode equivalence. This statement is stronger than ours, as Myhill-Nerode equivalence is concerned both with the language of the automaton and its complement. But similarly to our pumping lemma, the one for DPA applies to all possible suffixes z' .

Now, we apply the pumping lemma to compare the expressiveness of HDPa, DPA, and PA.

Theorem 1. *HDPa are more expressive than DPA, but less expressive than PA.*

Proof. First, we consider the separation between DPA and HDPa. The language N from Example 3, which is accepted by an HDPa, is known to be not accepted by any DPA: DPA are closed under complementation [29] while the complement of N is not even accepted by any PA [6].

To show that PA are more expressive than HDPa, consider the language $E = \{a, b\}^* \cdot \{a^n b^n \mid n > 0\}$, which can easily be seen to be accepted by a PA. We show that E is not accepted by any HDPa² via an application of the pumping lemma.

To this end, assume there is some HDPa (\mathcal{A}, C) accepting E , and let p, ℓ as in the pumping lemma. We pick $w = (a^{p+1}b^{p+1})^\ell$, which we decompose as $uvxvz$ with the properties guaranteed by the pumping lemma. In particular, we have $|v| \leq p$, therefore $v \in a^*b^* + b^*a^*$. We consider two cases depending on the last letter of v . In each one, we show the existence of a word z' such that the word $uvxvz'$ is in the language E , yet either uv^2xz' or uxv^2z' is not. This yields the desired contradiction to the pumping lemma.

1. First, assume that the last letter of v is an a . Since $|x| > p$ and x appears between two copies of v in $(a^{p+1}b^{p+1})^\ell$, the infix xv contains at least one full b -block: we have $xv = x'b^{p+1}a^k$ with $x' \in \{a, b\}^*$ and $0 < k \leq p+1$. We set $z' = a^{p+1-k}b^{p+1}$. Hence, $uvxvz' = uvx'b^{p+1}a^{p+1}b^{p+1} \in E$. We show that $uv^2xz' \notin E$ by differentiating two cases:
 - (a) If $v = a^i$ for some i , which must satisfy $0 < i \leq k$, then uv^2xz' is not in E as it ends with $b^{p+1}a^{p+1-i}b^{p+1}$.
 - (b) Otherwise, we must have $v = b^i a^k$ with $0 < i < p$. Then, uv^2xz' is not in E as it ends with $b^{p+1-i}a^{p+1-k}b^{p+1}$.
2. Otherwise, the last letter of v is a b . Since $|x| > p$ and x appears between two copies of v in $(a^{p+1}b^{p+1})^\ell$, the infix xv contains at least one full a -block: we have $xv = x'a^{p+1}b^k$ with $x' \in \{a, b\}^*$ and $0 < k \leq p+1$. This time we set $z' = b^{p+1-k}$. Thus, $uvxvz' = uvx'a^{p+1}b^{p+1} \in E$, and we differentiate two cases to show that $uxv^2z' \notin E$:
 - (a) If $v = b^i$ for some i , which must satisfy $0 < i \leq p$, then uxv^2z' ends with b^{p+i+1} . However, each of its a -blocks has length $p+1$, as moving $v = b^i$ with $i \leq p$ does not merge any a -blocks. Hence, uxv^2z' is not in E .
 - (b) Otherwise, we must have $v = a^i b^k$ with $0 < i < p$. Then, uxv^2z' is not in E as it ends with $v^2z' = a^i b^k a^i b^{p+1}$. \square

4.1 History-determinism vs. Unambiguity

After having placed history-deterministic Parikh automata strictly between deterministic and nondeterministic ones, we now compare them to unambiguous Parikh automata, another class of automata whose expressiveness lies strictly between that of DPA and PA. In the literature, there are two (nonequivalent) forms of unambiguous Parikh automata. We consider both of them here.

²Note that the related language $\{a, b\}^* \cdot \{a^n \# a^n \mid n \in \mathbb{N}\}$ is not accepted by any DPA [5].

Cadilhac et al. studied unambiguity in Parikh automata in the guise of unambiguous constrained automata (UCA) [6]. Constrained automata are a related model and effectively equivalent to PA. Intuitively, an UCA (\mathcal{A}, C) over an alphabet Σ consists of an unambiguous ε -NFA \mathcal{A} over Σ , say with d transitions, and a semilinear set $C \subseteq \mathbb{N}^d$, i.e., C has one dimension for each transition in \mathcal{A} . It accepts a word $w \in \Sigma^*$ if \mathcal{A} has an accepting run processing w (due to unambiguity this run must be unique) such that the Parikh image of the run (recording the number of times each transition occurs in the run) is in C .

On the other hand, Bostan et al. introduced so-called weakly-unambiguous Parikh automata (WUPA) [4]. Intuitively, a WUPA (\mathcal{A}, C) over Σ is a classical PA as introduced here where every input over Σ has at most one accepting run (in the sense of the definition in Section 2). Bostan et al. discuss the different definitions of unambiguity and in particular show that every UCA is a WUPA, but that WUPA are strictly more expressive [4]. Here, we compare the expressiveness of HDPA to that of UCA and WUPA.

The language E from the proof of Theorem 1 is accepted by an UCA [6] and a WUPA [4], but not by any HDPA (see Theorem 1). On the other hand, the language

$$T = \{c^{n_0}dc^{n_1}d \cdots c^{n_k}d \mid k \geq 1, n_0 = 1, \text{ and } n_{j+1} \neq 2n_j \text{ for some } 0 \leq j < k\}$$

is not accepted by any WUPA, and hence also not by any UCA [4], but there is an HDPA accepting it. Hence, these two languages show that these three classes of automata have incomparable expressiveness.

Theorem 2. *The expressiveness of HDPA is neither comparable with that of UCA nor with that of WUPA.*

Finally, we show that all intersections between the different classes introduced above are nonempty.

Theorem 3.

1. *There is a language that is accepted by an HDPA and by an UCA, but not by any DPA.*
2. *There is a language that is accepted by an HDPA and by a WUPA, but not by any UCA.*
3. *There is a language that is accepted by a PA, but not by any HDPA nor by any WUPA.*
4. *There is a language that is accepted by a WUPA, but not by any HDPA nor by any UCA.*

Here, we give proof sketches, full proofs can be found in [13].

Recall that the language $E = \{a, b\}^* \cdot \{a^n b^n \mid n > 0\}$ from Theorem 1 is accepted by an UCA, but not by any HDPA. However, a slight modification allows it to be accepted by both types of automata, but not by a deterministic automaton: We show that

$$E' = \{c^m \{a, b\}^{m-1} b a^n b^n \mid m, n > 0\}$$

has the desired property. Intuitively, the prefix c^m allows us to resolve the nondeterminism on-the-fly, but nondeterminism is still required.

The second separation relies on a similar trick: The language

$$N' = \{c^n w \mid w = a_0 \cdots a_k \in \{0, 1\}^*, |w| \geq n > 0, \text{ and } a_0 \cdots a_{n-1} \text{ is non-Dyck}\},$$

which is a variation of the language N of words that have a non-Dyck prefix, was shown to be accepted by a WUPA, but not by any UCA [4]. It is straightforward to show that it is also accepted by an HDPA.

The last two results follow easily from closure properties: The union $T \cup E$ is neither accepted by any HDPA nor by any WUPA, as both models are closed under intersection³, i.e., $(T \cup E) \cap \{a, b\}^* = E$ and $(T \cup E) \cap \{c, d\}^* = T$ yield the desired separation. A similar argument works for $N' \cup E$, which is accepted by some WUPA (as WUPA are closed under disjoint unions) but not by any HDPA nor by any UCA, as both classes are closed under intersection.

³For WUPA, this was shown by Bostan et al. [4], for HDPA this is shown in the next section.

4.2 History-deterministic Reversal-bounded Counter Machines

There is one more automaton model that is closely related to Parikh automata, i.e., reversal-bounded counter machines, originally introduced by Ibarra [27]. These are, in their most general form, two-way automata with multiple counters that can be incremented, decremented, and tested for zero, but there is a constant bound on the number of reversals of the reading head *and* on the number of switches between increments and decrements (on each counter). It is known that Parikh automata and nondeterministic reversal-bounded counter machines are equivalent [29], while deterministic reversal-bounded counter machines are strictly more expressive than deterministic Parikh automata [5]. Here, we compare history-deterministic reversal-bounded counter machines and history-deterministic Parikh automata (and, for technical reasons, also history-deterministic Parikh automata with ε -transitions).

We begin by introducing counter machines and then their reversal-bounded variant.

A (two-way) counter machine is a tuple $\mathcal{M} = (k, Q, \Sigma, \triangleright, \triangleleft, q_I, \Delta, F)$ where $k \in \mathbb{N}$ is the number of counters, Q is the finite set of states, Σ is the alphabet, $\triangleright, \triangleleft \notin \Sigma$ are the left and right endmarkers respectively, $q_I \in Q$ is the initial state,

$$\Delta \subseteq (Q \times \Sigma_{\triangleright\triangleleft} \times \{0, 1\}^k) \times (Q \times \{-1, 0, 1\} \times \{-1, 0, 1\}^k)$$

is the transition relation, and $F \subseteq Q$ is the set of accepting states. Here, we use the shorthand $\Sigma_{\triangleright\triangleleft} = \Sigma \cup \{\triangleright, \triangleleft\}$. Intuitively, a transition $((q, a, \vec{g}), (q', m, \vec{v}))$ is enabled if the current state is q , the current letter on the tape is a , and for each $0 \leq j \leq k-1$, the j -th entry in the guard \vec{g} is nonzero if and only if the current value of counter j is nonzero. Taking this transition updates the state to q' , moves the head in direction m , and adds the j -th entry of \vec{v} to counter j .

We require that all transitions $((q, a, \vec{g}), (q', m, \vec{v})) \in \Delta$ satisfy the following properties:

- If $a = \triangleright$, then $m \geq 0$: the head never leaves the tape to the left.
- If $a = \triangleleft$, then $m \leq 0$: the head never leaves the tape to the right.
- \vec{g} and \vec{v} are *compatible*, i.e. if the j -th entry of \vec{g} is zero, then the j -th entry of \vec{v} is nonnegative: a zero counter is not decremented.

For the sake of brevity, we refer the formal definition of the semantics to the full version [13].

In the following, we just need the definition of configurations: A configuration of \mathcal{M} on an input $w \in \Sigma^*$ is of the form $(q, \triangleright w \triangleleft, h, \vec{c})$ where $q \in Q$ is the current state, $\triangleright w \triangleleft$ is the content of the tape (which does not change during a run), $0 \leq h \leq |w| + 1$ is the current position of the reading head, and $\vec{c} \in \mathbb{N}^k$ is the vector of current counter values.

We say that a two-way counter machine is reversal-bounded, if there is a $b \in \mathbb{N}$ such that on each run, the reading head reverses its direction at most b times *and* each counter switches between incrementing and decrementing at most b times. We write RBCM for reversal-bounded counter machines and 1-RBCM for RBCM that do not make a reversal of the reading head (i.e., they are one-way). Their deterministic variants are denoted by DRBCM and 1-DRBCM, respectively.

Ibarra [27] has shown that every RBCM can be effectively turned into an equivalent 1-RBCM and that every RBCM can be effectively turned into an equivalent one where the number of reversals of each counter is bounded by 1. The latter construction preserves determinism and one-wayness. Hence, in the following, we assume that during each run of an RBCM, each counter reverses at most once.

In terms of expressiveness, Klaedtke and Rueß [29] showed that RBCM are equivalent to Parikh automata while Cadilhac et al. [5] showed that 1-DRBCM are strictly more expressive than DPA.

In the following, we determine the relation between history-deterministic RBCM and HDPA. To this end, we first have to define the notion of history-determinism for RBCM, which is slightly technical due to the two-wayness of these machines.

Let $\mathcal{M} = (k, Q, \Sigma, \triangleright, \triangleleft, q_I, \Delta, F)$ be an RBCM. Given a sequence $\tau_0 \cdots \tau_j$ of transitions inducing a run ρ , let $\text{pos}(\tau_0 \cdots \tau_j)$ be the position of the reading head at the end of ρ , so in particular $\text{pos}(\varepsilon) = 0$. Hence, $(\triangleright w \triangleleft)_{\text{pos}(\tau_0 \tau_1 \cdots \tau_j)}$ is the letter the reading head is currently pointing to.

A resolver for \mathcal{M} is a function $r: \Delta^* \times \Sigma_{\triangleright\triangleleft} \rightarrow \Delta$ such that if w is accepted by \mathcal{M} , there is a sequence of transitions $\tau_0\tau_1 \cdots \tau_{n-1}$ such that

- $\tau_{j+1} = r(\tau_0\tau_1 \cdots \tau_j, (\triangleright w \triangleleft)_{\text{pos}(\tau_0\tau_1 \cdots \tau_j)})$ for all $0 \leq j < n-1$, and
- the run of \mathcal{M} on w induced by the sequence of transitions $\tau_0\tau_1 \cdots \tau_{n-1}$ is accepting.

An RBCM \mathcal{M} is history-deterministic (an HDRBCM) if there exists a resolver for \mathcal{M} . One-way HDRBCM are denoted by 1-HDRBCM.

Now, we are able to state the main theorem of this subsection: History-deterministic two-way RBCM are as expressive as RBCM and PA while history-deterministic one-way RBCM are as expressive as history-deterministic PA.

Theorem 4.

1. HDRBCM are as expressive as RBCM, and therefore as expressive as PA.
2. 1-HDRBCM are as expressive as HDPA.

The proof of the first equivalence is very general and not restricted to RBCM: A two-way automaton over finite inputs can first read the whole input and then resolve nondeterministic choices based on the whole word. Spelt out more concisely: two-wayness makes history-determinism as powerful as general nondeterminism.

For the other equivalence, both directions are nontrivial: We show how to simulate a PA using an RBCM while preserving history-determinism, and how to simulate a 1-RBCM by a PA, again while preserving history-determinism. Due to the existence of transitions that do not move the reading head in a 1-RBCM, this simulation takes a detour via PA with ε -transitions.

Finally, let us remark that HDPA (or equivalently 1-HDRBCM) and deterministic RBCM have incomparable expressiveness. Indeed, the language E , which is not accepted by any HDPA (see Theorem 1), can easily be accepted by a deterministic RBCM while the language N (see Example 3) is accepted by an HDPA, but not by any deterministic RBCM [6]. The reason is that these machines are closed under complement, but the complement of N is not accepted by any PA as shown in [6], and therefore also not by any RBCM.

5 Closure Properties

In this subsection, we study the closure properties of history-deterministic Parikh automata, i.e., we consider Boolean operations, concatenation and Kleene star, (inverse) homomorphic image, and commutative closure. Let us begin by recalling the last three notions.

Fix some ordered alphabet $\Sigma = (a_0 < a_1 < \cdots < a_{d-1})$. The Parikh image of a word $w \in \Sigma^*$ is the vector $\Phi(w) = (|w|_{a_0}, |w|_{a_1}, \dots, |w|_{a_{d-1}})$ and the Parikh image of a language $L \subseteq \Sigma^*$ is $\Phi(L) = \{\Phi(w) \mid w \in L\}$. The commutative closure of L is $\{w \in \Sigma^* \mid \Phi(w) \in \Phi(L)\}$.

Now, fix some alphabets Σ and Γ and a homomorphism $h: \Sigma^* \rightarrow \Gamma^*$. The homomorphic image of a language $L \subseteq \Sigma^*$ is $h(L) = \{h(w) \mid w \in L\} \subseteq \Gamma^*$. Similarly, the inverse homomorphic image of a language $L \subseteq \Gamma^*$ is $h^{-1}(L) = \{w \in \Sigma^* \mid h(w) \in L\}$.

Theorem 5. *HDPA are closed under union, intersection, inverse homomorphic images, and commutative closure, but not under complement, concatenation, Kleene star, and homomorphic image.*

Proof Sketch. Closure under union and intersection is shown by a product construction, while closure under inverse homomorphic images is shown by lifting the construction for finite automata (just as for DPA and PA). Finally closure under commutative closure follows from previous work: Cadilhac et al. proved that the commutative closure of any PA (and therefore that of any HDPA) is accepted by some DPA, and therefore also by some HDPA.

The negative results follow from a combination of expressiveness results proven in Section 4 and nonexpressiveness results in the literature [5, 6]:

	\cup	\cap	$-$	\cdot	$*$	h	h^{-1}	c
DPA	Y	Y	Y	N	N	N	Y	Y
HDPA	Y	Y	N	N	N	N	Y	Y
UCA	Y	Y	Y	N	N	N	?	Y
PA	Y	Y	N	Y	N	Y	Y	Y

Table 1: Closure properties of history-deterministic Parikh automata (in grey) and comparison to other types of Parikh automata (results for other types are from [5, 6, 29]).

- Complement: In the first part of the proof of Theorem 1, we show that the language N is accepted by an HDPA, but its complement is known to not be accepted by any PA [6].
- Concatenation: The language E is the concatenation of the languages $\{a, b\}^*$ and $\{a^n b^n \mid n \in \mathbb{N}\}$, which are both accepted by a DPA, but itself is not accepted by any HDPA (see the proof of Theorem 1).
- Kleene star: There is a DPA (and therefore also an HDPA) such that the Kleene star of its language is not accepted by any PA [5], and therefore also by no HDPA.
- Homomorphic image: There is a DPA (and therefore also an HDPA) and a homomorphism such that the homomorphic image of the DPA's language is not accepted by any PA [5], and therefore also by no HDPA. \square

Table 1 compares the closure properties of HDPA with those of DPA, UCA, and PA. We do not compare to RBCM, as only deterministic ones differ from Parikh automata and results on these are incomplete: However, Ibarra proved closure under union, intersection, and complement [27].

6 Decision Problems

Next, we study various decision problems for history-deterministic PA. First, let us mention that nonemptiness and finiteness are decidable for HDPA, as these problems are decidable for PA [29, 5]. In the following, we consider universality, inclusion, equivalence, regularity, and model checking. We start with the universality problem.

Theorem 6. *The following problem is undecidable: Given an HDPA (\mathcal{A}, C) over Σ , is $L(\mathcal{A}, C) = \Sigma^*$?*

Proof Sketch. The result is shown by turning (deterministic) Minsky machines into HDPA such that the machine does not terminate if and only if the automaton is universal. As nontermination of Minsky machines is undecidable [32], the same is true for HDPA universality.

Intuitively, the automaton processes sequences of instructions of the Minsky machine and checks whether they are prefixes of the unique run of the machine or not, employing the counters of the automaton to simulate the counter of the Minsky machine. Finally, history-determinism can be employed to find a position in the sequence of instructions where it differs from the unique run of the machine. \square

The next results follow more or less immediately from the undecidability of universality.

Theorem 7. *The following problems are undecidable:*

1. *Given two HDPA (\mathcal{A}_0, C_0) and (\mathcal{A}_1, C_1) , is $L(\mathcal{A}_0, C_0) \subseteq L(\mathcal{A}_1, C_1)$?*
2. *Given two HDPA (\mathcal{A}_0, C_0) and (\mathcal{A}_1, C_1) , is $L(\mathcal{A}_0, C_0) = L(\mathcal{A}_1, C_1)$?*
3. *Given an HDPA (\mathcal{A}, C) , is $L(\mathcal{A}, C)$ regular?*
4. *Given an HDPA (\mathcal{A}, C) , is $L(\mathcal{A}, C)$ context-free?*

Proof Sketch. The first two items follow directly from the undecidability of universality (cf. Theorem 6), so let us consider the latter two. They are both shown by a variation of the construction proving Theorem 6: Given a Minsky machine we construct an HDPA such that the machine terminates if and only if the automaton accepts a regular language (respectively, a context-free language). \square

Table 2 compares the decidability of standard problems for HDPA with those of DPA, UCA, and PA.

Finally, we consider the problems of deciding whether a Parikh automaton is history-deterministic and whether it is equivalent to some HDPA. Both of our proofs follow arguments developed for similar results for history-deterministic pushdown automata [31].

Theorem 8. *The following problems are undecidable:*

1. *Given a PA (\mathcal{A}, C) , is it history-deterministic?*
2. *Given a PA (\mathcal{A}, C) , is it equivalent to some HDPA?*

Finally, let us introduce the model-checking problem (for safety properties): A transition system $\mathcal{T} = (V, v_I, E, \lambda)$ consists of a finite set V of vertices containing the initial state $v_I \in V$, a transition relation $E \subseteq V \times V$, and a labeling function $\lambda: V \rightarrow \Sigma$ for some alphabet Σ . A (finite and initial) path in \mathcal{T} is a sequence $v_0 v_1 \cdots v_n \in V^+$ such that $v_0 = v_I$ and $(v_i, v_{i+1}) \in E$ for all $0 \leq i < n$. Infinite (initial) paths are defined analogously. The trace of a path $v_0 v_1 \cdots v_n$ is $\lambda(v_0)\lambda(v_1) \cdots \lambda(v_n) \in \Sigma^+$. We denote the set of traces of paths of \mathcal{T} by $\text{tr}(\mathcal{T})$.

The model-checking problem for HDPA asks, given an HDPA \mathcal{A} and a transition system \mathcal{T} , whether $\text{tr}(\mathcal{T}) \cap L(\mathcal{A}) = \emptyset$? Note that the automaton specifies the set of *bad prefixes*, i.e., \mathcal{T} satisfies the specification encoded by \mathcal{A} if no trace of \mathcal{T} is in $L(\mathcal{A})$.

As the model-checking problem for PA is decidable, so is the model-checking problem for HDPA, which follows from the fact that a transition system \mathcal{T} can be turned into an NFA and hence into a PA $\mathcal{A}_{\mathcal{T}}$ with $L(\mathcal{A}_{\mathcal{T}}) = \text{tr}(\mathcal{T})$. Then, closure under intersection and decidability of nonemptiness yields the desired result.

Theorem 9. *The model-checking problem for HDPA is decidable.*

Let us conclude by mentioning that the dual problem, i.e., given a transition system \mathcal{T} and an HDPA \mathcal{A} , does every infinite path of \mathcal{T} have a prefix whose trace is in $L(\mathcal{A})$, is undecidable. This follows from recent results on Parikh automata over infinite words [22], i.e., that model checking for Parikh automata with reachability conditions is undecidable. Such automata are syntactically equal to Parikh automata over finite words and an (infinite) run is accepting if it has a prefix ending in an accepting state whose extended Parikh image is in the semilinear set of the automaton.

7 Conclusion

In this work, we have introduced and studied history-deterministic Parikh automata. We have shown that their expressiveness is strictly between that of deterministic and nondeterministic PA, incomparable to that of unambiguous PA, but equivalent to history-deterministic 1-RBCM. Furthermore, we showed that they

	$\neq \emptyset?$	finite?	$= \Sigma^*?$	$\subseteq?$	$=?$	regular?	MC
DPA	Y	Y	Y	Y	Y	Y	Y
HDPA	Y	Y	N	N	N	N	Y
UCA	Y	Y	Y	Y	Y	Y	Y
PA	Y	Y	N	N	N	N	Y

Table 2: Decision problems for history-deterministic Parikh automata (in grey) and comparison to other types of Parikh automata (results are from [5, 6, 29]).

have almost the same closure properties as DPA (complementation being the notable difference), and enjoy some of the desirable algorithmic properties of DPA.

An interesting direction for further research concerns the complexity of resolving nondeterminism in history-deterministic Parikh automata. It is straightforward to show that every HDPA has a positional resolver (i.e., one whose decision is only based on the last state of the run constructed thus far and on the extended Parikh image induced by this run) and that HDPA that have finite-state resolvers (say, implemented by a Mealy machine) can be determinized by taking the product of the HDPA and the Mealy machine. In fact, both proofs are simple adaptations of the corresponding ones for history-deterministic pushdown automata [23, 31]. A more interesting question is whether there is a notion of Parikh transducer such that every HDPA has a resolver implemented by such a transducer. Note that the analogous result for history-deterministic pushdown automata fails: not every history-deterministic pushdown automaton has a pushdown resolver [23, 31].

Good-for-gameness is another notion of restricted nondeterminism that is very tightly related to history-determinism. In fact, both terms were used interchangeably until very recently, when it was shown that they do not always coincide [2]. Formally, an automaton \mathcal{A} is good-for-games if every two-player zero-sum game with winning condition $L(\mathcal{A})$ has the same winner as the game where the player who wins if the outcome is in $L(\mathcal{A})$ additionally has to construct a witnessing run of \mathcal{A} during the play. This definition comes in two forms, depending on whether one considers only finitely branching (weak compositionality) or all games (compositionality).

Recently, the difference between being history-deterministic and both types of compositionality has been studied in detail for pushdown automata [21]. These results are very general and can easily be transferred to PA and 1-RBCM. They show that for PA, being history-deterministic, compositionality, and weak compositionality all coincide, while for 1-RBCM, being history-deterministic and compositionality coincide, but not weak compositionality.

The reason for this difference can be traced back to the fact that 1-RBCM may contain transitions that do not move the reading head (which are essentially ε -transitions), but that have side-effects beyond state changes, i.e., the counters are updated. This means that an unbounded number of configurations can be reached by processing a single letter, which implies that the game composed of an arena and a 1-RBCM may have infinite branching. So, while HDPA and 1-HDRBCM are expressively equivalent, they, perhaps surprisingly, behave differently when it comes to compositionality. We plan to investigate these differences in future work.

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Appendix

In our proofs, it is sometimes convenient to work with semilinear sets defined by formulas of Presburger arithmetic, i.e., first-order formulas over the structure $\mathcal{N} = (\mathbb{N}, +, <, 0, 1,)$.

Proposition 2. *A set $C \subseteq \mathbb{N}^d$ is semilinear if and only if there is such a formula $\varphi(x_0, \dots, x_{d-1})$, i.e., with d free variables, such that $C = \{\vec{v} \in \mathbb{N}^d \mid \mathcal{N} \models \varphi(\vec{v})\}$ [17].*

A Proofs omitted in Section 4

A.1 Proof of Theorem 2

Proof. The language E from the proof of Theorem 1 is accepted by an UCA [6] and thus also by a WUPA, as every UCA can be turned into an equivalent WUPA [4]. However, E is not accepted by any HDPA, as shown in that proof. This yields the first two separations.

Conversely, consider the language

$$T = \{c^{n_0}dc^{n_1}d \cdots c^{n_k}d \mid k \geq 1, n_0 = 1, \text{ and } n_{j+1} \neq 2n_j \text{ for some } 0 \leq j < k\}.$$

Baston et al. proved that T is not accepted by any WUPA, and therefore also not by any UCA. We show that it is accepted by an HDPA, yielding the other two separations.

We start by giving some intuition. Let $w = c^{n_0}dc^{n_1}d \cdots c^{n_k}d \in T$ and let $j' \in \mathbb{N}$ be minimal with $n_{j'+1} \neq 2n_{j'}$. Then, we have $n_j = 2^j$ for all $j \leq j'$. Our automaton sums up the n_j for even and odd j and relies on some basic facts about sums of powers of two to accept the language.

Consider, for example, the sums of the 2^j for even and odd $j \leq 5$, respectively. The former is $e_{\leq 5} = 2^0 + 2^2 + 2^4 = 21$ and the latter is $o_{\leq 5} = 2^1 + 2^3 + 2^5 = 42$. We have $2 \cdot e_{\leq 5} = o_{\leq 5}$ as the terms of the second sum are obtained by doubling the terms of the former one. Similarly, we have $e_{\leq 6} = 85 = 2 \cdot 42 + 1 = 2 \cdot o_{\leq 6} + 1$. Obviously, these equations hold for arbitrary bounds, i.e., if j is odd then we have $2 \cdot e_{\leq j} = o_{\leq j}$ and if j is even then we have $e_{\leq j} = 2 \cdot o_{\leq j} + 1$.

Recall that j' was chosen minimally with $n_{j'+1} \neq 2n_{j'}$. So, the equations described above hold for all $j \leq j'$, but they fail to hold for $j = j' + 1$.⁴

Figure 4 depicts an HDPA \mathcal{A} that we show to accept T . Intuitively, it sums up the n_j for even and odd j and nondeterministically decides to stop the summation at the end of one such block. In addition to summing up the n_j , \mathcal{A} also keeps track of j by counting the d 's. Thus, we equip it with the semilinear set

$$C = \{(e, o, j) \in \mathbb{N}^3 \mid 2 \cdot e \neq o \text{ and } j \text{ is odd}\} \cup \{(e, o, j) \in \mathbb{N}^3 \mid e \neq 2 \cdot o + 1 \text{ and } j \text{ is even}\},$$

i.e., we check that the above equations were violated.

First, let us argue that (\mathcal{A}, C) accepts T . If w is in T , then there is an accepting run processing w that moves to the accepting state as soon as the $(j' + 1)$ -th d is processed, where j' is defined as above. Whether this is the case only depends on the prefix ending at that position, i.e., the nondeterminism can be resolved by a resolver. Finally, if w is not in T , then there are three cases. Either, w does not start with cd , w does not end with a d , or it is of the form $c^1dc^2dc^4d \cdots c^{2^j}d$ for some $j \geq 1$. In the first two cases, there is no accepting run of the underlying automaton \mathcal{A} that processes w . In the last case, the equations described above are never violated when processing a d , so whenever a run ends in the accepting state, the extended Parikh image of the run is not in C . So, (\mathcal{A}, C) does indeed accept T and we have argued above that the only nondeterministic choice during any run can be made by a resolver, i.e., (\mathcal{A}, C) is an HDPA. \square

⁴Note that the equation might be satisfied again, e.g., if the error $n_{j'+1} - 2n_{j'}$ is compensated by $n_{j'+2}$. However, this is irrelevant for our argument.

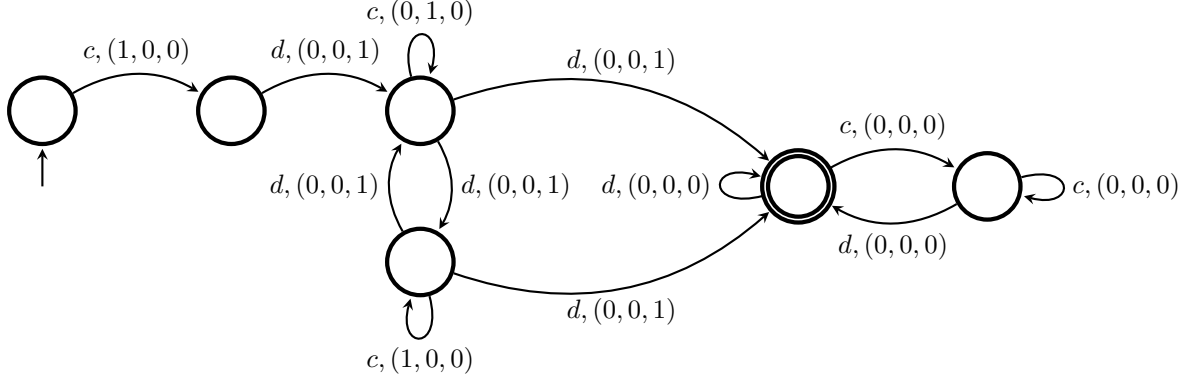


Figure 4: The automaton for language T .

A.2 Proof of Theorem 3

Proof. 1.) Consider the language

$$E' = \{c^m \{a, b\}^{m-1} b a^n b^n \mid m, n > 0\}$$

and compare it to the language $E = \{a, b\}^* \cdot \{a^n b^n \mid n > 0\}$ from Theorem 1, which is not accepted by any HDPA and thus also not by any DPA. The intuitive reason is that such an automaton has to guess when the suffix of the form $a^n b^n$ starts, which cannot be done by a resolver. However, by adding the c 's, which encode the length of the infix before the suffix of the form $a^n b^n$ starts, a resolver can determine when the suffix starts. Note that we also, for reasons that we discuss below, require that the last letter before the suffix of the form $a^n b^n$ is a b .

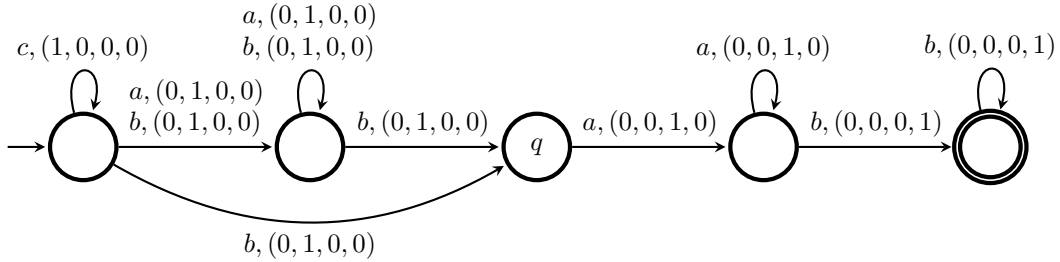


Figure 5: The automaton for language E' .

The automaton (\mathcal{A}, C) with \mathcal{A} in Figure 5 and

$$C = \{(m, m, n, n) \mid m, n \in \mathbb{N}\}$$

is an HDPA accepting E' . The automaton accepts E' (when viewed as a PA) and the only nondeterministic choice, i.e., when to move to q can be made based on the prefix processed thus far, i.e., q has to be reached with the (unique) prefix of the form $c^m \{a, b\}^{m-1} b$.

Furthermore, consider the NFA \mathcal{A}' over $\{a, b, c\}$ obtained from \mathcal{A} in Figure 5 by projecting away the vectors on the transitions, which is unambiguous: Every word accepted by \mathcal{A}' must end with ba^+b^+ and the first b of that suffix has to lead to q . As the only nondeterminism in the automaton is the choice to go to q or not, this implies that \mathcal{A}' has indeed at most one accepting run on every word.

Now, as every vector projected away from \mathcal{A} is a unit vector, it is straightforward to give an eleven-dimensional⁵ semilinear set C' such that (\mathcal{A}', C') is an UCA accepting E' , i.e., C' simulates C : C' ensures

⁵Note that the automaton in Figure 5 has eleven transitions.

that the initial c -labeled self-loop occurs in a run as often as transitions labeled by $(0, 1, 0, 0)$ in Figure 5 occur in the run (this simulates the first two components of vectors in C being equal). Similarly, C' ensures that the transitions labeled by $(0, 0, 1, 0)$ occur as often as transitions labeled by $(0, 0, 0, 1)$.

Note that requiring a b in front of the suffix $a^n b^n$ allows us to use the same *transition structure* for \mathcal{A} and \mathcal{A}' , as both automata make the same nondeterministic choice, i.e., they guess when the suffix of the form $a^n b^n$ starts. Thus, the separation between languages accepted by DPA and languages accepted both by HDPA and UCA can be witnessed by a language where the HDPA and the UCA are essentially the same automaton. Also, they both rely on the same nondeterministic guess, which can be made history-deterministically and unambiguously. Without the b , the unambiguous automaton would have to guess the start of the longest suffix of the form $a^n b^{n'}$ with $n \geq n'$, and thus an UCA accepting E would require a slightly different transition structure than the one shown in Figure 5.

Finally, it remains to argue that E' is not accepted by any DPA. First, let us remark that the pumping argument in the proof of Theorem 1 also shows that the language $E_b = \{a, b\}^* \cdot b \cdot \{a^n b^n \mid n > 0\}$ is not accepted by any HDPA and thus also not by any DPA. Now, we show that a DPA accepting E' can be turned into a DPA accepting E_b , which yields the desired result.

So, assume there is a DPA (\mathcal{A}, C) accepting E' , say with $\mathcal{A} = (Q, \{a, b, c\} \times D, q_I, \Delta, F)$. For every $q \in Q$ let R_q be the set of runs of \mathcal{A} starting in q_I , processing a word in c^+ , and ending in q . Furthermore, let

$$C_q = \left\{ \sum_{j=0}^{n-1} \vec{v}_j \mid (q_I, (c, \vec{v}_0), q_1)(q_1, (c, \vec{v}_1), q_2) \cdots (q_{n-1}, (c, \vec{v}_{n-1}), q) \in R_q \right\}$$

be the extended Parikh images of those runs, which is semilinear [34].

Furthermore, for each $q \in Q$ with nonempty R_q let $\mathcal{A}_q = (Q, \{a, b\} \times D, q, \Delta', F)$ where Δ' is obtained by removing all c -transitions from Δ . Note that each \mathcal{A}_q is still deterministic, as we have only changed the initial state and removed transitions. Finally, we define

$$C'_q = \{\vec{v} \mid \text{there exists } \vec{v}' \in C_q \text{ such that } \vec{v} + \vec{v}' \in C\},$$

which is again semilinear, as it can be defined by a Presburger formula constructed from Presburger formulas for C_q and C (see Proposition 2).

We claim $E_b = \bigcup_q L(\mathcal{A}_q, C'_q)$ where q ranges over all states such that R_q is nonempty. As DPA are closed under union, this yields the desired contradiction in the form of a DPA for E_b .

So, consider some $w \in E_b$, i.e., $w = w' b a^n b^n$ for some $n > 0$ and some $w' \in \{a, b\}^*$. Define $m = |w'b|$ and note that $c^m w$ is in E' , i.e., accepted by (\mathcal{A}, C) . Hence, let ρ be the unique run of (\mathcal{A}, C) processing $c^m w$, let ρ' be the prefix of ρ processing c^m , and let ρ'' be the suffix processing w . So, there is some q (the state ρ' ends in, which is equal to the state ρ'' begins with) such that $\rho' \in R_q$ and the extended Parikh image \vec{v}' induced by ρ' is in C_q . Also, ρ'' is a run of \mathcal{A}_q starting in q and ending in F . Let \vec{v} be the extended Parikh image induced by ρ'' . Note that $\vec{v} + \vec{v}'$ is the extended Parikh image induced by the full run $\rho = \rho' \rho''$ that witnesses $c^m w \in L(\mathcal{A}, C)$, and is therefore in C . From this we conclude $\vec{v} \in C'_q$ and therefore that ρ'' is an accepting run of (\mathcal{A}_q, C'_q) processing w , i.e., $w \in L(\mathcal{A}_q, C'_q)$ and R_q is nonempty as witnessed by ρ' .

For the other direction, consider a $w \in L(\mathcal{A}_q, C'_q)$ for some q with nonempty R_q . Then there is an accepting run ρ'' of (\mathcal{A}, C'_q) processing w , say with induced extended Parikh image $\vec{v} \in C'_q$. By construction, there is also a $\vec{v}' \in C_q$ such that $\vec{v} + \vec{v}' \in C$. Furthermore, \vec{v}' is the extended Parikh image induced by some run ρ' of (\mathcal{A}, C) processing some word of the form c^m . Now, $\rho' \rho''$ is a run of (\mathcal{A}, C) starting in the initial state, processing $c^m w$, ending in F , and with extended Parikh image $\vec{v} + \vec{v}' \in C$, i.e., it is an accepting run. Hence, $c^m w \in L(\mathcal{A}, C) = E'$. As w does not contain any c ((\mathcal{A}_q, C'_q) has no c -transitions), this implies that w must be of the form $w' b a^n b^n$ with $w' \in \{a, b\}^{m-1}$, i.e., $w \in E_b$ as required.

2.) Recall that we say that $w \in \{0, 1\}^*$ is non-Dyck if $|w|_0 < |w|_1$. Now, consider the language

$$N' = \{c^n w \mid w = a_0 \cdots a_k \in \{0, 1\}^*, |w| \geq n, \text{ and } a_0 \cdots a_{n-1} \text{ is non-Dyck}\},$$

which is a variation of the language N of words that have a non-Dyck prefix. In N' , the length of that prefix is given by the number of c 's in the beginning of the word, which makes accepting the language easier. Nevertheless, Bostan et al. showed that N' is not accepted by any UCA, but by a WUPA [4].

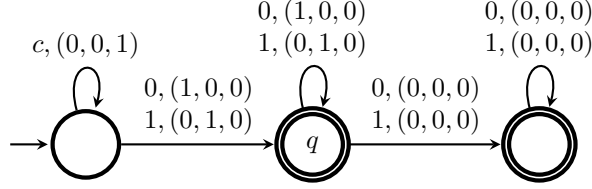


Figure 6: The automaton for language N' .

In particular, it is accepted by the PA (\mathcal{A}, C) where \mathcal{A} is depicted in Figure 6 and with

$$C = \{(n, n', n + n') \mid n < n'\}.$$

Note that (\mathcal{A}, C) is both an HDPA and a WUPA for $L(\mathcal{A}, C)$: every word $c^n w$ in N' has at most one accepting run, the one which leaves q with the $(2n + 1)$ -th letter of $c^n w$ (if this letter exists). Furthermore, this choice can be made by a resolver, as the number of c at the start of the word uniquely determines when this nondeterministic choice has to be made.

3.) and 4.) These follow immediately from closure properties, as shown in the main body. \square

A.3 Proof of Theorem 4

Before we prove the equivalence result, we first define the semantics of two-way counter machines and the formal definition of reversal-boundedness and history-determinism. To this end, let $\mathcal{M} = (k, Q, \Sigma, \triangleright, \triangleleft, q_I, \Delta, F)$ be a two-way counter machine.

Recall that a configuration of \mathcal{M} is of the form $(q, \triangleright w \triangleleft, h, \vec{c})$ where $q \in Q$ is the current state, $\triangleright w \triangleleft$ is the content of the tape (which does not change during a run), $0 \leq h \leq |w| + 1$ is the current position of the reading head, and $\vec{c} \in \mathbb{N}^k$ is the vector of current counter values. The initial configuration on $w \in \Sigma^*$ is $(q_I, \triangleright w \triangleleft, 0, \vec{0})$, where $\vec{0}$ is the k -dimensional zero vector.

We say that a vector $\vec{c} \in \mathbb{N}^k$ satisfies a guard $\vec{g} \in \{0, 1\}^k$ if the following is satisfied for every $1 \leq j \leq k$: the j -th entry of \vec{c} is zero if and only if the j -th entry of \vec{g} is zero. Now, we write $(q, \triangleright w \triangleleft, h, \vec{c}) \Rightarrow (q', \triangleright w \triangleleft, h + m, \vec{c} + \vec{v})$ if there is a transition $((q, a, \vec{g}), (q', m, \vec{v})) \in \Delta$ such that \vec{c} satisfies \vec{g} , where a is the h -th letter of $\triangleright w \triangleleft$.

A run of \mathcal{M} on an input w is a sequence of configurations $\rho = S_0 S_1 \cdots S_n$ such that S_0 is the initial configuration on w and $S_j \Rightarrow S_{j+1}$ for every $0 \leq j < n$. The run ρ is accepting if the state of S_n is in F . The language accepted by \mathcal{M} , denoted by $L(\mathcal{M})$, is the set of words $w \in \Sigma^*$ such that there exists an accepting run of \mathcal{M} on w .

Remark 2. Note that a run on some given $w \in \Sigma^*$ is fully described by the sequence of transitions between its configurations. However, not every sequence of transitions induces a run.

A counter machine \mathcal{M} is deterministic if, for every $q \in Q$, $a \in \Sigma_{\triangleright \triangleleft}$ and $\vec{g} \in \{0, 1\}^k$, there is at most one triple $(q', m, \vec{v}) \in Q \times \{-1, 0, 1\} \times \{-1, 0, 1\}^k$ such that $((q, a, \vec{g}), (q', m, \vec{v})) \in \Delta$.

The problem with counter machines is that even their deterministic variants are Turing-complete [27]. Therefore, one must impose some restrictions in order to obtain decidability of some decision problems. One way to do that is to introduce bounds on the number of reversals of the direction the reading head moves and on the number of reversals of the stack height of each counter.

More formally, consider a run ρ given by a finite sequence $((q_n, a_n, \vec{g}_n), (q_{n+1}, m_n, \vec{v}_n))_n$ of transitions. The number of reversals of the reading head during ρ is the number of sign alternations in the sequence $(m_n)_n$, ignoring the 0's. The number of reversals of the j -th counter during ρ is the number of sign alternations in the sequence $(v_{n,j})_n$, where $v_{n,j}$ is the j -th entry of \vec{v}_n , again ignoring the 0's. A counter machine \mathcal{M} is reversal-bounded if there exists a $b \in \mathbb{N}$ such that every accepting run has at most b reversals of the reading head and at most b reversals of each counter.

Proof. 1.) It is immediate from the definition that RBCM are at least as expressive as HDRBCM. Thus, we show that for any RBCM, one can construct an equivalent HDRBCM.

Let \mathcal{M} be an RBCM. We construct an RBCM \mathcal{M}' from \mathcal{M} as follows: first, the reading head moves right up to the right endmarker. Then, it moves back left to the left endmarker, and then behaves like \mathcal{M} on the input. During the initial scan, it leaves the counters unchanged. It is clear that \mathcal{M}' is equivalent to \mathcal{M} , as once the initial scan is finished, it behaves like \mathcal{M} .

Furthermore, \mathcal{M}' is history-deterministic since, after the initial scan, which is deterministic, a resolver has access to the whole input and can, for accepted words, fix one accepting run for the input, and then resolve the nondeterminism accordingly. This is possible, since the sequence of transitions seen during the initial scan is unique for every possible input, and the resolver can base its choices on that information.

2.) We first show how to turn an HDPA into an equivalent 1-RBCM and then show that it is history-deterministic. This construction is inspired by the constructions turning a PA (a DPA) into an equivalent 1-RBCM (an equivalent 1-DRBCM) [5, 29].

Intuitively, the 1-RBCM simulates the given HDPA by keeping track of the extended Parikh image of a run in its counters, i.e., there is a bijection between the runs of the HDPA and the 1-RBCM. As the HDPA is one-way by definition, so is the simulating RBCM. When the end of the input word is reached, acceptance depends on whether the extended Parikh image (the counter values) are in C . The 1-RBCM can do so using the following result due to Cadilhac et al. [5]: membership in a given semilinear set can be tested by a DRBCM that does not move its reading head. More formally they showed that for every semilinear set $C \subseteq \mathbb{N}^d$, there exists a DRBCM \mathcal{M}_C such that, for any configuration $S_0 = (q_I, \triangleright w \triangleleft, i, \vec{v} \cdot \vec{0})$ with $\vec{v} \in \mathbb{N}^d$, the unique run starting from S_0 is accepting if and only if $\vec{v} \in C$. Moreover, \mathcal{M}_C does not move its reading head. Note that \mathcal{M}_C will in general use more than d counters. In the configuration S_0 , the first d counters contain the vector to be checked, and the remaining ones are initialized with zero. This is exactly the situation after the simulation of the HDPA has finished.

It remains to argue that the resulting 1-DRBCM is indeed history-deterministic. However, this is straightforward, since it simulates the history-deterministic PA (recall that there is a bijection between the runs, which can be used to transfer the resolver) and then deterministically checks membership in a semilinear set.

So, let us consider the other direction, i.e., we turn an 1-HDRBCM \mathcal{M} into an equivalent HDPA (\mathcal{A}, C) . To do so, we proceed in three steps:

Step 1) We turn \mathcal{M} into a 1-HDRBCM \mathcal{M}' in a normal form that requires an RBCM to only terminate with the reading head at the right endmarker and all counters being zero (recall that a PA can only test for membership in the semilinear set at the end of a run).

Step 2) We turn the 1-HDRBCM \mathcal{M}' into an equivalent ε -HDPA.

Step 3) We show how to eliminate ε -transitions from PA while preserving history-determinism.

Step 1) A 1-RBCM \mathcal{M} is in normal form if the following conditions are satisfied:

1. Let $((q, a, \vec{g}), (q', m, \vec{v}))$ be a transition of \mathcal{M} such that q' is not final. Then, $((q, a, \vec{g}'), (q', m, \vec{v}))$ is also a transition of \mathcal{M} for every $\vec{g}' \in \{0, 1\}^k$ that is compatible with \vec{v} , i.e. \mathcal{M} does not test the counters during the run, but transitions that would decrement from a zero counter are still not allowed.
2. Let $((q, a, \vec{g}), (q', m, \vec{v}))$ be a transition of \mathcal{M} such that q' is final. Then, $a = \triangleleft$, $\vec{g} = \vec{0}$, and $\vec{v} = \vec{0}$, i.e. \mathcal{M} only accepts an input on the right endmarker and with all counters equal to zero.
3. Accepting states do not have any outgoing transitions.

We show how to turn a 1-RBCM into an equivalent one in normal form while preserving history-determinism.

It is easy to ensure the second and third conditions, since every 1-RBCM can be turned into an equivalent one that, upon reaching an accepting state, deterministically moves to the right endmarker and empties its

counters, and then reaches a fresh accepting state without outgoing transitions. This also preserves history-determinism.

For the first condition, let \mathcal{M} be a 1-RBCM meeting the other two conditions. Also, recall that we assume, again without loss of generality, that every counter reverses at most once. We construct \mathcal{M}' that simulates \mathcal{M} and instead of testing counters for zero it guesses the outcome of a test and verifies the guesses at the end of the run. The ability to postpone these tests crucially relies on the fact that every counter reverses at most once.

For every counter, \mathcal{M}' stores in its states one of the following statuses:

INI the counter has neither been increment nor decremented.

INC the counter has been incremented but not yet decremented.

DEC the counter has been incremented, decremented, but is still assumed to be nonzero.

ZERO the counter has been guessed to be zero.

At the beginning, all the counters are in status INI. Then, when a counter has status INI or INC and is incremented, then, its new status is INC. If the status is INC or DEC and the counter is decremented, its status is nondeterministically updated to DEC or ZERO. Intuitively, it should be updated to DEC as long as the true counter value is nonnegative and to ZERO if the true value is zero. A counter with status ZERO cannot be decremented (as it is assumed to be zero) nor incremented (as there is at most one reversal, and the status cannot be changed anymore. This implies that the value of the corresponding counter can no longer be updated as soon as its status is changed to ZERO.

Now, \mathcal{M}' simulates \mathcal{M} , accounting for tests in guards as follows:

- If a counter has status INI or ZERO, then its value is assumed to be equal to zero.
- If a counter has status INC or DEC, then its value is assumed to be positive.

As a counter is no longer updated once its status is ZERO, the guess that the counter is zero is correct if and only if the value of the counter is indeed zero at the end of the run.

As \mathcal{M} satisfies the second condition of the normal form, every accepting run ends with a transition testing all counters for zero. Thus, an accepting state of \mathcal{M} is only reached if all counters are zero (and stay zero after the transition has been taken). So, we equip \mathcal{M}' with a transition that checks, after the simulation of \mathcal{M} has finished in an accepting state, whether all counters are zero and have status INI or ZERO. It only accepts if this is the case.

Therefore, \mathcal{M}' accepts the same language as \mathcal{M} . In addition, the only nondeterminism introduced in the construction are the guesses whether a counter is zero after a decrement. These can be made history-deterministically since a resolver can keep track of the current values of the counters. Hence, \mathcal{M}' is history-deterministic if \mathcal{M} is.

Step 2) Here, we show how to turn a 1-RBCM into an equivalent ε -HDPA while preserving history-determinism. So, let us first introduce the latter type of automaton.

An ε -PA over an alphabet Σ and of dimension d is a tuple (\mathcal{A}, C) , where \mathcal{A} is a finite automaton over $(\Sigma \cup \{\varepsilon\}) \times D$, where ε is the empty word, and where $C \subseteq \mathbb{N}^d$ is semilinear. Note that this definition does not coincide with the classical notion of ε -NFA, as ε -transitions in \mathcal{A} are still labeled by a vector in the second component. The language accepted by (\mathcal{A}, C) is $L(\mathcal{A}, C) = \{p_\Sigma(w) \mid w \in L(\mathcal{A}) \text{ and } \Phi_\varepsilon(w) \in C\}$. Here, we treat p_Σ as a homomorphism, which implies $p_\Sigma(\varepsilon) = \varepsilon$. Note that ε -PA are not more expressive than PA, since PA are closed under homomorphisms: Hence, one can treat ε as a ordinary letter and then apply a homomorphism that deletes exactly that letter.

Let (\mathcal{A}, C) be an ε -PA with $\mathcal{A} = (Q, (\Sigma \cup \{\varepsilon\}) \times D, q_I, \Delta, F)$. Let Δ_ε be the set of ε -transitions and Δ_Σ be the set of Σ -transitions of \mathcal{A} . A resolver for (\mathcal{A}, C) is a function $r : \Sigma^+ \rightarrow \Delta_\varepsilon^* \cdot \Delta_\Sigma \cdot \Delta_\varepsilon^*$ such that the image $r^*(w)$ of r 's iteration (defined analogously to the case of Parikh automata without ε -transitions in Section 3) is an accepting run processing w , for every $w \in L(\mathcal{A}, C)$. An ε -PA (\mathcal{A}, C) is said to be history-deterministic (an ε -HDPA) if there exists a resolver for it.

Now, fix a 1-RBCM \mathcal{M} in normal form, say with k counters. We construct an equivalent ε -PA (\mathcal{A}, C) and show that the transformation preserves history-determinism.

Intuitively, \mathcal{A} simulates \mathcal{M} and the semilinear set is used to check that all counters of \mathcal{M} are zero after the simulation has ended. However, recall that a PA can only increment its counter while an RBCM can increment and decrement its counters. Hence, \mathcal{A} has two counters for each counter j of \mathcal{M} , one (counter $2j$) counting the increments and the other one (counter $2j + 1$) counting the decrements during the simulation. As \mathcal{M} does not test its counters during a run (due to the second condition of the normal form), \mathcal{A} only has to test whether the counters are equal to zero in the last configuration of a simulated run. This is the case if and only if the value of counter $2j$ of (\mathcal{A}, C) is equal to the value of counter $2j + 1$, for every counter j of machine. This can easily be expressed by a semilinear set.

As (\mathcal{A}, C) simulates a run of \mathcal{M} and accepts if all the counters of \mathcal{M} are equal to zero at the end, they both recognise the same language. In addition, this transformation does not add any nondeterminism as the PA simply simulates the run of the 1-RBCM. Therefore, (\mathcal{A}, C) is history-deterministic if \mathcal{M} is.

Step 3) Finally, we need to show how to eliminate ε -transitions in ε -PA while preserving history-determinism. The construction is similar to one used by Klaedtke and Rueß [28] to show that Parikh automata are closed under homomorphisms, which also requires the removal of ε -transitions. For the sake of completeness, we present the construction here, as we need to show that it allows us to preserve history-determinism.

Let (\mathcal{A}, C) be an ε -PA of dimension d and let $\varphi(x_0, \dots, x_{d-1})$ be a Presburger formula defining C (recall Proposition 2). We construct an equivalent PA (\mathcal{A}', C') by replacing paths that are labeled by $\varepsilon^* a \varepsilon^*$ by a single a -transition. However, taking ε -transitions has side-effects, i.e., the extended Parikh image is updated along them. This requires us to account for the effect of taking ε -cycles in the new semilinear set C' . To do so correctly, we need to keep track of (in the extended Parikh image) which ε -cycles could have been traversed during the run of (\mathcal{A}, C) being simulated by (\mathcal{A}', C') .

A finite run infix ρ is called reduced if no state is repeated in it. Let $a \in \Sigma$. An ε - a - ε -path is a run infix of the form $\rho_0 \tau \rho_1$ where ρ_0 and ρ_1 are (possibly empty) sequences of ε -transitions and where τ is a transition labeled by a . We say that $\rho_0 \tau \rho_1$ is reduced if ρ_0 and ρ_1 are reduced. Note that there are only finitely many reduced ε - a - ε -paths for every a . Finally, let $\{K_d, K_{d+1}, \dots, K_m\}$ be the set of ε -cycles of (\mathcal{A}, C) , which is again finite as we only consider simple cycles.

Consider an arbitrary, not necessarily reduced, ε - a - ε -path $\rho_0 \tau \rho_1$. If it is not reduced, then some ρ_0 or ρ_1 contains an ε -cycle K_j . Removing this cycle yields a shorter ε - a - ε -path starting and ending in the same state (but possibly with different extended Parikh image). By repeating this operation (always removing the first cycle in case there are several ones), we turn every ε - a - ε -path ρ into a reduced ε - a - ε -path $\text{red}(\rho)$ that still has the same first state and the same last state as the original one, but possibly a different extended Parikh image.

Now, we define \mathcal{A}' as follows: it has the same states, the same initial state, the same set of accepting states as \mathcal{A} , and the same alphabet Σ . Furthermore, for every ε - a - ε -path

$$\rho = (q_0, a_0, q_1) \cdots (q_{n-1}, a_{n-1}, q_n)$$

in \mathcal{A} , the automaton \mathcal{A}' has the transition $(q_0, (a, \vec{v}) \cdot \chi(\rho), q_n)$, where \vec{v} is the extended Parikh image of the reduced run $\text{red}(\rho)$ of ρ , and where

$$\chi(\rho) = (b_d, b_{d+1}, \dots, b_m)$$

is such that b_j is equal to 1 if K_j has been removed at least once from ρ to obtain $\text{red}(\rho)$. Otherwise, b_j is equal to 0. Note that this results in finitely many transitions, as there are only finitely many reduced ε - a - ε -paths and only finitely many choices for the $\chi(\rho)$.⁶

We then define C' by the Presburger formula

$$\varphi'(x_0, \dots, x_m) = \exists y_d \exists y_{d+1} \cdots \exists y_m \bigwedge_{j=d}^m (x_j > 0 \Leftrightarrow y_j > 0) \wedge \varphi((x_0, \dots, x_{d-1}) + \sum_{j=d}^m y_j \cdot \vec{v}_j)$$

⁶Furthermore, it is not hard to see that the construction can be made effective, as one does not have to consider all ε - a - ε -path: every transition in \mathcal{A}' is witnessed by an ε - a - ε -path of bounded length.

where \vec{v}_j is the extended Parikh image of K_j . The idea is that if a run could have went through the cycle K_j , then y_j captures how many times it would have been used. Note that this formula is not a first-order formula, due to the summation of the vectors, but can easily be turned into one by making the addition componentwise. We refrain from doing so for the sake of readability. Also, the multiplication in the formula is not an issue, as we only multiply with constants, which is just repeated addition.

Next, we show that (\mathcal{A}, C) and (\mathcal{A}', C') are equivalent. Let us first remark that either both accept the empty word or both reject it. This is because they share the same initial state and same accepting states and because the zero vector is in C if and only if it is in C' . So, in the following we only have to consider nonempty words.

Let $w = a_0 \cdots a_{n-1} \in L(\mathcal{A}, C)$ with $n > 0$, say with accepting run ρ . Then, ρ can be decomposed into a sequence $\rho_0 \cdots \rho_{n-1}$, where each ρ_i is an ε - a_i - ε -path (the exact splits are irrelevant). Consider one of these ρ_i , say it leads from q to q' . By construction, (\mathcal{A}', C') has a a_i -transition from q to q' . These transitions form a run ρ' of (\mathcal{A}', C') processing w . Note that both runs end in the same state, so ρ' ends in an accepting state as well.

Now, $\chi(\rho_i)$ encodes which cycles have been removed from ρ_i to obtain $\text{red}(\rho_i)$. Hence, taking the first d components of the extended Parikh image of ρ' and adding to it the extended Parikh images of the removed cycles (with their appropriate multiplicity $y_j > 0$) yields the Parikh image of ρ , which is in C . Hence, the extended Parikh image of ρ' is in C' .

Now, assume $a_0 \cdots a_{n-1} \in L(\mathcal{A}', C')$ with $n > 0$, say with accepting run $\rho' = \tau_0 \cdots \tau_{n-1}$. The transition τ_i is witnessed by some reduced ε - a_i - ε -path ρ_i . These form a run ρ of \mathcal{A} processing w , which ends in an accepting state, as ρ' does. Furthermore, we know that the extended Parikh image of ρ' is in C' , so there are values $y_j > 0$ for the ε -cycles K_j such that $(x_0, \dots, x_{d-1} + \sum_{j=d}^m y_j \cdot \vec{v}_j)$ is in C . Hence, we can add each K_j y_j times to ρ in order to obtain another accepting run of (\mathcal{A}, C) that processes w and has an extended Parikh image in C .

Finally, we need to show that (\mathcal{A}', C') is history-deterministic if (\mathcal{A}, C) is. So, let $r: \Sigma^+ \rightarrow \Delta_\varepsilon^* \Delta_\Sigma \Delta_\varepsilon^*$ be a resolver for (\mathcal{A}, C) . We construct a resolver $r': \Sigma^+ \rightarrow \Delta'$ for (\mathcal{A}', C') . We can assume without loss of generality that $r(wa)$ is an ε - a - ε -path for every input wa : if not, then this output cannot be part of an accepting run, which means we can redefine r arbitrarily so that it satisfies our assumption.

Now, let $wa \in \Sigma^+$ and let $\rho = r(wa)$, which is an ε - a - ε -path, say from q to q' . The reduced ε - a - ε -path $\text{red}(\rho)$ leads from q to q' and induces a transition τ of \mathcal{A}' processing a . We define $r'(wa) = \tau$ for this transition. Then, an induction on the length of an input word w shows that the run of (\mathcal{A}', C') constructed by r' on an input w is accepting if and only if the run of (\mathcal{A}, C) constructed by r on w is accepting. \square

B Proofs omitted in Subsection 5

B.1 Proof of Theorem 5

We need to show closure of HDPAs under union, intersection, inverse homomorphic images, and commutative closure.

Proof. For $i \in \{1, 2\}$, let (\mathcal{A}_i, C_i) be an HDPA with $\mathcal{A}_i = (Q_i, \Sigma \times D_i, q_I^i, \Delta_i, F_i)$, say with resolver r_i . Furthermore, let d_i be the dimension of D_i . As in the proof of Lemma 1, we assume without loss of generality that $r_i^*(w)$ is a run of \mathcal{A}_i processing w , for every $w \in \Sigma^*$.

First, we consider closure under union. Intuitively, we use a product construction to simulate a run of \mathcal{A}_1 and a run of \mathcal{A}_2 simultaneously. A naive approach would be to take the classical product of the \mathcal{A}_i where we concatenate the vectors labelling the transitions, and then use $C = C_1 \cdot \mathbb{N}^{d_2} \cup \mathbb{N}^{d_1} \cdot C_2$. However, this is not correct, as this automaton can accept if an accepting state of \mathcal{A}_1 is reached while the extended Parikh image is in $\mathbb{N}^{d_1} \cdot C_2$ or vice versa. To overcome this issue, we reflect in the extended Parikh image which one of the simulated runs ends in an accepting state. To simplify this, we assume, without loss of generality, that the initial states of both \mathcal{A}_i do not have incoming transitions.

Now, we define the product $\mathcal{A} = (Q_1 \times Q_2, \Sigma \times D, (q_I^1, q_I^2), \Delta, Q_1 \times Q_2)$ where

- $D = \{1, 2\} \cdot \{1, 2\} \cdot D_1 \cdot D_2$, and
- $\Delta = \{((q_1, q_2), (a, (v_1, v_2) \cdot \vec{v}_1 \cdot \vec{v}_2), (q'_1, q'_2)) \mid (q_i, (a, \vec{v}_i), q'_i) \in \Delta_i \text{ for } i \in \{1, 2\}\}$, where v_i for $i \in \{1, 2\}$ is defined as follows.
 - If q_i is the initial state of \mathcal{A}_i : If $q'_i \in F_i$ then $v_i = 2$, otherwise $v_i = 1$.
 - If q_i is not the initial state of \mathcal{A}_i : If $q_i \in F_i \Leftrightarrow q'_i \in F_i$ then $v_i = 2$, otherwise $v_i = 1$.

Note that this satisfies the following invariant for every nonempty run of \mathcal{A} : If $w \in (\Sigma \times D)^+$ is the input processed by the run, $\Phi_e(w) = (v_1, v_2, \dots)$, and the run ends in state $(q_1, q_2) \in Q_1 \times Q_2$, then we have $v_i \bmod 2 = 0$ if and only if $q_i \in F_i$. Thus, it is the value v that reflects in the extended Parikh image which of the simulated runs end in an accepting state. Note however, that this only holds for nonempty runs, as we need to initialize the reflection.

Let C_ε be the set containing the $(1 + d_1 + d_2)$ -dimensional zero vector if $\varepsilon \in L(\mathcal{A}_1, C_1) \cup L(\mathcal{A}_2, C_2)$, and C_ε be the empty set otherwise. Then, we define

$$C = \{n \in \mathbb{N} \mid n \bmod 2 = 0\} \cdot \mathbb{N} \cdot C_1 \cdot \mathbb{N}^{d_2} \cup \\ \mathbb{N} \cdot \{n \in \mathbb{N} \mid n \bmod 2 = 0\} \cdot \mathbb{N}^{d_1} \cdot C_2 \cup C_\varepsilon,$$

which is semilinear due to Proposition 1. Then, we have $L(\mathcal{A}, C) = L(\mathcal{A}_1, C_1) \cup L(\mathcal{A}_2, C_2)$ and the following function r is a resolver for (\mathcal{A}, C) : let $r_i(w) = (q_i, (a, \vec{v}_i), q'_i)$ and define $r(w) = ((q_1, q_2), (a, (v_1, v_2) \cdot \vec{v}_1 \cdot \vec{v}_2), (q'_1, q'_2))$, where (v_1, v_2) is defined as above.

Now, consider closure under intersection. Here, we take the product $\mathcal{A}' = (Q_1 \times Q_2, \Sigma \times (D_1 \cdot D_2), (q_I^1, q_I^2), \Delta', F_1 \times F_2)$ with

$$\Delta' = \{((q_1, q_2), (a, \vec{v}_1 \cdot \vec{v}_2), (q'_1, q'_2)) \mid (q_i, (a, \vec{v}_i), q'_i) \in \Delta_i \text{ for } i \in \{1, 2\}\}$$

and define $C' = C_1 \cdot C_2$, which is is semilinear due to Proposition 1. Then, $L(\mathcal{A}', C') = L(\mathcal{A}_1, C_1) \cap L(\mathcal{A}_2, C_2)$ and the following function r is a resolver for (\mathcal{A}, C) : let $r_i(w) = (q_i, (a, \vec{v}_i), q'_i)$ and define $r(w) = ((q_1, q_2), (a, \vec{v}_1 \cdot \vec{v}_2), (q'_1, q'_2))$.

Now, consider closure under inverse homomorphic images. In [29], it has been shown that DPA and PA are effectively closed under inverse homomorphic images. We follow the same construction as in [29] to show that HDPA are closed under inverse homomorphic images. In fact, a similar construction is also used to show that regular languages are also closed under inverse homomorphic images [26].

Consider a homomorphism $h : \Sigma^* \rightarrow \Gamma^*$. Given an HDPA (\mathcal{A}, C) with $\mathcal{A} = (Q, \Gamma \times D, q_I, \Delta, F)$, we construct another HDPA (\mathcal{A}', C) with $\mathcal{A}' = (Q, \Sigma \times D', q_I, \Delta', F)$, such that $L(\mathcal{A}', C) = h^{-1}(L(\mathcal{A}, C))$. Note that the set C is the same in both automata. Intuitively, \mathcal{A}' processes a letter (a, \vec{v}) by simulating a sequence of transitions in \mathcal{A} processing $(b_1, \vec{v}_1), \dots, (b_m, \vec{v}_m)$ with $h(a) = b_1 \cdots b_m$, and $\vec{v} = \sum_{i=1}^m \vec{v}_i$, for $m \in \mathbb{N}$. The NFA \mathcal{A}' has the same set Q of states as \mathcal{A} , and \mathcal{A}' has a transition from state p to state q with label $(a, \vec{v}) \in \Sigma \times D'$ if and only if there is a sequence of transitions labeled with $(b_1, \vec{v}_1), \dots, (b_m, \vec{v}_m) \in (\Gamma \times D)^*$ taking \mathcal{A} from state p to state q , and $h(a) = b_1 \cdots b_m$, and $\vec{v} = \sum_{i=1}^m \vec{v}_i$. It is easy to see that the set D' thus obtained is finite, and that (\mathcal{A}', C) accepts the language $h^{-1}(L(\mathcal{A}, C))$.

We are now left to prove that (\mathcal{A}', C) is an HDPA. For notational convenience, we lift the definition of Φ_e to the runs of a PA as $\Phi_e(\rho) = \Phi_e(w)$, where w is the word processed by ρ . Since (\mathcal{A}, C) is an HDPA, there exists a resolver r of (\mathcal{A}, C) . We define a resolver r' for (\mathcal{A}', C) as follows. Consider a word $w \in L(\mathcal{A}', C)$, and let $x = h(w)$. We have that $x \in L(\mathcal{A}, C)$. Let $w = a_1 \cdots a_m$ such that $a_j \in \Sigma$ for $1 \leq j \leq m$. We have that $h(w) = h(a_1) \cdots h(a_m)$, and let $h(a_j) = x_{j,1} \cdots x_{j,k_j}$ such that each $x_{j,i} \in \Gamma$ for $1 \leq i \leq k_j$. We define $\delta_{j,i} = r(h(a_1) \cdots h(a_{j-1})x_{j,1} \cdots x_{j,i})$, for $1 \leq j \leq m$ and $1 \leq i \leq k_j$, to be the transition induced by r after processing the prefix $h(a_1) \cdots h(a_{j-1})x_{j,1} \cdots x_{j,i}$ of $h(w)$. Further, let $\delta_{j,i} = (q_{j,i-1}, (x_{j,i}, \vec{v}_{j,i}), q_{j,i})$. In particular, we have that $q_{j,0}$ is the state in \mathcal{A} before processing $x_{j,1}$ and q_{j,k_j} is the state in \mathcal{A} after processing x_{j,k_j} . Then, we define $r'(a_1 \cdots a_j) = (q_{j,0}, (a_j, \vec{v}_j), q_{j,k_j})$, where $\vec{v}_j = \sum_{i=1}^{k_j} \vec{v}_{j,i}$.

Recall that $w \in L(\mathcal{A}', C)$ and $h(w) = x \in L(\mathcal{A}, C)$. If $q_f \in F$ is the state reached after processing x in (\mathcal{A}, C) following the accepting run $r^*(x)$, then the same state q_f is reached after processing w in (\mathcal{A}', C) in

the run $r'^*(w)$. Besides, the extended Parikh images corresponding to both $r^*(x)$ and $r'^*(w)$ are the same, that is $\Phi_e(r^*(x)) = \Phi_e(r'^*(w))$. Now since $\Phi_e(r^*(x))$ belongs to C , we have that $\Phi_e(r'^*(w))$ also belongs to C , and thus $r'^*(w)$ is an accepting run. Hence r' is indeed a resolver for (\mathcal{A}', C) .

Now, let us consider commutative closure. Cadilhac et al. proved that the commutative closure of any PA (and therefore that of any HDPA) is accepted by some DPA [5], and therefore also by some HDPA. \square

C Proofs omitted in Subsection 6

C.1 Minsky Machines and Parikh Automata

Our undecidability proofs are reductions from nontermination problems for two-counter machines. Such a machine \mathcal{M} is a sequence

$$(0 : \mathbf{I}_0)(1 : \mathbf{I}_1) \cdots (k-2 : \mathbf{I}_{k-2})(k-1 : \mathbf{STOP}),$$

where the first element of a pair $(\ell : \mathbf{I}_\ell)$ is the line number and \mathbf{I}_ℓ for $0 \leq \ell < k-1$ is an instruction of the form

- $\mathbf{INC}(X_i)$ with $i \in \{0, 1\}$,
- $\mathbf{DEC}(X_i)$ with $i \in \{0, 1\}$, or
- $\mathbf{IF } X_i=0 \mathbf{ GOTO } \ell' \mathbf{ ELSE GOTO } \ell''$ with $i \in \{0, 1\}$ and $\ell', \ell'' \in \{0, \dots, k-1\}$.

A configuration of \mathcal{M} is of the form (ℓ, c_0, c_1) with $\ell \in \{0, \dots, k-1\}$ (the current line number) and $c_0, c_1 \in \mathbb{N}$ (the current contents of the counters). The initial configuration is $(0, 0, 0)$ and the unique successor configuration of a configuration (ℓ, c_0, c_1) is defined as follows:

- If $\mathbf{I}_\ell = \mathbf{INC}(X_i)$, then the successor configuration is $(\ell + 1, c'_0, c'_1)$ with $c'_i = c_i + 1$ and $c'_{1-i} = c_{1-i}$.
- If $\mathbf{I}_\ell = \mathbf{DEC}(X_i)$, then the successor configuration is $(\ell + 1, c'_0, c'_1)$ with $c'_i = \max\{c_i - 1, 0\}$ and $c'_{1-i} = c_{1-i}$.
- If $\mathbf{I}_\ell = \mathbf{IF } X_i=0 \mathbf{ GOTO } \ell' \mathbf{ ELSE GOTO } \ell''$ and $c_i = 0$, then the successor configuration is (ℓ', c_0, c_1) .
- If $\mathbf{I}_\ell = \mathbf{IF } X_i=0 \mathbf{ GOTO } \ell' \mathbf{ ELSE GOTO } \ell''$ and $c_i > 0$, then the successor configuration is (ℓ'', c_0, c_1) .
- If $\mathbf{I}_\ell = \mathbf{STOP}$, then (ℓ, c_0, c_1) has no successor configuration.

The unique run of \mathcal{M} (starting in the initial configuration) is defined as expected. It is either finite (line $k-1$ is reached) or infinite (line $k-1$ is never reached). In the former case, we say that \mathcal{M} terminates.

Proposition 3 ([32]). *The following problem is undecidable: Given a two-counter machine \mathcal{M} , does \mathcal{M} terminate?*

In the following, we assume without loss of generality that each two-counter machine satisfies the *guarded-decrement property*: Every decrement instruction $(\ell : \mathbf{DEC}(X_i))$ is preceded by $(\ell - 1 : \mathbf{IF } X_i=0 \mathbf{ GOTO } \ell + 1 \mathbf{ ELSE GOTO } \ell)$ and decrements are never the target of a goto instruction. As the decrement of a zero counter has no effect, one can modify each two-counter machine \mathcal{M} into an \mathcal{M}' satisfying the guarded-decrement property such that \mathcal{M} terminates if and only if \mathcal{M}' terminates: One just adds the the required guard before every decrement instruction and changes each target of a goto instruction that is a decrement instruction to the preceding guard.

The guarded-decrement property implies that decrements are only executed if the corresponding counter is nonzero. Thus, the value of counter i after a finite sequence of executed instructions (starting with value zero in the counters) is equal to the number of executed increments of counter i minus the number of executed decrements of counter i . Note that the number of executed increments and decrements can be tracked by a Parikh automaton.

Consider a finite or infinite word $w = a_0 a_1 a_2 \dots$ over the set $\{0, 1, \dots, k-1\}$ of line numbers. We now describe how to characterize whether w is (a prefix of) the projection to the line numbers of the unique run of \mathcal{M} starting in the initial configuration. This characterization is designed to be checkable by a Parikh automaton. Note that w only contains line numbers, but does not encode values of the counters. These will be kept track of by the Parikh automaton by counting the number of increment and decrement instructions in the input, as explained above (this explains the need for the guard-decrement property). Formally, we say that w contains an *error* at position $n < |w| - 1$ if either $a_n = k - 1$ (the instruction in line a_n is **STOP**), or if one of the following two conditions is satisfied:

1. The instruction I_{a_n} in line a_n of \mathcal{M} is an increment or a decrement and $a_{n+1} \neq a_n + 1$, i.e., the letter a_{n+1} after a_n is not equal to the line number $a_n + 1$, which it should be after an increment or decrement.
2. I_{a_n} has the form **IF** $X_i=0$ **GOTO** ℓ **ELSE** **GOTO** ℓ' , and one of the following cases holds: Either, we have

$$\sum_{j: I_j=\text{INC}(X_i)} |a_0 \dots a_n|_j = \sum_{j: I_j=\text{DEC}(X_i)} |a_0 \dots a_n|_j$$

and $a_{n+1} \neq \ell$, i.e., the number of increments of counter i is equal to the number of decrements of counter i in $a_0 \dots a_n$ (i.e., the counter is zero) but the next line number in w is not the target of the if-branch. Or, we have

$$\sum_{j: I_j=\text{INC}(X_i)} |a_0 \dots a_n|_j \neq \sum_{j: I_j=\text{DEC}(X_i)} |a_0 \dots a_n|_j,$$

and $a_{n+1} \neq \ell'$, i.e., the number of increments of counter i is not equal to the number of decrements of counter i in $a_0 \dots a_n$ (i.e., the counter is nonzero) but the next line number in w is not the target of the else-branch.

Note that the definition of error (at position n) refers to the number of increments and decrements in the prefix $a_0 \dots a_n$, which does not need to be error-free itself. However, if a sequence of line numbers does not have an error, then the guarded-decrement property yields the following result.

Lemma 2. *Let $w \in \{0, 1, \dots, k-1\}^+$ with $a_0 = 0$. Then, w has no errors at positions $\{0, 1, \dots, |w| - 2\}$ if and only if w is a prefix of the projection to the line numbers of the run of \mathcal{M} .*

Proof. If w has no errors at positions $\{0, 1, \dots, |w| - 2\}$, then an induction shows that (a_n, c_0^n, c_1^n) with

$$c_i^n = \sum_{j: I_j=\text{INC}(X_i)} |a_0 \dots a_{n-1}|_j - \sum_{j: I_j=\text{DEC}(X_i)} |a_0 \dots a_{n-1}|_j$$

is the n -th configuration of the run of \mathcal{M} .

On the other hand, projecting a prefix of the run of \mathcal{M} to the line numbers yields a word w without errors at positions $\{0, 1, \dots, |w| - 2\}$. \square

The existence of an error can be captured by a Parikh automaton, leading to the undecidability of the safe word problem for Parikh automata, which we now prove. Let (\mathcal{A}, C) be a PA accepting finite words over Σ . A *safe* word of (\mathcal{A}, C) is an infinite word in Σ^ω such that each of its prefixes is in $L(\mathcal{A}, C)$.

Lemma 3. *The following problem is undecidable: Given a deterministic PA, does it have a safe word?*

Proof. Our proof proceeds by a reduction from the nontermination problem for decrement-guarded two-counter machines. Given such a machine $\mathcal{M} = (0 : I_0) \dots (k-2 : I_{k-2})(k-1 : \text{STOP})$ let $\Sigma = \{0, \dots, k-1\}$ be the set of its line numbers. We construct a deterministic PA $(\mathcal{A}_{\mathcal{M}}, C_{\mathcal{M}})$ that accepts a word $w \in \Sigma^*$ if and only if $w = \varepsilon$, $w = 0$, or if $|w| \geq 2$ and w does not contain an error at position $|w| - 2$ (but might contain errors at earlier positions). Intuitively, the automaton checks whether the second-to-last instruction is executed properly. The following is then a direct consequence of Lemma 2: $(\mathcal{A}_{\mathcal{M}}, C_{\mathcal{M}})$ has a safe word if and only if \mathcal{M} does not terminate.

The deterministic PA (\mathcal{A}_M, C_M) keeps track of the occurrence of line numbers with increment and decrement instructions of each counter (using four dimensions) and two auxiliary dimensions to ensure that the two cases in Condition 2 of the error definition on Page 26 are only checked when the second-to-last letter corresponds to a goto instruction. More formally, we construct \mathcal{A}_M such the unique run processing some input $w = a_0 \dots a_{n-1}$ has the extended Parikh image $(v_{\text{inc}}^0, v_{\text{dec}}^0, v_{\text{goto}}^0, v_{\text{inc}}^1, v_{\text{dec}}^1, v_{\text{goto}}^1)$ where

- v_{inc}^i is equal to $\sum_{j: \mathbb{I}_j = \text{INC}(x_i)} |a_0 \dots a_{n-2}|_j$, i.e., the number of increment instructions read so far (ignoring the last letter),
- v_{dec}^i is equal to $\sum_{j: \mathbb{I}_j = \text{DEC}(x_i)} |a_0 \dots a_{n-2}|_j$, i.e., the number of decrement instructions read so far (ignoring the last letter), and
- $v_{\text{goto}}^i \bmod 4 = 0$ if the second-to-last instruction $\mathbb{I}_{a_{n-2}}$ is not a goto testing counter i ,
- $v_{\text{goto}}^i \bmod 4 = 1$ if the second-to-last instruction $\mathbb{I}_{a_{n-2}}$ is a goto testing counter i and the last letter a_{n-1} is equal to the target of the if-branch of this instruction, and
- $v_{\text{goto}}^i \bmod 4 = 2$ if the second-to-last instruction $\mathbb{I}_{a_{n-2}}$ is a goto testing counter i and the last letter a_{n-1} is equal to the target of the else-branch of this instruction.
- $v_{\text{goto}}^i \bmod 4 = 3$ if the second-to-last instruction $\mathbb{I}_{a_{n-2}}$ is a goto testing counter i and the last letter a_{n-1} is neither equal to the target of the if-branch nor equal to the target of the else-branch of this instruction. Note that this constitutes an error at position $n - 2$.

Note that $v_{\text{goto}}^i \bmod 4 \neq 0$ can be true for at most one of the i at any time (as a goto instruction $\mathbb{I}_{a_{n-2}}$ only refers to one counter) and that the v_{inc}^i and v_{dec}^i are updated with a delay of one transition (as the last letter of w is ignored). This requires to store the previously processed letter in the state space of \mathcal{A}_M .

Further, C_M is defined such that $(v_{\text{inc}}^0, v_{\text{dec}}^0, v_{\text{goto}}^0, v_{\text{inc}}^1, v_{\text{dec}}^1, v_{\text{goto}}^1)$ is in C_M if and only if

- $v_{\text{goto}}^i \bmod 4 = 0$ for both i , or if
- $v_{\text{goto}}^i \bmod 4 = 1$ for some i (recall that i is unique then) and $v_{\text{inc}}^i = v_{\text{dec}}^i$, or if
- $v_{\text{goto}}^i \bmod 4 = 2$ for some i (again, i is unique) and $v_{\text{inc}}^i \neq v_{\text{dec}}^i$.

All other requirements, e.g., Condition 1 of the error definition on Page 26, the second-to-last letter not being $k - 1$, and the input being in $\{\varepsilon, 0\}$, can be checked using the state space of \mathcal{A}_M . \square

C.2 Proof of Theorem 6

We need to prove that universality for HDPa is undecidable.

Proof. By reduction from the nontermination problem for decrement-guarded Minsky machines: Given such a machine \mathcal{M} with line numbers $0, 1, \dots, k - 1$ where $k - 1$ is the stopping instruction, fix $\Sigma = \{0, 1, \dots, k - 1\}$ and consider $L_{\mathcal{M}} = L_{\mathcal{M}}^0 \cup L_{\mathcal{M}}^1$ with

$$L_{\mathcal{M}}^0 = \{w = a_0 \dots a_m \in \Sigma^* \mid a_0 \neq 0 \text{ or } |w|_{k-1} = 0\} \text{ and}$$

$$L_{\mathcal{M}}^1 = \{w = a_0 \dots a_m \in \Sigma^* \mid a_0 = 0, |w|_{k-1} \geq 1, \text{ and } w \text{ contains an error before the first } k - 1\}$$

Here, errors are defined as in Section C.1. We claim that \mathcal{M} does not terminate if and only if $L_{\mathcal{M}}$ is universal.

First, assume \mathcal{M} does terminate. Then, projecting the terminating run to its line numbers yields a sequence $w \in \Sigma^*$ starting with 0, ending with $k - 1$, and with no errors. This word is not in $L_{\mathcal{M}}$, so $L_{\mathcal{M}}$ is not universal.

Now, assume \mathcal{M} does not terminate and consider some $w \in \Sigma^*$. If w does not start with 0 or contains no $k - 1$, then it is in $L_{\mathcal{M}}$. Thus, now consider the case where w starts with 0 and contains a $k - 1$. Towards

a contradiction, assume that the sequence up to the first $k - 1$ does not contain an error. Then, Lemma 2 implies that \mathcal{M} terminates, i.e., we have derived the desired contradiction. Hence, w contains an error before the first $k - 1$, i.e., w is in $L_{\mathcal{M}}$. So, $L_{\mathcal{M}}$ is indeed universal.

So, it remains to show that $L_{\mathcal{M}}$ is accepted by an HDPA which can be effectively constructed from \mathcal{M} . As HDPA are closed under union and generalize finite automata, we only have to consider $L_{\mathcal{M}}^1$. In the proof of Lemma 3, we have constructed a DPA accepting a word if it does not have an error at the second-to-last position. Thus, using complementation and intersection with a regular language, we obtain a DPA that accepts a word if there is an error at the second-to-last position.

But the previous automaton only checks whether the error occurs at the second-to-last position. To find an error at any position, we use history-determinism to guess the prefix with the error and then stop updating the counters and the state, so that their current values can be checked at the end of the run (cf. Example 3). This automaton can easily be turned into one that additionally checks that the first letter is a 0 and that there is at least one $k - 1$ in the input, but not before the error. Thus, an HDPA accepting $L_{\mathcal{M}}^1$ can be effectively constructed from \mathcal{M} . \square

Note that there is no DPA accepting $L_{\mathcal{M}}$, as universality is decidable for DPA. Thus, the guessing described in the proof above is not avoidable.

C.3 Proof of Theorem 7

We need to prove that inclusion, equivalence, and regularity are undecidable for HDPA.

Proof. Undecidability of inclusion and equivalence follows immediately from Theorem 6 (and even holds if the input (\mathcal{A}_0, C_0) is replaced by a DFA accepting Σ^*), so let us consider the regularity problem.

Let $L'_{\mathcal{M}} = L_{\mathcal{M}}^0 \cup L_{\mathcal{M}}^1 \cup L_{\mathcal{M}}^2$ where $L_{\mathcal{M}}^0$ and $L_{\mathcal{M}}^1$ are defined as in the proof of Theorem 6 and with

$$L_{\mathcal{M}}^2 = \{w = a_0 \cdots a_m \in \Sigma^* \mid a_0 = 0 \text{ and } |w|_{k-1} \geq 1 \text{ and the suffix of } w \text{ after the first occurrence of the letter } k-1 \text{ is not of the form } 0^n 1^n \text{ for some } n > 0\}.$$

Note that we assume without loss of generality $k - 1 \geq 1$, i.e., \mathcal{M} has at least one non-stopping instruction.

A DPA for $L_{\mathcal{M}}^2$ is straightforward to construct. Hence, given \mathcal{M} , one can effectively construct an HDPA for $L'_{\mathcal{M}}$ as HDPA are closed under union. Now, we claim that $L'_{\mathcal{M}}$ is regular if and only if \mathcal{M} does not terminate.

First, assume that \mathcal{M} does not terminate. Then as shown in the proof of Theorem 6, $L_{\mathcal{M}} \subseteq L'_{\mathcal{M}}$ is equal to Σ^* . Hence, $L'_{\mathcal{M}} = \Sigma^*$ as well, which is regular.

Now, assume that \mathcal{M} terminates and let $w_t \in \Sigma^*$ be the projection of the run of \mathcal{M} to its line numbers. Note that w_t starts with 0, ends with $k - 1$, and does not contain an error. We show that $L'_{\mathcal{M}} = \Sigma^* \setminus w_t \cdot \{0^n 1^n \mid n > 0\}$, which is not regular.

First, if w is in $w_t \cdot \{0^n 1^n \mid n > 0\}$, then it is not in $L'_{\mathcal{M}}$. Now, if w is not in $L'_{\mathcal{M}}$, then it has to start with 0 and has to contain a $k - 1$ (otherwise, w would be in $L_{\mathcal{M}}^0 \subseteq L'_{\mathcal{M}}$). Furthermore, w cannot contain an error before the first $k - 1$ (otherwise, w would be in $L_{\mathcal{M}}^1 \subseteq L'_{\mathcal{M}}$). Thus, w_t is a prefix of w . Finally, the suffix of w after the first $k - 1$ has to be of the form $0^n 1^n$ for some $n > 0$ (otherwise, w would be in $L_{\mathcal{M}}^2 \subseteq L'_{\mathcal{M}}$). Altogether, w is in $w_t \cdot \{0^n 1^n \mid n > 0\}$.

Finally, using $0^n 1^n 2^n$ instead of $0^n 1^n$ in the definition of $L_{\mathcal{M}}^2$ in the previous proof, which is not context-free, allows us to show that the context-freeness problem is also undecidable: Given an HDPA (\mathcal{A}, C) , is $L(\mathcal{A}, C)$ context-free? \square

C.4 Proof of Theorem 8

We need to prove that deciding whether a PA is history-deterministic and whether it is equivalent to an HDPA are both undecidable. We begin with the former problem.

Proof. 1.) We say that a PA (\mathcal{A}, C) is *length-complete* if for every $n \in \mathbb{N}$, there is some word of length n in $L(\mathcal{A}, C)$. Furthermore, let $h: \Sigma^* \rightarrow \{\#\}^*$ be the homomorphism induced by mapping each $a \in \Sigma$ to $\#$. If (\mathcal{A}, C) is length-complete, then $\{h(w) \mid w \in L(\mathcal{A}, C)\} = \{\#\}^*$. Note that the DPA (\mathcal{A}_M, C_M) we have constructed in the proof of Lemma 3 are length-complete. Thus, we have actually shown that the following problem is undecidable: Given a length-complete DPA (\mathcal{A}, C) , does it have a safe word?

We reduce from the safe word problem for length-complete DPA. Given such a DPA (\mathcal{A}, C) with $L(\mathcal{A}, C) \subseteq \Sigma^*$, let (\mathcal{A}', C) be the PA obtained from (\mathcal{A}, C) by replacing each letter $a \in \Sigma$ by $\#$. Note that (\mathcal{A}', C) accepts $\{h(w) \mid w \in L(\mathcal{A}, C)\}$, which is equal to $\{\#\}^*$ due to the length-completeness of (\mathcal{A}, C) . We show that (\mathcal{A}', C) is history-deterministic if and only if (\mathcal{A}, C) has a safe word, which completes our proof.

So, let (\mathcal{A}', C) be history-deterministic, i.e., it has a resolver $r: \{\#\}^* \rightarrow \Delta_{\mathcal{A}'}$, where $\Delta_{\mathcal{A}'}$ is the set of transitions of \mathcal{A}' . Note that $r^*(\#^n)$ is a prefix of $r^*(\#^{n'})$ whenever $n' \geq n$. Hence, there is a unique limit

$$(q_0, (\#, \vec{v}_0), q_1)(q_1, (\#, \vec{v}_1), q_2)(q_2, (\#, \vec{v}_2), q_3) \cdots$$

such that $r^*(\#^n) = (q_0, (\#, \vec{v}_0), q_1) \cdots (q_{n-1}, (\#, \vec{v}_{n-1}), q_n)$. Each of the $r^*(\#^n)$ is an accepting run of \mathcal{A}' processing $\#^n$, as $\#^n$ is accepted by (\mathcal{A}', C) and r is a resolver.

Now, as each transition of \mathcal{A}' is obtained from a transition of \mathcal{A} , there is some infinite word $a_0 a_1 a_2 \cdots \in \Sigma^\omega$ such that $(q_0, (a_0, \vec{v}_0), q_1) \cdots (q_{n-1}, (a_{n-1}, \vec{v}_{n-1}), q_n)$ is an accepting run of \mathcal{A} for every $n \geq 1$. Thus, we have $a_0 \cdots a_{n-1} \in L(\mathcal{A}, C)$ for every $n \geq 1$. Lastly, we have $\varepsilon \in L(\mathcal{A}, C)$, as $\varepsilon \in L(\mathcal{A}', C)$. Altogether, $a_0 a_1 a_2 \cdots$ is indeed a safe word for (\mathcal{A}, C) .

Now, assume (\mathcal{A}, C) has a safe word, say $a_0 a_1 a_2 \cdots$. Thus, ε is in $L(\mathcal{A}, C)$ which implies that the initial state of \mathcal{A} is accepting and the zero vector is in C . Further, as (\mathcal{A}, C) is deterministic, there is a sequence $(q_0, (a_0, \vec{v}_0), q_1)(q_1, (a_1, \vec{v}_1), q_2)(q_2, (a_2, \vec{v}_2), q_3) \cdots$ such that each prefix $(q_0, (a_0, \vec{v}_0), q_1) \cdots (q_{n-1}, (a_{n-1}, \vec{v}_{n-1}), q_n)$ is an accepting run of \mathcal{A} .

By construction of (\mathcal{A}', C) , $(q_0, (\#, \vec{v}_0), q_1) \cdots (q_{n-1}, (\#, \vec{v}_{n-1}), q_n)$ is an accepting run of \mathcal{A}' on $\#^n$, for each $n \geq 1$.

Now, we define $r(\#^n) = (q_{n-1}, (\#, \vec{v}_{n-1}), q_n)$ for each $n \geq 1$. Then, we have $r^*(\#^n) = (q_0, (\#, \vec{v}_0), q_1) \cdots (q_{n-1}, (\#, \vec{v}_{n-1}), q_n)$, i.e., it is an accepting run of (\mathcal{A}', C) processing $\#^n$. Hence, r is a resolver for (\mathcal{A}', C) , i.e., (\mathcal{A}', C) is history-deterministic.

2.) We reduce from the universality problem for PA, which is undecidable [29], via a series of automata transformations.

First, given a PA (\mathcal{A}, C) processing words over Σ , we construct a PA (\mathcal{A}', C') with

$$L(\mathcal{A}', C') = L(\mathcal{A}, C) \cup L(\mathcal{A}, C) \cdot \# \cdot (\Sigma_\#)^*$$

where $\Sigma_\# = \Sigma \cup \{\#\}$ for some fresh $\# \notin \Sigma$. Note that (\mathcal{A}, C) is universal, i.e., $L(\mathcal{A}, C) = \Sigma^*$, if and only if (\mathcal{A}', C') is universal, i.e., $L(\mathcal{A}', C') = (\Sigma_\#)^*$.

Recall that the language $E \subseteq \{a, b\}^*$ from the proof of Theorem 1 is accepted by some PA, but not by any HDPA. Thus, we can construct a PA (\mathcal{A}_V, C_V) with

$$L(\mathcal{A}_V, C_V) = \{w \in (\Sigma_\# \times \{a, b\})^* \mid \pi_1(w) \in L(\mathcal{A}', C') \text{ or } \pi_2(w) \in E\}.$$

Here, π_i is the projection to the i -th component. We show that $L(\mathcal{A}, C)$ is universal if and only if $L(\mathcal{A}_V, C_V)$ is accepted by some HDPA.

First, assume that $L(\mathcal{A}, C)$ is universal. Then, $L(\mathcal{A}', C')$ and $L(\mathcal{A}_V, C_V)$ are universal as well. Hence, $L(\mathcal{A}_V, C_V)$ is accepted by some DFA, and therefore also by an HDPA.

Now, assume that $L(\mathcal{A}, C)$ is not universal, i.e., there is some $u = a_0 \cdots a_{n-1} \notin L(\mathcal{A}, C)$. Towards a contradiction, we assume there is an HDPA (\mathcal{A}_a, C_a) with $L(\mathcal{A}_a, C_a) = L(\mathcal{A}_V, C_V)$, say with resolver r_a . We show that the language

$$P = \left\{ w \in \{a, b\}^{\geq |u|} \mid \binom{u\#^{|w|-|u|}}{w} \in L(\mathcal{A}_V, C_V) \right\}$$

is also accepted by some HDPA (\mathcal{A}_P, C_a) . Note that P contains exactly the words $w \in \{a, b\}^{\geq |u|} \cap E$, as $u\#^{|w|-|u|}$ is not in $L(\mathcal{A}', C')$ for any w . The fact that P is accepted by some HDPA yields the desired

contradiction: We have $P \subseteq E$ and $E \setminus P$ is finite. Hence, $E \setminus P$ is accepted by some DFA and therefore also by some HDPA. Thus, due to closure of HDPA under union, if P is accepted by some HDPA, so is E . This contradicts that E is not accepted by any HDPA (see Theorem 1).

So, let $\mathcal{A}_a = (Q, (\Sigma_{\#} \times \{a, b\}) \times D, q_I, \Delta, F)$. We define $\mathcal{A}_P = (Q \times \{0, 1, \dots, |u|\}, \{a, b\} \times D, (q_I, 0), \Delta', F \times \{|u|\})$ with

$$\Delta' = \left\{ ((q, j), (a, \vec{v}), (q', j+1)) \mid (q, \binom{a_j}{a}, \vec{v}, q') \in \Delta \text{ and } j < |u| \right\} \cup \left\{ ((q, |u|), (a, \vec{v}), (q', |u|)) \mid (q, \binom{\#}{a}, \vec{v}, q') \in \Delta \right\}$$

Intuitively, to obtain \mathcal{A}_P , we hardcode u into the state space in order to restrict the runs of \mathcal{A}_a to those that process words of the form $u\#^*$ in the first component (which is projected away). Hence, (\mathcal{A}_P, C_a) does indeed accept P .

Now it remains to observe that (\mathcal{A}_P, C_a) has a resolver: Turning (\mathcal{A}_a, C_a) into (\mathcal{A}_P, C_a) does not introduce nondeterminism, i.e., a resolver r for (\mathcal{A}_a, C_a) can easily be turned into one for (\mathcal{A}_P, C_a) . So, (\mathcal{A}_P, C_a) is an HDPA. \square