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Extending Modal Transition Systems with Structured Labels[†]

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We introduce a novel formalism of label-structured modal transition systems that combines the classical may/must modalities on transitions with structured labels that represent quantitative aspects of the model. On the one hand, the specification formalism is general enough to include models like weighted modal transition systems and allows the system developers to employ more complex label refinement than in the previously studied theories. On the other hand, the formalism maintains the desirable properties required by any specification theory supporting compositional reasoning. In particular, we study modal and thorough refinement, determinization, parallel composition, conjunction, quotient, and logical characterization of label-structured modal transition systems.

1. Introduction

Modern computing systems are often large and complex assemblies of numerous reactive and interacting components. The components are often designed by independent teams, working under a common agreement what the interface of each component should be. Consequently, the search for mathematical foundations which support *compositional*

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reasoning about interfaces is a major research goal. The framework should support inferring properties of the global implementation, and designing and advisedly reusing components.

In a logical setting, interfaces are specifications and components that implement an interface are understood as models/implementations. Specification theories should support various features including (1) *refinement*, which allows to compare specifications as well as to replace a specification by another one in a larger design, (2) *structural composition*, which allows to combine specifications of different components, (3) *logical conjunction*, expressing the intersection of the set of requirements expressed by two or more specifications, and last but not least, (4) a *quotient operator* that, given two specifications S and T, synthesizes the largest (w.r.t. refinement) specification that can be composed with S in order to refine T.

For sequential systems the classical notion of Denotational Semantics, founded by Scott and Strachey, provides a rich mathematical foundation for successfully describing the semantics of many sequential programming languages and systems (Gordon, 1979; Stoy, 1977) where components, i.e. programs, are basically modelled as computable functions from the domain of input values to the domain of output values. Most importantly, the semantics of a composite program is expressed in terms of the semantics of its components thus supporting compositional reasoning. A similar well-established specification theory for sequential systems is that of Hoare Logic (Hoare, 1969), where a program is specified by pairs of pre- and post-conditions on states. In particular, Hoare Logic comes equipped with all the ingredients required and described above for a specification theory, with *"strongest postcondition"* and *"weakest precondition"* transformers providing the means for composing and quotienting specifications with respect to sequential composition.

Process algebras such as CCS (Milner, 1980) and CSP (Hoare, 1985) provide a corresponding mathematical foundation for concurrent and reactive systems. Here systems are semantically understood as labelled transition systems (Plotkin, 1981) describing their interaction capabilities and dynamic evolution. Based on the labelled transition system semantics, several *equivalences* and *preorders* have been proposed (van Glabbeek, 1990) in order to capture different aspects of the extensional behaviour of a process. This results in specification theories where both the specification and the implementation are expressed within the same formalism, e.g. CCS, and with a preferred preorder or equivalence determining the *satisfaction* of an implementation with respect to a specification. To achieve the goal of compositional analysis great care has normally been taken to ensure that the preorders and equivalences are substitutive with respect to the various process constructions, e.g. parallel composition, including the notions of *observational equivalence* (Milner, 1980) and *bisimulation equivalence* (Park, 1981; Milner, 1983) used in CCS.

The specification theory of *modal transition systems* (Larsen and Thomsen, 1988b) grew out of a series of attempts to achieve a more flexible and easy-to-use compositional development methodology for CCS. For the initial motivation consider the so-called *stepwise refinement method* to be carried out in CCS. A specification (in CCS) S of some desired system is given. The task is to find an implementation I of S such that $I \equiv S$, where $\equiv$ may be observational (or bisimulation) equivalence. In a first refinement-step S

might be refined to a composite specification of the form $C[S_1, S_2]$, where the context C is some CCS-construct (e.g. parallel composition) and S_1 and S_2 are subspecifications. Now we may have different teams working independently towards implementations I_1 and I_2 of the subspecifications S_1 and S_2. Given the congruence property of observational equivalence, it will now suffice to establish the equivalences below to conclude, in a compositional manner, that the assembled implementation $C[I_1, I_2]$ satisfies the original specification S:

$$\begin{aligned} C[S_1, S_2] &\equiv S & (1)\\ I_1 &\equiv S_1 & (2)\\ I_2 &\equiv S_2 & (3) \end{aligned}$$

However, looking more carefully at the stepwise refinement above, we notice that (2) and (3) require S_i and I_i $(i = 1, 2)$ to be proved congruent, i.e. interchangeable, in *any* context and not just interchangeable in the context of C in which they are actually going to be placed. We are therefore asked to prove more than what seems necessary. Moreover, the subspecifications S_i $(i = 1, 2)$ may have to specify behaviours that are not at all relevant in the context C. Again it seems that the above compositional analysis can be substantially harder than necessary.

In order to reduce the work of (1-3), the notion of *context-dependent* or *relativized* bisimulation was introduced in (Larsen, 1985; Larsen, 1987). Here, in order to reduce the overall effort, the observational equivalence $\equiv$ is relativized with information about the context C. The required proofs $I_i \equiv S_i$ can thus be replaced with proofs of the more specific $I_i \equiv_e S_i$ where e is some partial information (stated as a labelled transition system) about the context C. The work (Larsen and Milner, 1987; Larsen and Milner, 1992) applies the relativized bisimulation to the compositional verification of the Alternating Bit Protocol, and (Pierce and Sangiorgi, 2000) introduces a proof technique for polymorphic Pi-Calculus based on polymorphic types which can be seen as a "disciplined instance" of relativized bisimulation. A more recent usage of relativized bisimulation includes the use of environment information to produce environment-specific (reduced) code from embedded system specifications (Larsen et al., 2005) and to generate relevant test sequences from real-time specifications (Larsen et al., 2004).

The introduction of modal transition system was pre-dated by the simpler formalisms of *partial specifications* (Larsen and Thomsen, 1988a) and the corresponding notion of partial bisimulation. Roughly speaking, partial specifications are labelled transition systems with certain (specification) states being interpreted as completely unspecified. As such, partial specifications are very similar to that of processes with divergence and the so-called pre-bisimulation (Stirling, 1987; Stirling and Walker, 1989). However, though allowing for simple and intuitive subspecifications on several examples, the specification theory constituted by partial specifications is closed under neither conjunction nor quotienting.

Compared with partial specifications, the introduction of modal transition systems (Larsen and Thomsen, 1988b) resulted in a specification theory much closer to logic (see (Boudol and Larsen, 1992) for a logical characterization of the expressive power of modal transition systems), thus still with a behavioural semantics allowing for easy composition with respect to process constructions. In short, modal transition systems are labelled transition systems equipped with two types of transitions: *must* transitions that are mandatory for any implementation, and *may* transitions which are optional for an implementation. Refinement of modal transition systems now essentially consists in iteratively resolving the unsettled status of may transitions: either by removing them or by turning them into must transitions.

It is well admitted that modal transition systems and their extensions (e.g., (Raclet, 2008)) match all the requirements of a good specification theory. There is also no doubt that the formalism is expressive enough to encode complex industrial problems (see e.g., (COMBEST, 2011; SPEEDS, 2010)). Moreover, the model has applications in other contexts, which include the verification of product lines (Fischbein et al., 2006; Gruler et al., 2008; Larsen et al., 2007) and a counterexample-guided abstraction refinement technique for transition systems (Godefroid et al., 2001).

While searching for a specification theory for embedded systems, it is not only the functional requirements (Larsen et al., 2007; Feuillade and Pinchinat, 2007; Bauer et al., 2010) of system behaviours that are of importance. The theory should be also capable of expressing constraints for several non-functional properties such as timing, energy-consumption, band-width etc. Recently such efforts have been of high interest in the theory community (Caillaud et al., 2010; Katoen et al., 2009; Delahaye et al., 2011; Bertrand et al., 2009a; Bertrand et al., 2009b; David et al., 2010).

For different non-functional extensions it is common that similar proof techniques are used to argue about the specification formalisms. In this article, we present a specification theory that unifies several of the proof techniques described in the literature by introducing a general framework of *label-structured modal transition system.* Specializations of the framework include, apart from the well-known instances like unlabelled/labelled modal transition systems, also a new specification theory for weighted and multi-weighted transition systems that were studied only recently (Juhl et al., 2010). Other formalisms like timed modal transitions systems can be embedded into the framework as argued in (Bertrand et al., 2009a) where the authors show that operations defined on some classes of timed modal specifications can be reduced to questions on modal transition systems by using a region-based abstraction.

In this article, we study the classical questions related to the formalism of label-structured modal transition systems: (i) modal and thorough refinement, consistency and pruning, determinism and deterministic hull (in Section 2), (ii) parallel composition, conjunction and quotient (in Section 3) and (iii) logical characterization including generalized model checking (in Section 4).

As a result, we offer in a self-contained manner a full specification theory of label-structured modal transition systems. The theory specializes to some well-known formalisms studied earlier but at the same time also provides novel results for instances such as weighted and multi-weighted modal transition systems.

2. Label-Structured Modal Transition Systems

We shall now introduce the notion of label-structured modal transition systems and some basic properties of the formalism. Before that we need to define the notion of labels and label-sets used during the system design and specification refinement.

Definition 1 (Label-set). A *label-set* is a partially ordered set of labels $(K, \sqsubseteq)$ such that $\bot \in K$ (modelling inconsistency) is the least element of K.

A label $k \in K \setminus \{\bot\}$ is called an *implementation label* if $k' \sqsubseteq k$ implies $k' = k$ for all $k' \in K \setminus \{\bot\}$. In other words, implementation labels are all elements in K just above $\bot$. The set of all implementation labels of $(K, \sqsubseteq)$ is denoted by $\mathrm{Imp}(K, \sqsubseteq)$. To each label $k \in K$ we associate the set $[\![k]\!]$ of all implementation labels below k by

$$[\![k]\!] = \{k' \in \mathrm{Imp}(K, \sqsubseteq) \mid k' \sqsubseteq k\}.$$

Definition 2 (Well-formed label-set). A label-set $(K, \sqsubseteq)$ with the least element $\bot \in K$ is called *well-formed* if $[\![k]\!] \neq \emptyset$ for every $k \in K \setminus \{\bot\}$.

Well-formedness of label-sets ensures consistency of the label refinement relation $\sqsubseteq$, in other words it should always be possible to refine any label into an implementation label. We can now define label-structured modal transition systems that combine the underlying may/must transition relation known from modal transition systems with the label structure defined above.

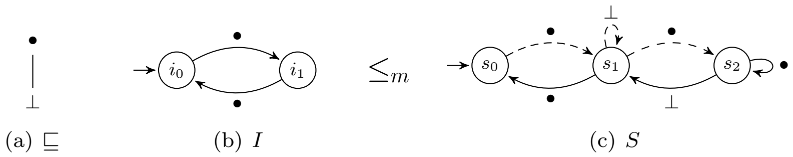
Definition 3 (Label-structured modal transition system). A *label-structured modal transition system (LSMTS)* is a tuple $(S, s_0, (K, \sqsubseteq), \dashrightarrow, \longrightarrow)$ where S is a set of states with the initial state $s_0 \in S$, $(K, \sqsubseteq)$ is a well-formed label-set, $\dashrightarrow \subseteq S \times K \times S$ is the *may* transition relation, and $\longrightarrow \subseteq S \times K \times S$ is the *must* transition relation such that $\longrightarrow \subseteq \dashrightarrow$.

We write $s \overset{k}{\dashrightarrow} s'$ if $(s, k, s') \in \dashrightarrow$. If for some $k \in K$ no state $s' \in S$ exists such that $s \overset{k}{\dashrightarrow} s'$ we write $s \overset{k}{\not\dashrightarrow}$, and if there exists some $s' \in S$ such that $s \overset{k}{\dashrightarrow} s'$ we write $s \overset{k}{\dashrightarrow}$. The aforementioned notations apply also to $\longrightarrow$. By abuse of notation, we use S to denote an LSMTS $(S, s_0, (K_S, \sqsubseteq_S), \dashrightarrow_S, \longrightarrow_S)$ and the subscripts are omitted if they are clear from the context. The notation (s, S) denotes the LSMTS S with the initial state s_0 replaced by s. The class of all LSMTSs with the well-formed label-set $(K, \sqsubseteq)$ is denoted by $\mathcal{M}_{(K,\sqsubseteq)}$, and we typically use capital letters S, T, U to range over this class.

An LSMTS S is called an *implementation* if $\longrightarrow = \dashrightarrow$ and all labels on the transitions are implementation labels, that is, for all $s \xrightarrow{k} s'$ in S we have $k \in \mathrm{Imp}(K, \sqsubseteq)$. The class of all implementations with well-formed label-set $(K, \sqsubseteq)$ is denoted by $\mathcal{I}_{(K,\sqsubseteq)}$, and we typically use capital letters I and J to range over this class.

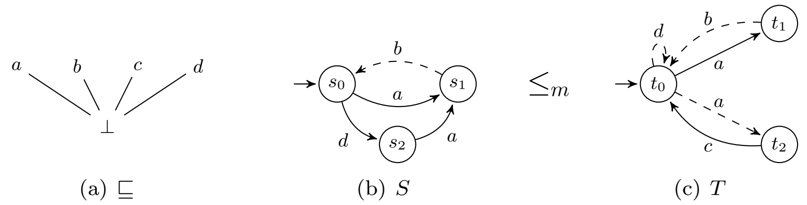
In the following, LSMTSs will be often represented as graphs with the convention that whenever two states are connected by both a must and a may transition under the same label, then we draw only the must transition.

Example 1. The most trivial instance of LSMTSs is obtained by choosing the well-formed label-set $K_{unlabelled} = (\{\bot, \bullet\}, \sqsubseteq)$ where $\sqsubseteq = \{(\bot, \bot), (\bot, \bullet), (\bullet, \bullet)\}$ illustrated



(a) $\sqsubseteq$ (b) I (c) S

Fig. 1. Unlabelled modal transition system over the label-set $K_{unlabelled}$



(a) $\sqsubseteq$ (b) S (c) T

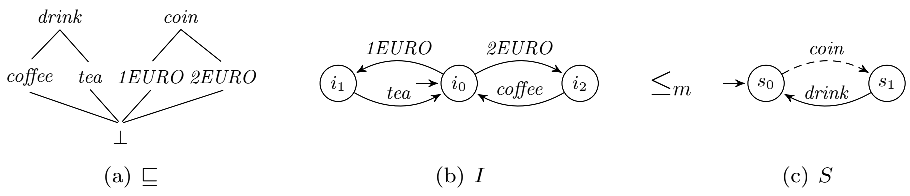
Fig. 2. Modal transition systems over the label-set K_{action}

in Figure 1(a). This label-set gives rise to unlabelled modal transition systems where $\bullet$ models a single implementation label and $\bot$ is the inconsistency label. An example is shown in Figure 1(b) and (c). The LSMTS I in Figure 1(b) is an implementation because every label is an implementation label and the may and must transition relations coincide. Note that the LSMTS in Figure 1(c) is not an implementation as (i) there are transitions labelled with $\bot$ and (ii) there are several may transitions without the corresponding must ones. The definition and explanation of modal refinement, denoted $\leq_m$, is deferred to Section 2.2. ■

Example 2. A well-known instance of the framework is obtained by considering a finite set of actions Σ and defining a well-formed label-set K_{action} by $K_{action} = (\Sigma \cup \{\bot\}, \sqsubseteq)$ where $a \sqsubseteq b$ if and only if $a = \bot$ or $a = b$. Here all labels (apart from $\bot$) are implementation labels and this setting corresponds exactly to the class of modal transition systems (Larsen and Thomsen, 1988b). Illustration of the label-set K_{action} and two examples of modal transition systems are given in Figure 2. ■

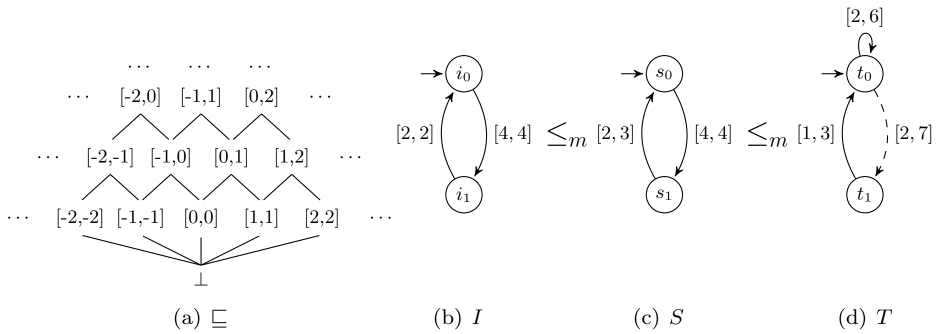
Example 3. As another example of a well-formed label-set demonstrating a more interesting label refinement, we can consider the following structure $K_{machine} = (\{\mathit{drink}, \mathit{coffee}, \mathit{tea}, \mathit{coin}, \mathit{1EURO}, \mathit{2EURO}, \bot\}, \sqsubseteq)$ where the ordering $\sqsubseteq$ is given in Figure 3(a). Here it is possible to provide a high-level specification of a vending machine by using the labels *drink* and *coin* that can be later in a concrete implementation refined into the implementation labels *coffee* and *tea*, and *1EURO* and *2EURO*, respectively. ■

Example 4. Another instance of the framework is called weighted modal automata. Here the well-formed label-set $K_{weighted}$ is given as a set of integer intervals with the natural inclusion ordering, formally $K_{weighted} = (K, \sqsubseteq)$ where $K = \{[a, b] \mid a, b \in \mathbb{Z} \text{ s.t. } a \leq b\} \cup \{\bot\}$ and $[a', b'] \sqsubseteq [a, b]$ if $a \leq a'$ and $b' \leq b$, $\bot \sqsubseteq [a, b]$, and $\bot \sqsubseteq \bot$, for all $a, a', b, b' \in \mathbb{Z}$. It follows that implementation labels are singleton sets of the form $[a, a]$ where $a \in \mathbb{Z}$.



(a) $\sqsubseteq$ (b) I (c) S

Fig. 3. Vending machines over the label-set $K_{machine}$



(a) $\sqsubseteq$ (b) I (c) S (d) T

Fig. 4. Weighted modal automata over $K_{weighted}$

Consult Figure 4 for the illustration of $K_{weighted}$ and for three examples of weighted modal automata. The automaton I is an implementation while S and T are not. ■

2.1. *Product of Labels*

In this subsection we will discuss a product construction on labels which will allow us to form (by a general construction) new instances of the framework from existing ones.

Definition 4 (Product). Let $(K_1, \sqsubseteq_1)$ and $(K_2, \sqsubseteq_2)$ be two label-sets with the least elements $\bot_1$ and $\bot_2$, respectively. The *product* $(K_1, \sqsubseteq_1) \otimes (K_2, \sqsubseteq_2)$ of the two label-sets is a label-set $(K, \sqsubseteq)$ where $K = \big((K_1 \smallsetminus \{\bot_1\}) \times (K_2 \smallsetminus \{\bot_2\})\big) \cup \{\bot\}$ and $(k_1', k_2') \sqsubseteq (k_1, k_2)$ if $k_1' \sqsubseteq_1 k_1$ and $k_2' \sqsubseteq_2 k_2$ for all $k_1, k_1' \in K_1 \smallsetminus \{\bot_1\}$ and all $k_2, k_2' \in K_2 \smallsetminus \{\bot_2\}$, and $\bot \sqsubseteq \ell$ for all $\ell \in K$.

It is easy to observe that the product construction preserves well-formedness and implementations are derived component-wise as stated in the following lemma.

Lemma 1. Let $(K_1, \sqsubseteq_1)$ and $(K_2, \sqsubseteq_2)$ be well-formed label-sets. Then

1. $(K_1, \sqsubseteq_1) \otimes (K_2, \sqsubseteq_2)$ is a well-formed label-set, and
2. $\mathrm{Imp}((K_1, \sqsubseteq_1) \otimes (K_2, \sqsubseteq_2)) = \mathrm{Imp}(K_1, \sqsubseteq_1) \times \mathrm{Imp}(K_2, \sqsubseteq_2)$.

Using the product construction of label-sets, we can e.g. combine the previously introduced well-formed label-sets K_{action} and $K_{weighted}$ from Examples 2 and 4 into weighted modal transition systems using the label-set $K_{action} \otimes K_{weighted}$ or into multi-weighted modal transition systems using the label-set $K_{action} \otimes K_{weighted} \otimes K_{weighted} \otimes \ldots \otimes K_{weighted}$ and further combine these with other quantitative aspects.

2.2. *Refinement*

We shall now define the notion of modal refinement that combines the label refinement, given by the partial ordering on the label-set, with the allowed transitions that may be present and required transitions that must be present. It is a generalization of the original notion of modal refinement over classical modal transition systems (Larsen and Thomsen, 1988b).

Definition 5 (Modal refinement). Let $S, T \in \mathcal{M}_{(K,\sqsubseteq)}$ be two LSMTSs with initial states s_0 and t_0, respectively. We say that S *modally refines* T, written $S \leq_m T$, if there exists a relation $R \subseteq S \times T$ with $(s_0, t_0) \in R$ such that for every $(s,t) \in R$:

1. whenever $s \overset{k}{\dashrightarrow} s'$ then there is $t \overset{\ell}{\dashrightarrow} t'$ such that $k \sqsubseteq \ell$ and $(s', t') \in R$, and
2. whenever $t \xrightarrow{\ell} t'$ then there is $s \xrightarrow{k} s'$ such that $k \sqsubseteq \ell$ and $(s', t') \in R$.

The *implementation semantics* of an LSMTS $S \in \mathcal{M}_{(K,\sqsubseteq)}$ is defined as the class $[\![S]\!]$ of all implementations refining S, i.e. $[\![S]\!] = \{I \in \mathcal{I}_{(K,\sqsubseteq)} \mid I \leq_m S\}$.

Example 5. Refinement of modal transition systems labelled with actions (see Example 2) is illustrated in Figure 2. The system S is a modal refinement of the system T, and the relation demonstrating this is given by $\{(s_0, t_0), (s_1, t_1), (s_2, t_0)\}$. Note that S is not an implementation yet, as it contains a may transition under b without a must transition under the same label. ■

Example 6. Consider the label-set $K_{machine}$ from Example 3. A specification of a vending machine is depicted in Figure 3(c). It allows to enter a coin and, should this happen, it requires that a drink is returned to the customer. One of the possible implementations (where all labels are implementation labels and the may and must transition relations coincide) of this specification is given in Figure 3(b). The modal refinement between the implementation and specification is easily demonstrated by the relation $\{(i_0, s_0), (i_1, s_1), (i_2, s_1)\}$. ■

Example 7. Refinement of weighted modal automata (see Example 4) is illustrated in Figure 4. The relation $\{(s_0, t_0), (s_1, t_0)\}$ is witnessing the modal refinement between Figure 4(d) and 4(c). Note that the refined specification in Figure 4(c) is not an implementation yet as it contains the label $[2, 3]$ which is not an implementation label. We can thus refine it further, ending up with an implementation as seen in Figure 4(b). The witnessing relation is $\{(i_0, s_0), (i_1, s_1)\}$. ■

Lemma 2. The modal refinement relation $\leq_m$ is a preorder.

Proof. Reflexivity is trivial. Transitivity can be seen as follows. Let $S, T, U \in \mathcal{M}_{(K,\sqsubseteq)}$ be three LSMTSs with their initial states s_0, t_0, u_0 such that $S \leq_m T \leq_m U$. From the assumption $S \leq_m T$ we know that there exists a witnessing relation $R_1 \subseteq S \times T$, and from the assumption $T \leq_m U$ we know that there exists a witnessing relation $R_2 \subseteq T \times U$. We define a relation $R \subseteq S \times U$ by the relational composition of R_1 and R_2, i.e.

$$R = \{(s,u) \mid \exists t \in T : (s,t) \in R_1 \text{ and } (t,u) \in R_2\}.$$

We show that R is proving $S \leq_m U$. Obviously $(s_0, u_0) \in R$. Now, let $(s,u) \in R$ be an arbitrary element of R. Let $t \in T$ be a state such that $(s,t) \in R_1$ and $(t,u) \in R_2$.

1. Assume $s \overset{k_s}{\dashrightarrow} s'$. From $(s,t) \in R_1$ it follows that there exists $t \overset{k_t}{\dashrightarrow} t'$ such that $k_s \sqsubseteq k_t$ and $(s',t') \in R_1$. Then, as $(t,u) \in R_2$, we get $u \overset{k_u}{\dashrightarrow} u'$ such that $k_t \sqsubseteq k_u$ and $(t',u') \in R_2$, hence $(s',u') \in R$ and by transitivity of $\sqsubseteq$ also $k_s \sqsubseteq k_u$.
2. Symmetric to the previous direction.

□

Modal refinement induces an equivalence relation on LSMTS. We say that S and T are *equivalent*, denoted by $S \equiv_m T$, if both $S \leq_m T$ and $T \leq_m S$ are satisfied.

Lemma 3. Let $I, J \in \mathcal{I}_{(K,\sqsubseteq)}$ be two implementations. Then $I \leq_m J$ implies $I \equiv_m J$.

Proof. Given a relation R witnessing $I \leq_m J$ we can use $R^{-1} = \{(j,i) \mid (i,j) \in R\}$ to prove that $J \leq_m I$. Let $(j,i) \in R^{-1}$.

1. Assume that $i \xrightarrow{k} i'$. Remember that then also $i \overset{k}{\dashrightarrow} i'$. From the fact that $(i,j) \in R$ it follows that there exists $j \overset{\ell}{\dashrightarrow} j'$ such that $k \sqsubseteq \ell$ and $(i',j') \in R$. Since J is an implementation, we get that $k = \ell$ and $j \xrightarrow{\ell} j'$. Obviously, $(j',i') \in R^{-1}$.
2. The other direction is symmetric.

□

The notion of modal refinement can be understood as refinement defined at the syntactical level as it directly relates the states of two specifications. A semantically motivated notion of refinement, usually called *thorough refinement*, says that S is a refinement of T if every implementation of S is also an implementation of T.

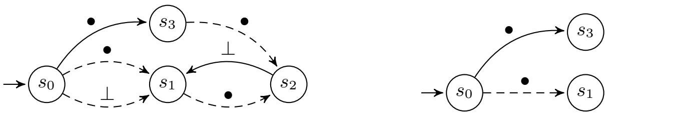
Definition 6 (Thorough refinement). Let $S, T \in \mathcal{M}_{(K,\sqsubseteq)}$ be two LSMTSs. We say that S *thoroughly refines* T, written $S \leq_t T$, if $[\![S]\!] \subseteq [\![T]\!]$.

It is an expected result that modal refinement implies thorough refinement, as stated in the following soundness theorem. The opposite implication does not hold in general and details are discussed in Section 2.4.

Theorem 1 (Soundness). Let $S, T \in \mathcal{M}_{(K,\sqsubseteq)}$ be two LSMTSs. Then $S \leq_m T$ implies $S \leq_t T$.

Proof. Follows immediately from the transitivity of modal refinement (see Lemma 2).

□



(a) A consistent, but locally inconsistent LSMTS S

(b) The locally consistent result $\rho(S)$ of pruning S

Fig. 5. Example of the pruning operator

2.3. *Consistency and Pruning*

Similar to the classical notion of consistency, an LSMTS S is consistent if it has at least one implementation.

Definition 7 (Consistency). Let $S \in \mathcal{M}_{(K,\sqsubseteq)}$. The LSMTS S is *consistent* if $[\![S]\!] \neq \emptyset$.

Consistency is a semantical notion and in the rest of this article it will be useful to introduce also a syntactical notion of consistency, called local consistency.

Definition 8 (Local consistency). Let $S \in \mathcal{M}_{(K,\sqsubseteq)}$. A state $s \in S$ is *locally consistent* if $s \not\xrightarrow{\perp}$. The LSMTS S is *locally consistent* if all states of S are locally consistent.

From our assumption of well-formedness of label-sets, it follows that any locally consistent S has at least one implementation, thus local consistency implies consistency. The converse is not true as explained in the following example.

Example 8. Consider the LSMTS S presented in Figure 5(a) with the label-set $K_{unlabelled}$ from Example 1. The system S is clearly not locally consistent but it is consistent as an implementation with just two states connected by a must (and may) transition labelled with • is an implementation of S. ■

We shall now define a pruning operator that removes locally inconsistent states.

Definition 9 (Pruning). Let $S \in \mathcal{M}_{(K,\sqsubseteq)}$ be an LSMTS and let $B \subseteq S$ be a subset of its states. Let

$$pre(B) = \{s \in S \mid s \xrightarrow{k} s' \text{ and } s' \in B \text{ for some } k \in K\}$$

and $pre^0(B) = B$, $pre^{j+1}(B) = pre(pre^j(B))$ for $j \geq 0$, and $pre^*(B) = \bigcup_{j\geq 0} pre^j(B)$. The *pruning* $\rho(S)$ of S is defined if $s_0 \notin pre^*(\mathsf{bad})$ where $\mathsf{bad} = \{s \in S \mid s \xrightarrow{\perp}\}$, and in this case, $\rho(S)$ is the LSMTS $(S^\rho, s_0, (K, \sqsubseteq), \dashrightarrow_\rho, \longrightarrow_\rho)$ where

$$\begin{aligned} S^\rho &= S \setminus pre^*(\mathsf{bad}), \\ \dashrightarrow_\rho &= \dashrightarrow \cap \big(S^\rho \times (K \setminus \{\perp\}) \times S^\rho\big), \text{ and} \\ \longrightarrow_\rho &= \longrightarrow \cap \big(S^\rho \times (K \setminus \{\perp\}) \times S^\rho\big). \end{aligned}$$

It is clear that for an LSMTS with n states one can compute $pre^*(\mathsf{bad})$ by finitely many iterations, more precisely $pre^*(\mathsf{bad}) = pre^n(\mathsf{bad})$ in this case. Also note that well-definedness of pruning is equivalent to the absence of a path of must transitions from the initial state to a locally inconsistent state (that enforces inconsistency via must transition labelled with $\bot$).

Figure 5 shows the application of the pruning operator ρ to the system S and one can easily observe that $\rho(S) \leq_m S$. Pruning also does not remove any implementation. A summary of the properties of pruning is given in the following proposition.

Proposition 1. Let $S \in \mathcal{M}_{(K,\sqsubseteq)}$. If $\rho(S)$ is defined, then

1. $\rho(S)$ is locally consistent,
2. $\rho(S) \leq_m S$,
3. $[\![\rho(S)]\!] = [\![S]\!]$, and
4. for any locally consistent $T \in \mathcal{M}_{(K,\sqsubseteq)}$, if $T \leq_m S$ then $T \leq_m \rho(S)$.

Moreover, $\rho(S)$ is defined if and only if S is consistent.

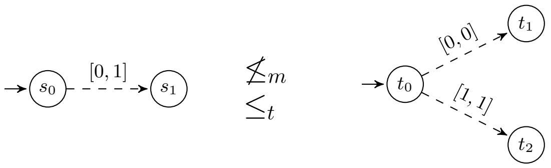
Proof.

1. As all labels $\bot$ were removed in $\rho(S)$, it is trivially locally consistent.
2. We will show that the relation $R = \{(s,s) \mid s \in S \setminus \mathit{pre}^*(\mathsf{bad})\}$ is a refinement relation in order to argue that $\rho(S) \leq_m S$. Let $(s,s) \in R$. If $s \overset{k}{\dashrightarrow}_\rho s'$ in $\rho(S)$ then this by the construction implies that $s' \in S \setminus \mathit{pre}^*(\mathsf{bad})$. Clearly we have also $s \overset{k}{\dashrightarrow} s'$ in S and $(s',s') \in R$. On the other hand, if $s \xrightarrow{k} s'$ in S then $s' \notin \mathit{pre}^*(\mathsf{bad})$ as $s \notin \mathit{pre}^*(\mathsf{bad})$, which means that $s \xrightarrow{k}_\rho s'$ also in $\rho(S)$ and $(s',s') \in R$.
3. The inclusion $[\![\rho(S)]\!] \subseteq [\![S]\!]$ follows from the fact that $\rho(S) \leq_m S$. Let $I \in [\![S]\!]$. This means that there is a refinement relation R demonstrating that $I \leq_m S$. We will argue that R is also a refinement relation demonstrating $I \leq_m \rho(S)$. However, this easily follows from the observation that R cannot contain any state from $\mathit{pre}^*(\mathsf{bad})$, because otherwise a must path to the label $\bot$ will be enforced in I too, but then I is not an implementation.
4. The same argumentation as in the previous point applies also here. Any refinement relation demonstrating $T \leq_m S$ can be used to establish also $T \leq_m \rho(S)$. In order to apply the reasoning as above, it is important that T does not contain any transition with the label $\bot$.

For the last claim, observe that if $\rho(S)$ is not defined then there is a must path from its initial state to a state requiring a transition under $\bot$. Any implementation then has to contain such a path to a state requiring a transition under $\bot$, but then it is not an implementation. On the other hand, if $\rho(S)$ is defined then we can change every transition in $\rho(S)$ to a must transition and replace every label by some implementation label below it (possible thanks to well-formedness of the label-set) in order to construct an implementation of $\rho(S)$, which is also an implementation of S as $\rho(S) \leq_m S$. □

2.4. *Determinism and Completeness of Refinement*

In general, thorough refinement does not imply modal refinement. A counterexample, using the label-set $K_{weighted}$ (see Example 7), is given in Figure 6. Clearly, the transition $s_0 \overset{[0,1]}{\dashrightarrow} s_1$ cannot be matched by any of the two transitions from t_0 as their labels are



(a) S

(b) T

Fig. 6. Incompleteness of modal refinement demonstrated by systems S and T

less general than $[0, 1]$. Hence $S \not\leq_m T$. On the other hand, any implementation of S is either empty or it is a tree of height one with the outgoing edges labelled by either $[0, 0]$ or $[1, 1]$. All such implementations are also refinements of the system T.

It is known that for classical modal transition systems thorough refinement implies the modal one, under the assumption of determinism (Beneš et al., 2009b). We can generalize this result to the class of label-structured modal transition systems. Before we define when an LSMTS is deterministic, we first define when two labels k_1, k_2 are unifiable, that is, if there is another label k which overlaps with k_1 and k_2 with respect to their sets of implementation labels.

Definition 10 (Unifiable labels). Two labels $k_1, k_2 \in K$ are called *unifiable* if there exists $k \in K$ such that $[\![k]\!] \cap [\![k_1]\!] \neq \emptyset$ and $[\![k]\!] \cap [\![k_2]\!] \neq \emptyset$.

Then, determinism expresses that for any two outgoing may transitions from the same state under two different labels k_1 and k_2, the labels k_1 and k_2 are not unifiable.

Definition 11 (Determinism). A LSMTS S is called *deterministic* if for any state $s \in S$ and any two transitions $s \overset{k_1}{\dashrightarrow} s'_1$ and $s \overset{k_2}{\dashrightarrow} s'_2$, if k_1 and k_2 are unifiable, then $k_1 = k_2$ and $s'_1 = s'_2$.

Returning to Figure 6 we can realize that the system T is not deterministic as there is a branching of the transitions with labels $[0, 0]$ and $[1, 1]$, while there exists a label $[0, 1]$ such that $[\![[0, 1]]\!] \cap [\![[0, 0]]\!] \neq \emptyset$ and $[\![[0, 1]]\!] \cap [\![[1, 1]]\!] \neq \emptyset$.

A very natural assumption that has to be imposed on the label-sets later on in order to show completeness of modal refinement, is completeness of label refinement: inclusion of implementation labels implies label refinement.

Definition 12 (Completeness of label refinement). Let $(K, \sqsubseteq)$ be a label-set. Label refinement $\sqsubseteq$ is *complete* if for all $k, \ell \in K$, $[\![k]\!] \subseteq [\![\ell]\!]$ implies $k \sqsubseteq \ell$.

All examples of label-sets provided in this article satisfy this property. Note that label refinement is always sound by definition, i.e. $k \sqsubseteq \ell$ implies $[\![k]\!] \subseteq [\![\ell]\!]$ by transitivity of label refinement.

Under the assumption of (1) completeness of label refinement, (2) determinism of the refined LSMTS, and (3) local consistency of the refining LSMTS, thorough refinement implies the modal one.

Theorem 2 (Completeness). Let $(K, \sqsubseteq)$ be a well-formed label-set for which label refinement $\sqsubseteq$ is complete. Let $S, T \in \mathcal{M}_{(K,\sqsubseteq)}$ with initial states s_0 and t_0, respectively, such that S is locally consistent and T is deterministic. Then $S \leq_t T$ implies $S \leq_m T$.

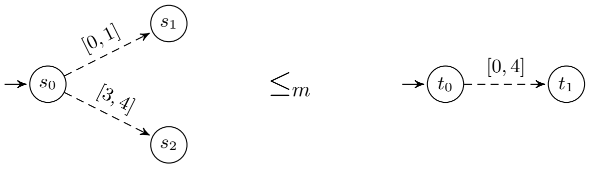
Proof. Assume that $S \leq_t T$. We define a relation $R \subseteq S \times T$ as the smallest relation satisfying:

1. $(s_0, t_0) \in R$,
2. if $(s,t) \in R$, $s \overset{k}{\dashrightarrow} s'$, $t \overset{\ell}{\dashrightarrow} t'$, and $\llbracket k \rrbracket \cap \llbracket \ell \rrbracket \neq \emptyset$ then $(s', t') \in R$.

First, we show a technical result (that we use later on) saying that any $(s,t) \in R$ satisfies $\llbracket (s,S) \rrbracket \subseteq \llbracket (t,T) \rrbracket$. For $(s_0,t_0) \in R$, we have $\llbracket (s_0,S) \rrbracket = \llbracket S \rrbracket \subseteq \llbracket T \rrbracket = \llbracket (t_0,T) \rrbracket$ from the assumption $S \leq_t T$. Now, let $(s,t) \in R$ such that $\llbracket (s,S) \rrbracket \subseteq \llbracket (t,T) \rrbracket$ and assume that $s \overset{k}{\dashrightarrow} s'$, $t \overset{\ell}{\dashrightarrow} t'$, and $\llbracket k \rrbracket \cap \llbracket \ell \rrbracket \neq \emptyset$. Let $I' \in \llbracket (s',S) \rrbracket$ and let $m \in \llbracket k \rrbracket \cap \llbracket \ell \rrbracket$ which exists by the construction. Then, since S is locally consistent, there exists an implementation $(i_0, I) \in \llbracket (s,S) \rrbracket$ such that $i_0 \xrightarrow{m} i'$ and $(i', I) \leq_m I'$. From $\llbracket (s,S) \rrbracket \subseteq \llbracket (t,T) \rrbracket$ it follows that $I \in \llbracket (t,T) \rrbracket$. Then there exists a transition $t \overset{\ell'}{\dashrightarrow} t''$ such that $(i', I) \in \llbracket (t'', T) \rrbracket$ and $m \in \llbracket \ell' \rrbracket$. Now, we have $t \overset{\ell}{\dashrightarrow} t'$ and $t \overset{\ell'}{\dashrightarrow} t''$ such that $m \in \llbracket \ell \rrbracket \cap \llbracket \ell' \rrbracket$, hence ℓ and ℓ' are unifiable. As T is deterministic it follows that $\ell = \ell'$ and $t' = t''$, so $(i', I) \in \llbracket (t', T) \rrbracket$. Finally, from $(i', I) \leq_m I'$ and Lemma 3 it follows that $(i', I) \equiv_m I'$ and hence $I' \in \llbracket (t', T) \rrbracket$.

Now we show that R is a relation witnessing $S \leq_m T$. Clearly $(s_0, t_0) \in R$. Let $(s,t) \in R$.

1. Assume $s \overset{k}{\dashrightarrow} s'$. By local consistency of S we can assume that $k \neq \perp$. Then, for each implementation label $m \in \llbracket k \rrbracket$, there exists an implementation $I_m \in \llbracket (s,S) \rrbracket$ such that $i_0 \xrightarrow{m} i'$. We also know that $I_m \in \llbracket (t,T) \rrbracket$ because $\llbracket (s,S) \rrbracket \subseteq \llbracket (t,T) \rrbracket$. Hence there exists a transition $t \overset{\ell_m}{\dashrightarrow} t'_m$ such that $m \in \llbracket \ell_m \rrbracket$. We have to show that, for all $m \in \llbracket k \rrbracket$, the labels ℓ_m are the same. Suppose that there are $m_1, m_2 \in \llbracket k \rrbracket$ and transitions $t \overset{\ell_{m_1}}{\dashrightarrow} t'_{m_1}$ and $t \overset{\ell_{m_2}}{\dashrightarrow} t'_{m_2}$ such that $m_1 \in \llbracket \ell_{m_1} \rrbracket$ and $m_2 \in \llbracket \ell_{m_2} \rrbracket$. Then, since $m_1 \in \llbracket \ell_{m_1} \rrbracket \cap \llbracket k \rrbracket$ and $m_2 \in \llbracket \ell_{m_2} \rrbracket \cap \llbracket k \rrbracket$ and T is deterministic, it follows that $\ell_{m_1} = \ell_{m_2}$ and $t'_{m_1} = t'_{m_2}$. It follows that there is a unique transition $t \overset{\ell}{\dashrightarrow} t'$ such that $m \in \llbracket \ell \rrbracket$ for all implementation labels $m \in \llbracket k \rrbracket$, this means $\llbracket k \rrbracket \subseteq \llbracket \ell \rrbracket$ which implies $k \sqsubseteq \ell$ by completeness of label refinement. Moreover, by the definition of R, we get $(s', t') \in R$.
2. Assume $t \xrightarrow{\ell} t'$. If $\ell = \perp$ then there is a transition $s \xrightarrow{\perp} s'$ in S, contradicting local consistency of S; hence $\ell \neq \perp$. Then, for each implementation $I \in \llbracket (t,T) \rrbracket$ we have that there exists a transition $i_0 \xrightarrow{m} i'$ for some label $m \in \llbracket \ell \rrbracket$. We know that $\llbracket (s,S) \rrbracket \subseteq \llbracket (t,T) \rrbracket$, so every implementation $(j_0, J) \in \llbracket (s,S) \rrbracket$ has a transition $j_0 \xrightarrow{m} j'$. It follows that S must have a transition $s \xrightarrow{k} s'$ such that $m \in \llbracket k \rrbracket$. Suppose that $\llbracket k \rrbracket \not\subseteq \llbracket \ell \rrbracket$, then there would exist an implementation $(\bar{j_0}, \bar{J}) \in \llbracket (s,S) \rrbracket$ having $\bar{j}_0 \xrightarrow{n} \bar{j}'$ with an implementation label $n \in \llbracket k \rrbracket$ not belonging to $\llbracket \ell \rrbracket$, which means that there must exist another transition in T, say $t \overset{\ell'}{\dashrightarrow} t''$ such that $n \in \llbracket \ell' \rrbracket$. But then we have $m \in \llbracket \ell \rrbracket \cap \llbracket k \rrbracket$ and $n \in \llbracket \ell' \rrbracket \cap \llbracket k \rrbracket$ which contradicts determinism of T.



(a) S (b) $\mathcal{D}(S)$

Fig. 7. Determinization

Thus, we have $[\![k]\!] \subseteq [\![\ell]\!]$, which implies $k \sqsubseteq \ell$ by completeness of label refinement. Moreover, by definition of R, we get $(s', t') \in R$.

□

2.5. *Determinization*

It is a well-known fact that, for almost all existing specification theories, deciding thorough refinement involves more complex decision procedures than the ones that can be used to decide modal refinement (see e.g. (Beneš et al., 2009a) dealing with the classical modal transition systems). In the previous subsection, we have seen that in case of deterministic systems modal and thorough refinements coincide and can be decided by efficient syntactical fixed-point based algorithms. In Section 3.1 it will become moreover evident that determinism plays an important role in establishing several soundness results.

It is thus worth studying a general procedure that, given a nondeterministic LSMTS S, computes its smallest deterministic over-approximation $\mathcal{D}(S)$, called deterministic hull.

Example 9. Consider the system in Figure 7(a) where S is a LSMTS with label-set $K_{weighted}$. It is nondeterministic since there is the label $[0,4]$ for which $[\![[0,4]]\!] \cap [\![[0,1]]\!] \neq \emptyset$ and $[\![[0,4]]\!] \cap [\![[3,4]]\!] \neq \emptyset$. The best we can do is to approximate S by a deterministic LSMTS in Figure 7(b). However, $\mathcal{D}(S)$ is an over-approximation as $[\![S]\!] \subsetneq [\![\mathcal{D}(S)]\!]$ witnessed by the implementation with a single must transition labelled with $[2,2]$.

In the following, we will propose an algorithm that computes, for a given LSMTS S, a deterministic LSMTS $\mathcal{D}(S)$ such that $\mathcal{D}(S)$ is an over-approximation of S with the property that it is a minimal one with respect to modal refinement. The construction is a generalization of the algorithm presented in (Beneš et al., 2009b) and dealing specifically with modal transition systems.

We impose the following assumption on the label-set $(K, \sqsubseteq)$ necessary for applying the determinization algorithm. For any set $L \subseteq K$ of pairwise unifiable labels we require the existence of the least upper bound $lub(L) \in K$ such that $\ell \sqsubseteq lub(L)$ for all $\ell \in L$ and whenever $\ell \sqsubseteq \ell'$ for all $\ell \in L$ then $lub(L) \sqsubseteq \ell'$.

Definition 13 (Deterministic hull). Let $S = (S, s_0, (K, \sqsubseteq), \dashrightarrow, \longrightarrow)$ be an LSMTS. The *deterministic hull of* S is defined by the LSMTS

$$\mathcal{D}(S) = (\mathcal{P}(S) \setminus \{\emptyset\}, \{s_0\}, (K, \sqsubseteq), \dashrightarrow_{\mathrm{D}}, \longrightarrow_{\mathrm{D}})$$

where the transition relations $\dashrightarrow_{\mathrm{D}}$ and $\longrightarrow_{\mathrm{D}}$ are defined as follows. Let $\mathcal{T} \in (\mathcal{P}(S) \setminus \{\emptyset\})$

be a state in $\mathcal{D}(S)$. For every maximal, nonempty set $L \subseteq \{k \mid s \overset{k}{\dashrightarrow},\ s \in \mathcal{T}\}$ of pairwise unifiable labels we have $\mathcal{T} \overset{\ell}{\dashrightarrow}_{\mathrm{D}} \mathcal{T}_\ell$ where $\ell = \mathit{lub}(L)$ and $\mathcal{T}_\ell = \{s' \in S \mid s \overset{k}{\dashrightarrow} s',\ s \in \mathcal{T},\ k \in L\}$. If, moreover, for each $s \in \mathcal{T}$ we have $s \xrightarrow{k} s'$ for some $s' \in \mathcal{T}_\ell$ and some $k \in K$ such that $k \sqsubseteq \ell$, then $\mathcal{T} \xrightarrow{\ell}_{\mathrm{D}} \mathcal{T}_\ell$.

Now we show that $\mathcal{D}(S)$ is the smallest deterministic over-approximation of S.

Theorem 3 (Soundness and minimality of determinization). Let $S \in \mathcal{M}_{(K,\sqsubseteq)}$. Then the following holds:

1. $\mathcal{D}(S)$ is deterministic,
2. $S \leq_m \mathcal{D}(S)$, and
3. for every deterministic $D \in \mathcal{M}_{(K,\sqsubseteq)}$, if $S \leq_m D$ then $\mathcal{D}(S) \leq_m D$.

Proof.

1. Let $\mathcal{T}$ be a state in $\mathcal{D}(S)$ and assume that there exist two different transitions $\mathcal{T} \overset{\ell_1}{\dashrightarrow} \mathcal{T}_{\ell_1}$ and $\mathcal{T} \overset{\ell_2}{\dashrightarrow} \mathcal{T}_{\ell_2}$ such that ℓ_1 and ℓ_2 are unifiable. It follows that there exists a least upper bound $\ell \in K$ of ℓ_1 and ℓ_2. Hence all labels below ℓ_1 are unifiable with all labels below ℓ_2. This contradicts the fact that ℓ_1 and ℓ_2 are the least upper bounds of maximal, unifiable sets of labels.
2. We define a relation $R \subseteq S \times \mathcal{D}(S)$ by $R = \{(s, \mathcal{T}) \mid s \in \mathcal{T}\}$. Clearly, $(s_0, \{s_0\}) \in R$ for the respective initial states. Now, let $(s, \mathcal{T}) \in R$.

   First, assume that $s \overset{k}{\dashrightarrow} s'$. Then, since $s \in \mathcal{T}$, there exists a maximal, nonempty set $L \subseteq \{k \mid s \overset{k}{\dashrightarrow},\ s \in \mathcal{T}\}$ of pairwise unifiable labels such that $k \in L$. So there exists a transition $\mathcal{T} \overset{\ell}{\dashrightarrow} \mathcal{T}_\ell$ where $\ell = \mathit{lub}(L)$, which in particular means that $k \sqsubseteq \ell$. Moreover, by the definition of $\mathcal{T}_\ell$, we have that $s' \in \mathcal{T}_\ell$, hence $(s', \mathcal{T}_\ell) \in R$.

   Second, assume that $\mathcal{T} \xrightarrow{\ell} \mathcal{T}_\ell$. Then we know that for all $t \in \mathcal{T}$ there is $t \xrightarrow{k} t'$ such that $k \sqsubseteq \ell$ and $t' \in \mathcal{T}_\ell$. Since we know that $s \in \mathcal{T}$, it follows that there is $s \xrightarrow{k} s'$ such that $k \sqsubseteq \ell$ and $s' \in \mathcal{T}_\ell$, hence $(s', \mathcal{T}_\ell) \in R$.
3. Let $D \in \mathcal{M}_{(K,\sqsubseteq)}$ be a deterministic LSMTS and assume that $S \leq_m D$ witnessed by the relation R. We want to show that $\mathcal{D}(S) \leq_m D$. We define a relation $R' \subseteq \mathcal{D}(S) \times D$ by

   $$(\mathcal{T}, d) \in R' \qquad \text{if and only if} \qquad \emptyset \neq \mathcal{T} \subseteq \{s \in S \mid (s, d) \in R\}.$$

   We show that R' is a relation witnessing $\mathcal{D}(S) \leq_m D$. Clearly, $(\{s_0\}, d_0) \in R'$ for the corresponding initial states. Let $(\mathcal{T}, d) \in R'$.

   — First, assume that $\mathcal{T} \overset{\ell}{\dashrightarrow} \mathcal{T}_\ell$ and $\ell = \mathit{lub}(L)$ for some maximal, nonempty set $L \subseteq \{k \mid s \overset{k}{\dashrightarrow},\ s \in \mathcal{T}\}$. We want to show that there exists $d \overset{\ell'}{\dashrightarrow} d'$ such that $\ell \sqsubseteq \ell'$ and $(\mathcal{T}_\ell, d') \in R'$. Let

   $$\mathcal{S} = \{s \in \mathcal{T} \mid s \overset{k}{\dashrightarrow},\ k \in L\}$$

   which is nonempty since L is nonempty and it is a subset of $\{k \mid s \overset{k}{\dashrightarrow},\ s \in \mathcal{T}\}$. From the assumption $(\mathcal{T}, d) \in R'$, we know that for all $s \in \mathcal{S}$ it holds that

$(s,d) \in R$ by the definition of R'. Let

$$\Delta = \{s \overset{k}{\dashrightarrow} s' \mid s \in \mathcal{S} \text{ and } k \in L\}.$$

For every transition $s \overset{k}{\dashrightarrow} s'$ in Δ, it follows from $(s,d) \in R$ that there exists $d \overset{k'}{\dashrightarrow} d_{k'}$ in D such that $k \sqsubseteq k'$ and $(s', d_{k'}) \in R$. Let L' denote the set of all such k'. Since L is a set of pairwise unifiable labels and for every $k' \in L'$ there is some $k \in L$ such that $k \sqsubseteq k'$, we know that L' is a set of pairwise unifiable labels, too. For every $k' \in L'$ we have a transition $d \overset{k'}{\dashrightarrow} d_{k'}$ for some $d_{k'} \in D$. From the determinism of D, it follows that there exists $d \overset{\ell'}{\dashrightarrow} d'$ such that $d' = d_{k'}$ and $\ell' = k'$ for all $k' \in L'$. Hence $k \sqsubseteq \ell'$ for all $k \in L$. Since ℓ is the least upper bound of L, we can conclude that $\ell \sqsubseteq \ell'$. Moreover, it holds that $(\mathcal{T}_\ell, d') \in R'$ because $\mathcal{T}_\ell = \{s' \in S \mid s \overset{k}{\dashrightarrow} s',\ s \in \mathcal{T},\ k \in L\}$ is nonempty, and for every $s' \in \mathcal{T}_\ell$ it holds that $(s', d') \in R$.

— Second, assume that $d \xrightarrow{\ell'} d'$. By the definition of R' we know that $(s,d) \in R$ for all $s \in \mathcal{T}$. Then it follows that, for all $s \in \mathcal{T}$, we have $s \xrightarrow{k} s'$ such that $k \sqsubseteq \ell'$ and $(s', d') \in R$. Note that $\mathcal{T}$ is nonempty, thus there is some maximal, nonempty set $L \subseteq \{k \mid s \overset{k}{\dashrightarrow},\ s \in \mathcal{T}\}$ of pairwise unifiable labels such that $k \in L$ for all labels k where $s \overset{k}{\dashrightarrow}$ and $k \sqsubseteq \ell'$. This implies that there exists $\mathcal{T} \xrightarrow{\ell} \mathcal{T}_\ell$ with $\ell = \mathit{lub}(L)$. Now, we have to show that $\ell \sqsubseteq \ell'$. Let $k \in L$, then there exists $s \overset{k}{\dashrightarrow} s'$ with $s \in \mathcal{T}$, and from $(s,d) \in R$ it follows that $k \sqsubseteq \ell'$ by the determinism of D, and $(s', d') \in R$. Since ℓ is the least upper bound of L, it follows that $\ell \sqsubseteq \ell'$. Finally, we show that $(\mathcal{T}_\ell, d') \in R'$. The set $\mathcal{T}_\ell$ is clearly nonempty, and moreover, it holds that for every $s' \in \mathcal{T}_\ell$ we have $(s', d') \in R$.

□

We can use the above theorem to prove that given two LSMTSs S and T, whenever S thoroughly refines T, then $\mathcal{D}(S)$ modally (syntactically) refines $\mathcal{D}(T)$.

Corollary 1. Let $(K, \sqsubseteq)$ be a well-formed label-set for which the label refinement $\sqsubseteq$ is complete. Let $S, T \in \mathcal{M}_{(K,\sqsubseteq)}$ be locally consistent LSMTSs. If $S \leq_t T$ then $\mathcal{D}(S) \leq_m \mathcal{D}(T)$.

Proof. By Theorem 3 we know that $T \leq_m \mathcal{D}(T)$ and by Theorem 1 also $T \leq_t \mathcal{D}(T)$. By transitivity and the assumption $S \leq_t T$, it follows that $S \leq_t \mathcal{D}(T)$, which implies $S \leq_m \mathcal{D}(T)$ by completeness of refinement (Theorem 2). By minimality of $\mathcal{D}(S)$ we can conclude that $\mathcal{D}(S) \leq_m \mathcal{D}(T)$. □

3. Specification Theory

In order to apply label-structured modal transition systems as a specification formalism for software components, we need to define several operators on LSMTSs essential for any specification theory supporting compositional reasoning. First, *structural composition* (or *parallel composition*) allows us to combine interacting specifications. Second, *logical composition* (or *conjunction*) of two or more specifications expresses the greatest specification satisfying all the requirements of the given set of specifications. And third, a *quotient operator* which is dual to parallel composition: given a specification T expressing a requirement that needs to be implemented, and a specification S of the components that already exist, the quotient of T by S is the smallest specification which, together with S, satisfies T. All these operators are an important part of any compositional specification theory.

3.1. *Operators on Labels and their Product*

Operators on LSMTSs naturally involve different ways of combining labels. In this subsection, we introduce *label operators* on well-formed label-sets which will become crucial ingredients for the operators on LSMTSs introduced later on. In the text to follow, label operators are consistently denoted by symbols in circles.

Definition 14 (Label operator). A (binary) *label operator* on a label-set $(K, \sqsubseteq)$ is a partial function $\odot : K \times K \rightharpoonup K$.

A label operator $\odot$ on $(K, \sqsubseteq)$ is called *commutative* if $k \odot \ell$ is defined iff $\ell \odot k$ is defined, and if they are defined then $k \odot \ell = \ell \odot k$. The operator is *associative* if $k \odot (\ell \odot m)$ is defined iff $(k \odot \ell) \odot m$, and if they are defined then $k \odot (\ell \odot m) = (k \odot \ell) \odot m$.

We can also form products of label operators which are operators on the product of their label-sets.

Definition 15 (Product of label operators). Given two well-formed label-sets $(K_1, \sqsubseteq_1)$ and $(K_2, \sqsubseteq_2)$, and two label operators $\odot_1$ and $\odot_2$, the product of $\odot_1$ and $\odot_2$ is given by the label operator $\odot_1 \times \odot_2$ on $K_1 \otimes K_2$ which is defined as follows:

$$(k_1, k_2)(\odot_1 \times \odot_2)(\ell_1, \ell_2) = \begin{cases} (k_1 \odot_1 \ell_1, k_2 \odot_2 \ell_2) & \text{if } \bot \neq k_i \odot_i \ell_i \text{ is defined for } i \in \{1, 2\} \\ \bot & \text{if } k_i \odot_i \ell_i \text{ is defined for } i \in \{1, 2\} \\ & \text{and either } k_1 \odot \ell_1 = \bot \text{ or } k_2 \odot \ell_2 = \bot \\ \text{undefined} & \text{otherwise} \end{cases}$$

It is easy to see that if two label operators, $\odot_1$ and $\odot_2$, are commutative and associative then so is the product operator $\odot_1 \times \odot_2$.

3.2. *Parallel Composition*

We start with the parallel composition, an operator that reflects the standard structural composition of implementations at the specification level.

Two LSMTSs with the same well-formed label-set $(K, \sqsubseteq)$ can be structurally composed with respect to a label operator $\oplus$ on $(K, \sqsubseteq)$. Two transitions (in different parallel components) labelled with $k \in K$ and $\ell \in K$ can synchronize if $k \oplus \ell$ is defined. The synchronized transition is then labelled with the label $k \oplus \ell$.

The desired property of parallel composition, crucial for any compositional specification theory, is called compositional refinement. It allows for a step-wise refinement of individual specifications while their parallel composition is guaranteed to refine the parallel composition of the original specifications. However, in order to achieve this, we have to impose a natural requirement on the label operator $\oplus$ for composing labels.

Definition 16 (Compositional label operator). A label operator $\oplus$ on a well-formed label-set $(K, \sqsubseteq)$ is *compositional* if whenever $k' \sqsubseteq k$ and $\ell' \sqsubseteq \ell$ then $k' \oplus \ell'$ is defined if and only if $k \oplus \ell$ is defined, and in the positive case moreover $k' \oplus \ell' \sqsubseteq k \oplus \ell$.

As expected, compositionality of label operators is preserved under their products.

Proposition 2. If $\oplus_1$ and $\oplus_2$ are compositional label operators on well-formed label-sets $(K_1, \sqsubseteq_1)$ and $(K_2, \sqsubseteq_2)$ respectively, then $\oplus_1 \times \oplus_2$ is a compositional label operator on $(K_1, \sqsubseteq_1) \otimes (K_2, \sqsubseteq_2)$.

The actual definition of $\oplus$ depends on the interpretation of parallel composition for the modelled quantity. We present several possible definitions of $\oplus$ for some of the examples seen so far. All of the defined operators are compositional label operators and they can be further combined using the product construction.

Example 10. In Example 2 we have instantiated LSMTSs to modal transition systems labelled with actions, with well-formed label-set $K_{action} = (\Sigma \cup \{\bot\}, \sqsubseteq)$ for a finite set of actions Σ. The label operator for K_{action} can be defined as follows, depending on the desired synchronization scheme.

— *Synchronization by shared actions:*

$$a \oplus b = \begin{cases} a & \text{if } a = b \neq \bot \\ \bot & \text{if } a = \bot \text{ or } b = \bot \\ \text{undefined} & \text{otherwise} \end{cases}$$

— *Complete interleaving:*

$$a \oplus b = \begin{cases} a & \text{if } a \neq \bot \text{ and } b = e \\ b & \text{if } a = e \text{ and } b \neq \bot \\ \bot & \text{if } a = \bot \text{ or } b = \bot \\ \text{undefined} & \text{otherwise} \end{cases}$$

Here we assume the existence of a special action $e \in K$ s.t. $s \overset{e}{\dashrightarrow} s$ for every state s.

■

Example 11. For weighted modal automata (see Example 4), the definition of $\oplus$ depends on how we want to interpret the weights. If the weights on transitions model e.g. *costs* (or *energy consumption*) then the composition operator may be defined as the sum of intervals:

$$k \oplus \ell = \begin{cases} [i_1 + i_2, j_1 + j_2] & \text{if } k = [i_1, j_1] \text{ and } \ell = [i_2, j_2] \\ \bot & \text{if } k = \bot \text{ or } \ell = \bot \end{cases}$$

Note that this label operator is total, so every transition in a weighted modal automaton will synchronize with each transition in the other automaton. Other options may include taking the interval intersection as the composition operator, should the weights represent e.g. (discrete) time intervals in which a transition can be executed. ■

For the rest of this subsection, let us fix a well-formed label-set $(K, \sqsubseteq)$ with a compositional label operator $\oplus$ on it.

Definition 17 (Parallel composition). Let $S, T \in \mathcal{M}_{(K,\sqsubseteq)}$ be two LSMTSs such that $S = (S, s_0, (K, \sqsubseteq), \dashrightarrow_S, \longrightarrow_S)$ and $T = (T, t_0, (K, \sqsubseteq), \dashrightarrow_T, \longrightarrow_T)$. The parallel composition of S and T is defined as the LSMTS

$$S \| T = (S \times T, (s_0, t_0), (K, \sqsubseteq), \dashrightarrow, \longrightarrow)$$

where the transition relations $\dashrightarrow$ and $\longrightarrow$ are defined by the following rules:

$$\frac{s \overset{k}{\dashrightarrow}_S s' \quad t \overset{\ell}{\dashrightarrow}_T t' \quad k \oplus \ell \text{ is defined}}{(s,t) \overset{k\oplus\ell}{\dashrightarrow} (s', t')} \qquad \frac{s \xrightarrow{k}_S s' \quad t \xrightarrow{\ell}_T t' \quad k \oplus \ell \text{ is defined}}{(s,t) \xrightarrow{k\oplus\ell} (s', t')}$$

As we assumed that the label operator $\oplus$ is compositional, we get the property of compositional refinement, also called independent implementability (de Alfaro and Henzinger, 2005). In other words, modal refinement is a precongruence with respect to parallel composition. This is formalized in the following theorem.

Theorem 4 (Independent Implementability). Let $S, S', T, T' \in \mathcal{M}_{(K,\sqsubseteq)}$ be LSMTSs and let $\oplus$ be a compositional label operator on $(K, \sqsubseteq)$. If $S' \leq_m S$ and $T' \leq_m T$ then $S' \| T' \leq_m S \| T$.

Proof. Assume that R_1 is a relation showing $S' \leq_m S$ and R_2 is a relation showing $T' \leq_m T$. We define a relation $R \subseteq (S' \times T') \times (S \times T)$ by $((s', t'), (s, t)) \in R$ if and only if $(s', s) \in R_1$ and $(t', t) \in R_2$. We show that R witnesses $S' \| T' \leq_m S \| T$.

Obviously $((s'_0, t'_0), (s_0, t_0)) \in R$ where s_0, s'_0, t_0, t'_0 are the initial states of S, S', T, T', respectively. Let $((s', t'), (s, t)) \in R$.

1. Assume $(s', t') \overset{k'\oplus\ell'}{\dashrightarrow} (\hat{s}', \hat{t}')$. By the rule of parallel composition, we have $s' \overset{k'}{\dashrightarrow} \hat{s}'$ and $t' \overset{\ell'}{\dashrightarrow} \hat{t}'$. Then, from $(s', s) \in R_1$ and $(t', t) \in R_2$ it follows that there exist $s \overset{k}{\dashrightarrow} \hat{s}$ and $t \overset{\ell}{\dashrightarrow} \hat{t}$ such that $k' \sqsubseteq k$, $\ell' \sqsubseteq \ell$, $(\hat{s}', \hat{s}) \in R_1$, and $(\hat{t}', \hat{t}) \in R_2$. From the fact that $\oplus$ is a compositional label operator it follows that $k' \oplus l' \sqsubseteq k \oplus \ell$, and then $(s, t) \overset{k\oplus\ell}{\dashrightarrow} (\hat{s}, \hat{t})$ and $((\hat{s}', \hat{t}'), (\hat{s}, \hat{t})) \in R$.
2. Assume $(s, t) \xrightarrow{k\oplus\ell} (\hat{s}, \hat{t})$. By the rule of parallel composition, we have $s \xrightarrow{k} \hat{s}$ and $t \xrightarrow{\ell} \hat{t}$. Then, from $(s', s) \in R_1$ and $(t', t) \in R_2$ it follows that there exist $s' \xrightarrow{k'} \hat{s}'$ and $t \xrightarrow{\ell'} \hat{t}'$ such that $k' \sqsubseteq k$, $\ell' \sqsubseteq \ell$, $(\hat{s}', \hat{s}) \in R_1$, and $(\hat{t}', \hat{t}) \in R_2$. From the compositionality

of the label operator it follows that $k' \oplus \ell' \sqsubseteq k \oplus \ell$, and then $(s', t') \xrightarrow{k' \oplus \ell'} (\hat{s}', \hat{t}')$ and $((\hat{s}', \hat{t}'), (\hat{s}, \hat{t})) \in R$.

□

Clearly, if $\oplus$ is commutative and associative, then so is the parallel composition (up to isomorphism).

3.3. *Conjunction*

Different component requirements can be often specified by independent teams. The issue of dealing with the aspects of multiple viewpoints/properties is thus essential. It should be possible to represent several specifications (viewpoints) for the same implementation and to combine them in a logical manner. This is the objective of the *conjunction* operation.

Two LSMTSs with the same label-set $(K, \sqsubseteq)$ can be conjoined with respect to a label operator $\owedge$ on $(K, \sqsubseteq)$. We first state a necessary condition on $\owedge$ such that the conjunction operator yields the greatest lower bound with respect to the modal refinement relation on LSMTSs.

Definition 18 (Greatest lower bound operator). A commutative label operator $\owedge$ on a well-formed set $(K, \sqsubseteq)$ is a *greatest lower bound operator* if the following is satisfied:

1. if $k \owedge \ell$ is defined, then $k \owedge \ell \sqsubseteq k$ and $k \owedge \ell \sqsubseteq \ell$,
2. if $m \neq \bot$, $m \sqsubseteq k$ and $m \sqsubseteq \ell$, then $k \owedge \ell$ is defined and $m \sqsubseteq k \owedge \ell$.

As in the case of the compositional label operator, it is easy to see that greatest lower bound operators are preserved by the product construction.

Proposition 3. Let $\owedge_1$ and $\owedge_2$ be greatest lower bound operators on $(K_1, \sqsubseteq_1)$ and $(K_2, \sqsubseteq_2)$, respectively. Then $\owedge_1 \times \owedge_2$ is a greatest lower bound operator on the product $(K_1, \sqsubseteq_1) \otimes (K_2, \sqsubseteq_2)$.

Again, the actual definition of $\owedge$ depends on the interpretation of conjunction for the modelled quantity.

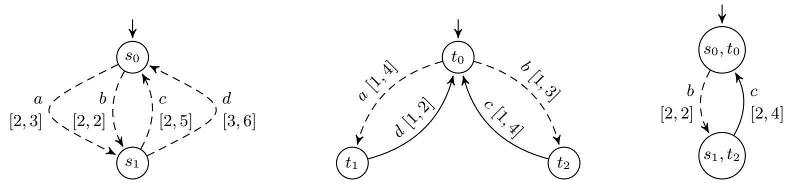
Example 12. Conjoining labels in $K_{action} = (\Sigma \cup \{\bot\}, \sqsubseteq)$, for a finite set of actions Σ (see Example 2), can be defined as follows:

$$a \owedge b = \begin{cases} a & \text{if } a = b,\ a \neq \bot,\ b \neq \bot \\ \bot & \text{if } a = \bot \text{ or } b = \bot \\ \text{undefined} & \text{otherwise} \end{cases}$$

■

Example 13. For the case of weighted modal automata (see Example 4), conjunction $\owedge$ can be defined as the intersection of the intervals, assuming that we consider the cost (energy) interpretation.

$$k \owedge \ell = \begin{cases} k \cap \ell & \text{if } k \neq \bot,\ \ell \neq \bot \text{ and } k \cap \ell \neq \emptyset \\ \bot & \text{otherwise} \end{cases}$$



(a) S (b) T (c) $\rho(S \wedge T)$

Fig. 8. Pruned conjunction of two LSMTSs S and T

■

It is easy to see that the label operators defined in Examples 12 and 13 are greatest lower bound operators on their respective label-sets.

Let us fix a well-formed label-set $(K, \sqsubseteq)$ and some greatest lower bound operator $\owedge$ on $(K, \sqsubseteq)$ for the rest of this subsection.

Definition 19 (Conjunction). Let $S, T \in \mathcal{M}_{(K,\sqsubseteq)}$ be two LSMTSs such that $\mathcal{S} = (S, s_0, (K, \sqsubseteq), \dashrightarrow_S, \longrightarrow_S)$ and $\mathcal{T} = (T, t_0, (K, \sqsubseteq), \dashrightarrow_T, \longrightarrow_T)$. The *conjunction of* S *and* T is defined by the LSMTS $S \wedge T = (S \times T, (s_0, t_0), (K, \sqsubseteq), \dashrightarrow, \longrightarrow)$ where the transition relations $\dashrightarrow$ and $\longrightarrow$ are defined by the following rules.

$$\frac{s \overset{k}{\longrightarrow}_S s' \quad t \overset{\ell}{\dashrightarrow}_T t' \quad k \owedge \ell \text{ is defined}}{(s,t) \overset{k \owedge \ell}{\longrightarrow} (s', t')} \qquad \frac{s \overset{k}{\dashrightarrow}_S s' \quad t \overset{\ell}{\longrightarrow}_T t' \quad k \owedge \ell \text{ is defined}}{(s,t) \overset{k \owedge \ell}{\longrightarrow} (s', t')}$$

$$\frac{s \overset{k}{\dashrightarrow}_S s' \quad t \overset{\ell}{\dashrightarrow}_T t' \quad k \owedge \ell \text{ is defined}}{(s,t) \overset{k \owedge \ell}{\dashrightarrow} (s', t')}$$

$$\frac{s \overset{k}{\longrightarrow}_S s' \quad \big(k \owedge \ell \text{ is not defined for any } \ell \text{ such that } t \overset{\ell}{\dashrightarrow}_T\big)}{(s,t) \overset{\perp}{\longrightarrow} (s,t)}$$

$$\frac{t \overset{\ell}{\longrightarrow}_T t' \quad \big(k \owedge \ell \text{ is not defined for any } k \text{ such that } s \overset{k}{\dashrightarrow}_S\big)}{(s,t) \overset{\perp}{\longrightarrow} (s,t)}$$

Clearly, conjunction $\wedge$ on LSMTSs is commutative (up to isomorphism) as $\owedge$ is commutative, and moreover, if $\owedge$ is an associative label operator, then so is conjunction.

Example 14. An example for conjoining specifications is given in Figure 8. Here S and T are LSMTSs with the label-set $K_{action} \otimes K_{weighted}$. Note that the state (s_1, t_1) does not appear in $\rho(S \wedge T)$ since it is locally inconsistent in $S \wedge T$; the LSMTS T requires a transition labelled with the action d and the weight interval $[1, 2]$, however, S only allows d with the weight interval $[3, 6]$. ■

We now propose a notion of determinism that is dedicated to the conjunction operation. This definition shall be used later on to prove that conjunction is the greatest lower bound with respect to the modal refinement ordering.

Definition 20 ($\owedge$-determinism). A LSMTS $S \in \mathcal{M}_{(K,\sqsubseteq)}$ is $\owedge$-*deterministic* if for all may transitions $s \overset{k'}{\dashrightarrow} s'$ and $s \overset{k''}{\dashrightarrow} s''$ in S with labels $k', k'' \in K \smallsetminus \{\bot\}$, whenever there is an $\ell \in K \smallsetminus \{\bot\}$ such that both $k' \owedge \ell$ and $k'' \owedge \ell$ are defined, then $k' = k''$ and $s' = s''$.

It is easy to see that if $\owedge$ is associative, then the construction for conjunction presented in Definition 19 preserves $\owedge$-determinism. The reader may also observe that for modal transition systems with the label-set K_{action}, the notions of $\owedge$-determinism and determinism (as defined in Section 2) coincide. In this case, we note that the determinization algorithm proposed in Section 2.5 can be applied to compute minimal deterministic LSMTSs of non-deterministic ones.

Under the assumption of $\owedge$-determinism, the conjunction construction yields the greatest lower bound with respect to modal refinement, as stated in the following theorem.

Theorem 5 (Greatest lower bound of conjunction). Let $S, T, U \in \mathcal{M}_{(K,\sqsubseteq)}$ be locally consistent LSMTSs such that S and T are $\owedge$-deterministic and $S \wedge T$ is consistent. Assume that $\owedge$ is a greatest lower bound label operator. Then

1. $\rho(S \wedge T) \leq_m S$ and $\rho(S \wedge T) \leq_m T$, and
2. if $U \leq_m S$ and $U \leq_m T$, then $U \leq_m \rho(S \wedge T)$.

Proof.

1. It suffices to show that $\rho(S \wedge T) \leq_m S$, the other assertion is symmetric. We define a relation $R \subseteq (S \times T) \times S$ by $R = \{((s,t),s) \mid s \in S, t \in T\}$. We will show that R is a relation witnessing $\rho(S \wedge T) \leq_m S$. Clearly $((s_0,t_0),s_0) \in R$ where s_0 is the initial state of S and t_0 is the initial state of T. Let $((s,t),s) \in R$.

   Let $(s,t) \overset{k \owedge \ell}{\dashrightarrow} (s',t')$. Since $\rho(S \wedge T)$ does not contain any transitions labelled with $\bot$, we know that $k \owedge \ell \neq \bot$. Then there are transitions $s \overset{k}{\dashrightarrow} s'$ and $t \overset{\ell}{\dashrightarrow} t'$. By assumption we know $k \owedge \ell \sqsubseteq k$, and from the definition of R we can conclude $((s',t'),s') \in R$.

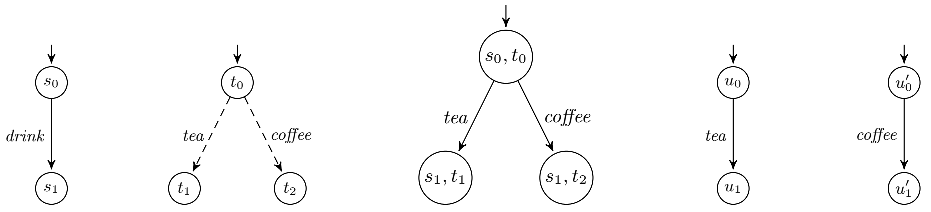
   Now, let $s \xrightarrow{k} s'$. By local consistency of S we can assume that $k \neq \bot$. Suppose that T does not have any $t \overset{\ell}{\dashrightarrow} t'$ such that $k \owedge \ell$ is defined, then $(s,t) \xrightarrow{\bot} (s',t)$ contradicting the local consistency of $\rho(S \wedge T)$. So there exists $t \overset{\ell}{\dashrightarrow} t'$ such that $k \owedge \ell$ is defined and then $(s,t) \xrightarrow{k \owedge \ell} (s',t')$. By the assumption about the label operator we know that $k \owedge \ell \sqsubseteq k$. By the definition of R we get $((s',t'),s') \in R$.
2. We can assume a relation R_1 witnessing $U \leq_m S$ and a relation R_2 witnessing $U \leq_m T$. We define a relation $R \subseteq U \times (S \times T)$ by

   $$R = \{(u,(s,t)) \mid (u,s) \in R_1 \text{ and } (u,t) \in R_2\}.$$

   We show that R is witnessing $U \leq_m \rho(S \wedge T)$. Clearly $(u_0,(s_0,t_0)) \in R$ for the initial states. Let $(u,(s,t)) \in R$.

   Assume that $u \overset{m}{\dashrightarrow} u'$. Then there exists $s \overset{k}{\dashrightarrow} s'$ such that $m \sqsubseteq k$, and $t \overset{\ell}{\dashrightarrow} t'$ such that $m \sqsubseteq \ell$. By Definition 18 part 2. we get that $k \owedge \ell$ is defined and $m \sqsubseteq k \owedge \ell$, hence $(s,t) \overset{k \owedge \ell}{\dashrightarrow} (s',t')$, and $(u',(s',t')) \in R$ by the definition of R.

   Assume that $(s,t) \xrightarrow{k \owedge \ell} (s',t')$. By local consistency of $\rho(S \wedge T)$ we can assume that



(a) S (b) T (c) $S \wedge T = \rho(S \wedge T)$ (d) U (e) U'

Fig. 9. Conjunction of two LSMTSs specifying a vending machine

$k \owedge \ell \neq \bot$. Then (w.l.o.g.) there exist $s \xrightarrow{k} s'$ with $k \neq \bot$, and $t \overset{\ell}{\dashrightarrow} t'$ with $\ell \neq \bot$ thanks to local consistency of S and T. It follows from $(u, s) \in R_1$ that there exists $u \xrightarrow{m} u'$ such that $(u', s') \in R_1$ and $m \sqsubseteq k$. Local consistency of U implies $m \neq \bot$. We have to show that $(u', t') \in R_2$. From $(u, t) \in R_2$ it follows that there exists $t \overset{\ell'}{\dashrightarrow} t''$ such that $m \sqsubseteq \ell'$ and $(u', t'') \in R_2$. Now, by $m \neq \bot$, $m \sqsubseteq k$ and $m \sqsubseteq \ell'$ we know that $k \owedge \ell'$ is defined (Definition 18 part 2.). From $\owedge$-determinism of T we get $\ell = \ell'$ and $t' = t''$. It follows that $(u', t') \in R_2$ and thus $(u', (s', t')) \in R$.

□

Corollary 2. Let $S, T, U \in \mathcal{M}_{(K,\sqsubseteq)}$ be locally consistent LSMTSs such that S and T are $\owedge$-deterministic and $S \wedge T$ is consistent. Then $U \leq_m S$ and $U \leq_m T$ if and only if $U \leq_m \rho(S \wedge T)$. In particular, $[\![S \wedge T]\!] = [\![S]\!] \cap [\![T]\!]$.

Proof. The implication from left to right is exactly the second part of Theorem 5. The other direction follows from the first part of Theorem 5 and the transitivity of $\sqsubseteq$. The additional assertion $[\![S \wedge T]\!] = [\![S]\!] \cap [\![T]\!]$ follows from $U \leq_m S$ and $U \leq_m T$ if and only if $U \leq_m \rho(S \wedge T)$ if we take U as an implementation, and by Proposition 1 showing that $[\![\rho(S \wedge T)]\!] = [\![S \wedge T]\!]$. □

For maximality of the conjunction (see Theorem 5, 2.), $\owedge$-determinism is necessary. To see this, consider the example shown in Figure 9 that shows two specifications of a vending machine with the label-set $K_{machine}$ from Example 3. The LSMTS S requires *drink*, and T allows *tea* and *coffee*. For the conjunction $S \wedge T$ we take $\owedge$ as the obvious greatest lower bound for $K_{machine}$ for which *drink* $\owedge$ *tea* = *tea* and *drink* $\owedge$ *coffee* = *coffee*. Note that T is *not* $\owedge$-deterministic in this case. It is easy to see that $(S \wedge T) \leq_m S$ and $(S \wedge T) \leq_m T$, however U and U' are both refining S and T, but $U \not\leq_m (S \wedge T) = \rho(S \wedge T)$. With a small extension of the proof for modal transition systems (Delahaye et al., 2010), it can easily be proved that there does not exist any LSMTS which is the greatest lower bound for S and T but the construction of conjunction in this case is at least safe.

3.4. *Quotient*

An essential operator in a complete specification theory is the one of *quotienting*. It allows for factoring out behaviours from a larger component. Given two component specifications S and T, the quotient of T by S, written $T \mathbin{\backslash\!\!\backslash} S$, is a specification of exactly those components that when composed with S refine T. In other words, the quotient is the largest specification that can be composed with S and still refines T.

As expected, we have to first state the required property for label operators used for quotienting.

Definition 21 (Dual label operators). Let $\ominus$ and $\oplus$ be two label operators on a given well-formed label-set $(K, \sqsubseteq)$. We say that the operator $\ominus$ is a *dual label operator to* $\oplus$ if $m \sqsubseteq \ell \ominus k$ if and only if $k \oplus m \sqsubseteq \ell$.

Example 15. Quotienting labels in $K_{action} = (\Sigma \cup \{\bot\}, \sqsubseteq)$ for a finite set of actions Σ as introduced in Example 2, together with the $\oplus$ operator for synchronization by shared actions given in Example 10 can be defined as identical to $\oplus$, namely:

$$a \ominus b = \begin{cases} a & \text{if } a = b \neq \bot \\ \bot & \text{if } a = \bot \text{ or } b = \bot \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Clearly, $\ominus$ is a dual label operator to $\oplus$. ■

Example 16. A more interesting example of $\ominus$ in $K_{weighted}$ (see Example 4) dual to the composition operator $\oplus$ from Example 11 that sums up weight intervals is given by

$$[a, b] \ominus [a', b'] = \begin{cases} [a - a', b - b'] & \text{if } [a - a', b - b'] \in K_{weighted} \\ \bot & \text{otherwise.} \end{cases}$$

■

Let us for the rest of this subsection fix two label operators $\ominus$ and $\oplus$ such that $\ominus$ is a dual label operator to $\oplus$ on a well-formed label-set $(K, \sqsubseteq)$.

Definition 22 (Quotient). Let $S, T \in \mathcal{M}_{(K,\sqsubseteq)}$ be two LSMTS such that $S = (S, s_0, (K, \sqsubseteq), \dashrightarrow_S, \longrightarrow_S)$ and $T = (T, t_0, (K, \sqsubseteq), \dashrightarrow_T, \longrightarrow_T)$. The *quotient of* T *by* S is defined as $T \mathbin{\backslash\!\!\backslash} S = \big((T \times S) \cup \{\mathsf{u}\}, (t_0, s_0), (K, \sqsubseteq), \dashrightarrow, \longrightarrow\big)$ where u is a new state modelling a universal state, and the transition relations $\dashrightarrow$ and $\longrightarrow$ are defined by the following rules:

$$\frac{t \overset{\ell}{\dashrightarrow}_T t' \quad s \overset{k}{\dashrightarrow}_S s' \quad \ell \ominus k \text{ is defined}}{(t, s) \overset{\ell \ominus k}{\dashrightarrow} (t', s')} \qquad \frac{t \xrightarrow{\ell}_T t' \quad s \xrightarrow{k}_S s' \quad \ell \ominus k \text{ is defined}}{(t, s) \xrightarrow{\ell \ominus k} (t', s')}$$

$$\frac{t \xrightarrow{\ell}_T \quad \big(\ell \ominus k \text{ is not defined for any } k \text{ s.t. } s \xrightarrow{k}_S\big)}{(t, s) \xrightarrow{\bot} (t, s)}$$

$$\frac{m \in K \text{ and } m \oplus k \text{ is not defined for any } s \overset{k}{\dashrightarrow}_S}{(t, s) \overset{m}{\dashrightarrow} \mathsf{u}}$$

$$\frac{m \in K}{\mathsf{u} \overset{m}{\dashrightarrow} \mathsf{u}}$$

The first two rules presented above are derived from the two rules of Definition 17, while the third one captures the inconsistency present when the larger system T has a must transition that cannot be mimicked by the smaller system S in parallel with any transition from the quotient. To achieve maximality of the quotient, we introduce a universal state u that allows for an arbitrary behaviour. Any allowed behaviour not structurally composable with the allowed behaviour of the smaller system can be safely added to the quotient (leading to the state u) as this will not affect the parallel composition with the smaller system. This is captured by the fourth rule.

Example 17. An example of quotienting is shown in Figure 10. Both specifications T and S have as the label-set $K_{action} \otimes K_{weighted}$ where $K_{action} = (\{a, b, \bot\}, \sqsubseteq)$, and $\rho(T \setminus\!\!\setminus S)$ is the result of the pruned quotient of T by S with respect to the product of the corresponding label operators from Examples 15 and 16 where the resulting operators are summarized in Figure 11 (these operators are already defined by the product construction, we just present their combined definitions for clarity reasons). Thereby, a may transition under $[-\infty, \infty]$ between two states stands for all may transitions between those states under any label in $K_{weighted}$.

- The quotient $\rho(T \setminus\!\!\setminus S)$, in state (t_0, s_0), may do the action b with any weight (abbreviated by $[-\infty, \infty]$) and afterwards show arbitrary behaviour (reflected by the universal state) since S has no corresponding transition for action b in state s_0.
- Note that the state (t_3, s_3) is not present in the pruned quotient. If $\rho(T \setminus\!\!\setminus S)$ would allow for a transition labelled with a (which would be possible with the weight interval $[0, 0]$), then $(b, [1, 4]) \ominus (b, [0, 4])$ yields $\bot$ (on a must transition), turning (t_3, s_3) into an inconsistent state with no implementation.

■

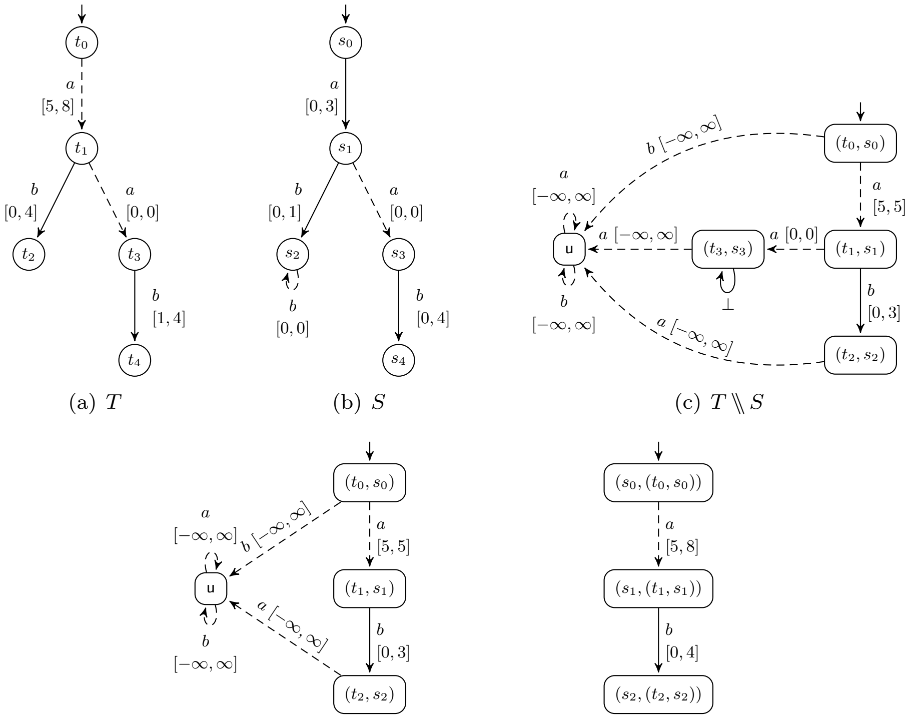
The quotient $T \setminus\!\!\setminus S$ intends to synthesize the largest component that can be composed with S in order to refine T. In existing theories such as modal transition systems, this maximality property only holds when the specifications are deterministic. As for conjunction, we now propose a general notion of determinism for quotienting.

Definition 23 ($\oplus/\ominus$-determinism). Let $S \in \mathcal{M}_{(K,\sqsubseteq)}$ be an LSMTS. We say that S is *$\oplus/\ominus$-deterministic* if, for any $k', k'' \in K \setminus \{\bot\}$,

1. whenever $s \overset{k'}{\dashrightarrow} s'$ and $s \overset{k''}{\dashrightarrow} s''$ and there exists $m \in K \setminus \{\bot\}$ such that $k' \oplus m$ and $k'' \oplus m$ are defined, then $k' = k''$ and $s' = s''$, and
2. whenever $s \xrightarrow{k'} s'$ and $s \xrightarrow{k''} s''$ and there exists $\ell \in K \setminus \{\bot\}$ such that $\ell \ominus k'$ and $\ell \ominus k''$ are defined, then $k' = k''$ and $s' = s''$.

The reader may again observe that for modal transition systems with the label-set K_{action}, the notions of $\oplus/\ominus$-determinism and determinism (as defined in Section 2) as well as $\owedge$-determinism coincide. Here, in case of non-deterministic LSMTSs, the determinization algorithm proposed in Section 2.5 can be applied to compute minimal deterministic versions of them.

Under the assumption of $\oplus/\ominus$-determinism, the quotient construction $T \setminus\!\!\setminus S$ yields the most general LSMTS that, composed with S, still refines T.



(a) T (b) S (c) $T \mathbin{\backslash\!\!\backslash} S$

(d) $\rho(T \mathbin{\backslash\!\!\backslash} S)$ (e) $S \| \rho(T \mathbin{\backslash\!\!\backslash} S)$

Fig. 10. Quotienting of two LSMTSs

$$(a,i) \oplus (a',i') = \begin{cases} a, [l+l', r+r'] & \text{if } a = a' \neq \perp,\ i = [l,r],\ i' = [l',r'] \\ \perp & \text{if } a = \perp \text{ or } a' = \perp \text{ or } i = \perp \text{ or } i' = \perp \\ \text{undefined} & \text{otherwise} \end{cases}$$

$$(a,i) \ominus (a',i') = \begin{cases} a, [l-l', r-r'] & \text{if } a = a' \neq \perp,\ i = [l,r],\ i' = [l',r'],\ [l-l', r-r'] \in K_{weighted} \\ \perp & \text{if } a = \perp \text{ or } a' = \perp \text{ or } i = \perp \text{ or } i' = \perp \\ & \text{or } i = [l,r],\ i' = [l',r'] \text{ and } [l-l', r-r'] \notin K_{weighted} \\ \text{undefined} & \text{otherwise} \end{cases}$$

Fig. 11. Label operators $\oplus$, $\ominus$ for quotienting with the label-set $K_{action} \otimes K_{weighted}$

Theorem 6 (Soundness and maximality of quotient). Let $S, T, X \in \mathcal{M}_{(K,\sqsubseteq)}$ be locally consistent LSMTSs such that $T \setminus\!\!\setminus S$ is consistent. Assume that S is $\oplus/\ominus$-deterministic, and assume that $\ominus$ is the dual label operator to $\oplus$. Then $X \leq_m \rho(T \setminus\!\!\setminus S)$ if and only if $S \| X \leq_m T$.

Proof. "$\Longrightarrow$": We assume a relation R_1 witnessing $X \leq_m \rho(T \setminus\!\!\setminus S)$. We define a relation $R_2 \subseteq (S \times X) \times T$ by

$$R_2 = \{((s,x),t) \mid (x,(t,s)) \in R_1\}.$$

We show that R_2 is a relation witnessing $S \| X \leq_m T$. Clearly $((s_0,x_0),t_0) \in R_2$ for the corresponding initial states. Let $((s,x),t) \in R_2$.

1. Assume $(s,x) \overset{k\oplus m}{\dashrightarrow} (s',x')$. Then there exist $s \overset{k}{\dashrightarrow} s'$ and $x \overset{m}{\dashrightarrow} x'$. Since X is locally consistent, we know $m \neq \bot$. From $(x,(t,s)) \in R_1$ and $x \overset{m}{\dashrightarrow} x'$ it follows that there exists $(t,s) \overset{\ell\ominus k'}{\dashrightarrow} (t',s'')$ such that $m \sqsubseteq \ell \ominus k'$ and $(x',(t',s'')) \in R_1$. From $m \sqsubseteq \ell \ominus k'$ we get $k' \oplus m \sqsubseteq \ell$ by assumption. Moreover, we have $t \overset{\ell}{\dashrightarrow} t'$ and $s \overset{k'}{\dashrightarrow} s''$. Then, from $\oplus/\ominus$-determinism of S it follows $k = k'$ and $s' = s''$, and so $k \oplus m \sqsubseteq \ell$. Thus $(x',(t',s')) \in R_1$ and hence $((s',x'),t') \in R_2$.
2. Assume $t \xrightarrow{\ell} t'$. By local consistency of T we can assume that $\ell \neq \bot$. Suppose that there does not exist a transition $s \xrightarrow{k} s'$ such that $\ell \ominus k$ is defined, then $(t,s) \xrightarrow{\bot}$ which contradicts $(t,s) \in \rho(T \setminus\!\!\setminus S)$. It follows that there exists $s \xrightarrow{k} s'$ such that $\ell \ominus k$ is defined, and $(t,s) \xrightarrow{\ell\ominus k} (t',s')$. By a similar argument as above we know that $\ell \ominus k \neq \bot$. Then from $(x,(t,s)) \in R_1$ it follows that there exists $x \xrightarrow{m} x'$ such that $(x',(t',s')) \in R_1$ and $m \sqsubseteq \ell \ominus k$ which implies $k \oplus m \sqsubseteq \ell$. We can conclude that $(s,x) \xrightarrow{k\oplus m} (s',x')$ and $((s',x'),t') \in R_2$.

"$\Longleftarrow$": We assume a relation R_2 witnessing $S \| X \leq_m T$. We define a relation $R_1 \subseteq X \times (T \times S)$ by

$$R_1 = \{(x,(t,s)) \mid ((s,x),t) \in R_2\} \cup \{(x,\mathsf{u})\}.$$

We show that R_1 is a relation witnessing $X \leq_m \rho(T \setminus\!\!\setminus S)$. Clearly $(x_0,(t_0,s_0)) \in R_1$ for the corresponding initial states. First, let $(x,\mathsf{u}) \in R_1$, then clearly for every transition $x \overset{m}{\dashrightarrow} x'$, it holds again that $(x',\mathsf{u}) \in R_1$ since the universal state u allows arbitrary behaviour. Second, let $(x,(t,s)) \in R_1$.

1. Assume $x \overset{m}{\dashrightarrow} x'$. If there is no transition $s \overset{k}{\dashrightarrow} s'$ such that $k \oplus m$ is defined, then $(t,s) \overset{m}{\dashrightarrow} \mathsf{u}$, and in this case, $(x,\mathsf{u}) \in R_1$, and clearly $m \sqsubseteq m$ by reflexivity of $\sqsubseteq$. If there is a transition $s \overset{k}{\dashrightarrow} s'$ such that $k \oplus m$ is defined, then $(s,x) \overset{k\oplus m}{\dashrightarrow} (s',x')$. From $((s,x),t) \in R_2$ it follows that there exists $t \overset{\ell}{\dashrightarrow} t'$ such that $((s',x'),t') \in R_2$ and $k \oplus m \sqsubseteq \ell$, implying $m \sqsubseteq \ell \ominus k$. Hence $(t,s) \overset{\ell\ominus k}{\dashrightarrow} (t',s')$ and $(x',(t',s')) \in R_1$.
2. Assume $(t,s) \xrightarrow{\ell\ominus k} (t',s')$. From local consistency of $\rho(T \setminus\!\!\setminus S)$ we know $\ell \ominus k \neq \bot$. Then there are $t \xrightarrow{\ell} t'$ and $s \xrightarrow{k} s'$. From $((s,x),t) \in R_2$ we can conclude that there exists $(s,x) \xrightarrow{k'\oplus m} (s'',x')$ such that $k' \oplus m \sqsubseteq \ell$ and $((s'',x'),t') \in R_2$. Then there exist $s \xrightarrow{k'} s''$ and $x \xrightarrow{m} x'$. The fact that $k' \oplus m \sqsubseteq \ell$ implies $m \sqsubseteq \ell \ominus k'$ by the duality of

the operators. From $\oplus/\ominus$-determinism of S it follows that $k = k'$ and $s' = s''$. Thus $m \sqsubseteq \ell \ominus k$ and $((s', x'), t') \in R_2$, hence $(x', (t', s')) \in R_1$.

□

4. Logical Characterization

It was shown in (Larsen, 1989) that Hennessy-Milner logic (Hennessy and Milner, 1985) can be used as a logical characterization for modal refinement of modal transition systems (the reader may also consult (Bruns and Godefroid, 2000)). In this section we shall extend this result to LSMTSs and study other related topics. For the rest of this section, we fix a well-formed label-set $(K, \sqsubseteq)$.

Let us first introduce LSHML, an extension of Hennessy-Milner logic (HML) that is interpreted over LSMTSs, taking into account their label structures. The syntax of the logic is given by the abstract syntax:

$$\varphi ::= \mathsf{true} \mid \mathsf{false} \mid \varphi_1 \wedge \varphi_2 \mid \varphi_1 \vee \varphi_2 \mid \langle k \rangle \psi \mid [k]\psi$$

where $k \in K$ is a label. We define the $\vee$-free fragment of LSHML as a set of formulae in LSHML not containing the disjunction operator.

Let $S \in \mathcal{M}_{(K,\sqsubseteq)}$ be an LSMTS. The satisfaction relation between a state $s \in S$ and a formula φ is defined inductively as follows.

$s \models \mathsf{true}$

$s \not\models \mathsf{false}$

$s \models \varphi_1 \wedge \varphi_2$ iff $S \models \varphi_1$ and $S \models \varphi_2$

$s \models \varphi_1 \vee \varphi_2$ iff $S \models \varphi_1$ or $S \models \varphi_2$

$s \models \langle k \rangle \varphi$ iff $\exists (s \xrightarrow{\ell} s') : [\![\ell]\!] \subseteq [\![k]\!]$ and $s' \models \varphi$

$s \models [k]\varphi$ iff $\forall (s \overset{\ell}{\dashrightarrow} s')$ s.t. $[\![\ell]\!] \cap [\![k]\!] \neq \emptyset : s' \models \varphi$

We write $S \models \varphi$ iff $s_0 \models \varphi$ where s_0 is the initial state of S.

Example 18. Consider the specification of a vending machine given in Figure 3(c). The specification satisfies the property that after inserting a 1€ coin, a drink is guaranteed, as we have $s_0 \models [\mathit{1EURO}]\langle \mathit{drink} \rangle\mathsf{true}$. On the other hand, we are not guaranteed to receive a cup of tea if a coin is inserted as $s_0 \not\models [\mathit{coin}]\langle \mathit{tea} \rangle\mathsf{true}$. ■

We are now ready to prove the soundness and completeness theorems for our logic. The following theorem ensures soundness of LSHML, i.e., if a formula holds for a specification, then it holds for any of its refinements.

Theorem 7 (Soundness). Let $T \in \mathcal{M}_{(K,\sqsubseteq)}$, and φ a LSHML-formula. Then

$$T \models \varphi \quad \Longrightarrow \quad \forall S \leq_m T : S \models \varphi \ .$$

Proof. Assume that $S \leq_m T$ and $T \models \varphi$. We prove by induction on the structure of φ that $S \models \varphi$ too. Let s_0 and t_0 be the initial states of S and T, respectively. The induction basis, where $\varphi = \mathsf{true}$ and $\varphi = \mathsf{false}$, is trivial.

$\varphi = \varphi_1 \wedge \varphi_2$. By the definition of $\models$ and then from the induction hypothesis.

$\varphi = \varphi_1 \vee \varphi_2$. As in the case above.

$\varphi = \langle k \rangle \psi$. From $T \models \langle k \rangle \psi$ it follows that there exists $t_0 \xrightarrow{\ell} t$ such that $[\![\ell]\!] \subseteq [\![k]\!]$ and $t \models \psi$. Since $S \leq_m T$ there exists $s_0 \xrightarrow{\ell'} s$ such that $\ell' \sqsubseteq \ell$ and $(s, S) \leq_m (t, T)$. By the induction hypothesis we get $s \models \psi$. From transitivity of $\sqsubseteq$ we have $[\![\ell']\!] \subseteq [\![k]\!]$ and therefore $S \models \varphi$.

$\varphi = [k]\psi$. Let $s_0 \overset{\ell}{\dashrightarrow} s$ such that $[\![k]\!] \cap [\![\ell]\!] \neq \emptyset$. Since $S \leq_m T$ we know that there exists $t_0 \overset{\ell'}{\dashrightarrow} t$ in T with $\ell \sqsubseteq \ell'$ and $(s, S) \leq_m (t, T)$. Clearly, $[\![k]\!] \cap [\![\ell']\!] \neq \emptyset$ and because $T \models [k]\psi$ we know that $t \models \psi$. By the induction hypothesis we get $s \models \psi$ and hence $S \models \varphi$.

□

We shall now focus on the issue of completeness. We consider two possible definitions:

1. *Completeness with respect to implementations*: if all implementations of a specification S satisfy a formula of the logic, then so does the specification S.
2. *Completeness with respect to modal refinement*: if all formulae satisfied by some specification S are satisfied also by another specification T, then $S \leq_m T$.

The latter, completeness with respect to modal refinement, is also known as logical characterization in the literature (Larsen, 1989).

We first study the completeness with respect to implementations and observe that LSHML-logic is not complete in this case.

Theorem 8. The logic LSHML is incomplete with respect to implementations.

Proof. Let T be an LSMTS consisting of a single transition $t_0 \overset{\bullet}{\dashrightarrow} t$ over the label-set $K_{unlabelled}$ from Example 1. Consider the formula $\varphi = \langle \bullet \rangle \mathsf{true} \vee [\bullet]\mathsf{false}$. Since there is no must transition from t_0 and at the same time there is a may transition, we get $t_0 \not\models \varphi$. On the other hand, any implementation of T either contains no transition at all (and then it satisfies $[\bullet]\mathsf{false}$) or it contains at least one outgoing transition (and then it satisfies $\langle \bullet \rangle \mathsf{true}$). Hence any implementation of T satisfies φ and we get the incompleteness result with respect to implementations. □

Inspecting the proof of the above theorem, one can notice that it is the disjunction that breaks the completeness property. In fact, we can show completeness if we consider the $\vee$-free fragment of LSHML.

Theorem 9 (Completeness with respect to implementations for $\vee$-free LSHML). Let $T \in \mathcal{M}_{(K,\sqsubseteq)}$ be a locally consistent specification, and let φ be a $\vee$-free LSHML-formula. Then

$$(\forall I \in [\![T]\!] : I \models \varphi) \quad \Longrightarrow \quad T \models \varphi \ .$$

Proof. We prove the contraposition. We show that for any $\vee$-free LSHML-formula φ

$$\text{if } T \not\models \varphi \text{ then there exists } I \in [\![T]\!] \text{ such that } I \not\models \varphi \ .$$

The proof is by induction on the structure of the formula φ and under the assumption that $T \not\models \varphi$ we construct its implementation (i_0, I) such that $i_0 \not\models \varphi$. During the construction we will write that we add a transition $i_0 \xrightarrow{n} (i'_0, I')$ for an implementation (i'_0, I'), meaning that together with this transition we implicitly add also a disjoint copy of I' rooted at i'_0 to the implementation I.

The induction basis, where $\varphi = \mathsf{true}$ and $\varphi = \mathsf{false}$, is trivial.

— Case 1: $\varphi = \varphi_1 \wedge \varphi_2$. By the definition of $\models$ either $T \not\models \varphi_1$ or $T \not\models \varphi_2$ Assume w.l.o.g. that $T \not\models \varphi_1$. By applying the induction hypothesis there is $I \in [\![T]\!]$ such that $I \not\models \varphi_1$ and we conclude that $I \not\models \varphi_1 \wedge \varphi_2$.

— Case 2: $\varphi = \langle k \rangle \psi$. Assume that $T \not\models \langle k \rangle \psi$, which is the case if for all $t_0 \xrightarrow{\ell} t$ we have either (1) $[\![\ell]\!] \not\subseteq [\![k]\!]$ or (2) $(t, T) \not\models \psi$. We construct an implementation $(i_0, I) \in [\![T]\!]$ as follows. For every $t_0 \xrightarrow{\ell} t$ such that (1) is satisfied, we add the transition $i_0 \xrightarrow{n} (i'_0, I')$ into I where $n \in [\![\ell]\!] \smallsetminus [\![k]\!]$ and $(i'_0, I') \leq_m (t, T)$ (the implementation (i'_0, I') exists by local consistency of T and well-formedness of the label-set). For every $t_0 \xrightarrow{\ell} t$ such that (2) is satisfied, we have by induction hypothesis an implementation $(i'_0, I') \in [\![(t, T)]\!]$ such that $i'_0 \not\models \psi$. We add $i_0 \xrightarrow{m} (i'_0, I')$ to I for some $m \in [\![\ell]\!]$. It is easy to see that $I \leq_m T$, and moreover $I \not\models \langle k \rangle \psi$ by the construction.

— Case 3: $\varphi = [k]\psi$. Assume that $T \not\models [k]\psi$. Then there exists $t_0 \overset{\ell}{\dashrightarrow} t$ such that $[\![\ell]\!] \cap [\![k]\!] \neq \emptyset$ and $(t, T) \not\models \psi$. By induction hypothesis there exists $(i'_0, I') \in [\![(t, T)]\!]$ such that $i'_0 \not\models \psi$. Let $(i_0, I) \in [\![T]\!]$ be some implementation of T (which exists by local consistency of T) where we add the transition $i_0 \xrightarrow{n} (i'_0, I')$ with $n \in [\![\ell]\!] \cap [\![k]\!]$. Clearly, we still have $I \leq_m T$ and moreover the transition $i_0 \xrightarrow{n} i'_0$ ensures that $i_0 \not\models [k]\psi$.

□

A similar completeness result in the setting of partial Kripke structures can be found also in (Antonik and Huth, 2006).

We now study completeness with respect to modal refinement, that is the completeness definition considered in (Larsen, 1989). In this article, it was shown that for classical modal transition systems (with the label-set K_{action} from Example 2), the LSHML logic is complete with respect to refinement. We first observe that the result does not extend to general LSMTSs.

Theorem 10. The logic LSHML is incomplete with respect to modal refinement.

Proof. Consider the systems S and T from Figure 6. By case analysis it is easy to verify that $s_0 \models \varphi$ if and only if $t_0 \models \varphi$ for any LSHML-formula φ. However, as argued before, $S \not\leq_m T$. □

On the other hand, if we consider only deterministic systems, LSHML is complete even with disjunction, as proved below. We let $\mathcal{F}(S) = \{\varphi \mid S \models \varphi\}$ denote the set of all LSHML-formulae satisfied by S.

Theorem 11 (Completeness with respect to modal refinement for determin-

istic LSMTSs). Let $S, T \in \mathcal{M}_{(K,\sqsubseteq)}$ be deterministic LSMTSs (see Definition 11) and assume that the label refinement relation $\sqsubseteq$ is complete. Then

$$\mathcal{F}(T) \subseteq \mathcal{F}(S) \quad \Longrightarrow \quad S \leq_m T \ .$$

Proof.
Assume that $\mathcal{F}(T) \subseteq \mathcal{F}(S)$. We define a relation $R \subseteq S \times T$ by

$$R = \{(s,t) \mid \mathcal{F}((t,T)) \subseteq \mathcal{F}((s,S))\}.$$

We show that R is a relation witnessing $S \leq_m T$. Clearly $(s_0, t_0) \in R$ for the respective initial states. Let $(s,t) \in R$.

- First, assume that $s \overset{k}{\dashrightarrow} s'$. Clearly, $t \overset{\ell}{\dashrightarrow} t'$ for some ℓ such that $[\![k]\!] \cap [\![\ell]\!] \neq \emptyset$, otherwise the formula $[k]\mathsf{false}$ is satisfied in (t,T) but not in (s,S), contradicting our assumption that $\mathcal{F}((t,T)) \subseteq \mathcal{F}((s,S))$. By the determinism of T there can only be one such ℓ with $[\![k]\!] \cap [\![\ell]\!] \neq \emptyset$.
  For the sake of contradiction assume that $k \not\sqsubseteq \ell$. By completeness of label refinement we get $[\![k]\!] \not\subseteq [\![\ell]\!]$. Thus, there exists some $m \in [\![k]\!] \setminus [\![\ell]\!]$. The formula $[m]\mathsf{false}$ holds in (t,T) due to the choice of m and the absence of any other may transition having any common implementation labels with $[\![k]\!]$, hence in particular also with $[\![m]\!] = \{m\}$. However, $[m]\mathsf{false}$ does not hold in (s,S), contradicting the assumption that $\mathcal{F}((t,T)) \subseteq \mathcal{F}((s,S))$. Thus, we can assume the existence of $t \overset{\ell}{\dashrightarrow} t'$ with $k \sqsubseteq \ell$.
  Now we need to argue that $(s',t') \in R$. Assume that this is not the case. Then we have $\mathcal{F}((t',T)) \not\subseteq \mathcal{F}((s',S))$, and therefore there is a formula φ' such that $(t',T) \models \varphi'$ and $(s',S) \not\models \varphi'$. Consider the formula $\varphi = [k]\varphi'$. Again $(t,T) \models \varphi$, but $(s,S) \not\models \varphi$. This contradicts the assumption that $\mathcal{F}((t,T)) \subseteq \mathcal{F}((s,S))$. Thus $(s',t') \in R$.
- Second, assume that $t \xrightarrow{\ell} t'$. As in the previous item, there must be a transition $s \xrightarrow{k} s'$ such that $[\![k]\!] \subseteq [\![\ell]\!]$, otherwise the formula $\langle \ell \rangle \mathsf{true}$ is satisfied in (t,T) but not in (s,S), contradicting the assumption that $\mathcal{F}((t,T)) \subseteq \mathcal{F}((s,S))$. By the determinism of S we know that $s \xrightarrow{k} s'$ is a unique transition such that $[\![k]\!] \subseteq [\![\ell]\!]$ and due to the completeness of label refinement we know that $k \sqsubseteq \ell$.
  Remains to argue that $(s',t') \in R$. Assume that this is not the case. The arguments are similar to the previous case by considering the formula $\langle \ell \rangle \varphi'$ where $t' \models \varphi'$ and $s' \not\models \varphi'$. Thus $(s',t') \in R$ and this completes the proof.

□

5. Conclusion

We introduced label-structured modal transition systems, a basis for a specification formalism that generalizes modal theories such as weighted and multi-weighted modal transition systems. Our work contributes to the long-term objective towards the unification of existing specification theories through a common framework. A full specification theory for label-structured modal transition systems was developed, including the notions of modal and thorough refinement, consistency, determinization and deterministic-hull and a number of completeness results, often conditioned (as expected) by the requirement of determinism. We showed soundness results for the operators of parallel composition, conjunction and quotient. The specification theory was concluded by suggesting an extension of Hennessy-Milner logic to handle quantitative aspects and by showing the interplay between the logic and the refinement theory in a similar way as known from the classical theory of labelled transition systems and bisimulation.

Most of the proof techniques were generalizations of the techniques developed for concrete instances of the framework like modal transition systems, however, the general theorems provide novel results for particular instances like weighted and multi-weighted modal transition systems. Finally, we consider the uniform and complete presentation of the main aspects of the suggested specification theory as a contribution on its own.

There are a few instances of recently studied extensions of modal transition systems that cannot be captured in our framework. Here we list some of them as possible directions for future research.

Abstract Probabilistic Automata (Delahaye et al., 2011) and constraint Markov Chains (Caillaud et al., 2010) are recently introduced stochastic extensions of modal transition systems. One of the major difficulties is how to capture and generalize the satisfaction relation which is (unlike to our framework) based on a redistribution of weights from one to several transitions.

The theories based on the optimistic approach introduced in Interface Automata (de Alfaro and Henzinger, 2001; de Alfaro et al., 2002) are also hard to capture by our framework. Here the semantics of a given specification is viewed as a two player game. The approach is optimistic in the sense that two specifications can be composed if and only if there exists at least one environment in which they can cooperate. The input and output modalities are orthogonal to may and must ones, which suggests that the label-structured modal transition systems need to be further extended to capture this phenomenon.

Finally, it would be of interest to consider models that manipulate data like in the spirit of sociable interfaces (de Alfaro et al., 2005; Adler et al., 2006) and see if they can be described in the framework of label-structured modal transition systems.

References

Adler, B. T., de Alfaro, L., da Silva, L. D., Faella, M., Legay, A., Raman, V., and Roy, P. (2006). Ticc: A tool for interface compatibility and composition. In *Proc. 18th Int. Conference on Computer Aided Verification (CAV)*, volume 4144 of *Lecture Notes in Computer Science*, pages 59–62. Springer.

Antonik, A. and Huth, M. (2006). Efficient patterns for model checking partial state spaces in ctl intersection ltl. *Electronic Notes in Theoretical Computer Science*, 158(0):41 – 57.

Bauer, S. S., Mayer, P., Schroeder, A., and Hennicker, R. (2010). On weak modal compatibility, refinement, and the mio workbench. In Esparza, J. and Majumdar, R., editors, *TACAS*, volume 6015 of *Lecture Notes in Computer Science*, pages 175–189. Springer.

Beneš, N., Křetínský, J., Larsen, K. G., and Srba, J. (2009a). Checking thorough refinement on modal transition systems is exptime-complete. In Leucker, M. and Morgan, C., editors, *ICTAC*, volume 5684 of *Lecture Notes in Computer Science*, pages 112–126. Springer.

Beneš, N., Křetínský, J., Larsen, K. G., and Srba, J. (2009b). On determinism in modal transition systems. *Theoretical Computer Science*, 410(41):4026–4043.

Bertrand, N., Legay, A., Pinchinat, S., and Raclet, J.-B. (2009a). A compositional approach on modal specifications for timed systems. In Breitman, K. and Cavalcanti, A., editors, *ICFEM*, volume 5885 of *Lecture Notes in Computer Science*, pages 679–697. Springer.

Bertrand, N., Pinchinat, S., and Raclet, J.-B. (2009b). Refinement and consistency of timed modal specifications. In Dediu, A. H., Ionescu, A.-M., and Martín-Vide, C., editors, *LATA*, volume 5457 of *Lecture Notes in Computer Science*, pages 152–163. Springer.

Boudol, G. and Larsen, K. G. (1992). Graphical versus logical specifications. *Theoretical Computer Science*, 106(1):3–20.

Bruns, G. and Godefroid, P. (2000). Generalized model checking: Reasoning about partial state spaces. In Palamidessi, C., editor, *Proceedings of the 11th International Conference on Concurrency Theory (CONCUR'00)*, volume 1877 of *Lecture Notes in Computer Science*, pages 168–182. Springer.

Caillaud, B., Delahaye, B., Larsen, K. G., Legay, A., Pedersen, M. L., and Wasowski, A. (2010). Compositional design methodology with constraint markov chains. In *Proceedings of 7th International Conference on Quantitative Evaluation of SysTems (QEST)*, page XXXX. IEEE Computer Society.

COMBEST (2008–2011). Combest. http://www.combest.eu.com.

David, A., Larsen, K. G., Legay, A., Nyman, U., and Wasowski, A. (2010). Timed i/o automata: a complete specification theory for real-time systems. In Johansson, K. H. and Yi, W., editors, *HSCC*, pages 91–100. ACM.

de Alfaro, L., da Silva, L. D., Faella, M., Legay, A., Roy, P., and Sorea, M. (2005). Sociable interfaces. In *Proc 5th Int. Conference on Frontiers of Combining Systems (FROCOS)*, volume 3717 of *Lecture Notes in Computer Science*, pages 81–105. Springer.

de Alfaro, L. and Henzinger, T. A. (2001). Interface automata. In *Proceedings of the 8th ESEC and 9th ACM SIGSOFT FSE*, ESEC/FSE-9, pages 109–120, New York, NY, USA. ACM.

de Alfaro, L. and Henzinger, T. A. (2005). Interface-based Design. In Broy, M., Grünbauer, J., Harel, D., and Hoare, C. A. R., editors, *Engineering Theories of Software-intensive Systems*, volume 195 of *NATO Science Series: Mathematics, Physics, and Chemistry*, pages 83–104. Springer.

de Alfaro, L., Henzinger, T. A., and Stoelinga, M. (2002). Timed interfaces. In Sangiovanni-Vincentelli, A. L. and Sifakis, J., editors, *EMSOFT*, volume 2491 of *Lecture Notes in Computer Science*, pages 108–122. Springer.

Delahaye, B., Katoen, J.-P., Larsen, K. G., Legay, A., Pedersen, M. L., Sher, F., and Wasowski, A. (2011). Abstract probabilistic automata. In *Proceedings of 12th International Conference on Verification, Model Checking, and Abstract Interpretation (VMCAI)*, Lecture Notes in Computer Science. Springer. To appear.

Delahaye, B., Larsen, K., Legay, A., and Wasowski, A. (2010). On greatest lower bound of modal transition systems. Technical report, INRIA.

Feuillade, G. and Pinchinat, S. (2007). Modal specifications for the control theory of discrete event systems. *Discrete Event Dynamic Systems*, 17(2):211–232.

Fischbein, D., Uchitel, S., and Braberman, V. (2006). A foundation for behavioural conformance in software product line architectures. In *Proceedings of the ISSTA 2006 Workshop on Role of Software Architecture for Testing and Analysis (ROSATEA'06)*, pages 39–48. ACM.

Godefroid, P., Huth, M., and Jagadeesan, R. (2001). Abstraction-based model checking using modal transition systems. In Larsen, K. G. and Nielsen, M., editors, *CONCUR*, volume 2154 of *Lecture Notes in Computer Science*, pages 426–440. Springer.

Gordon, M. (1979). *The Denotational Description of Programming Languages*. Springer-Verlag.

Gruler, A., Leucker, M., and Scheidemann, K. D. (2008). Calculating and modeling common parts of software product lines. In *SPLC*, pages 203–212. IEEE Computer Society.

Hennessy, M. and Milner, R. (1985). Algebraic laws for nondeterminism and concurrency. *J. ACM*, 32(1):137–161.

Hoare, C. A. R. (1969). An axiomatic basis for computer programming. *Commun. ACM*, 12:576–580.

Hoare, C. A. R. (1985). *Communicating Sequential Processes*. Prentice-Hall.

Juhl, L., Larsen, K. G., and Srba, J. (2010). Modal Transition Systems with Weight Intervals. To appear in JLAP.

Katoen, J.-P., Klink, D., and Neuhäußer, M. R. (2009). Compositional abstraction for stochastic systems. In Ouaknine, J. and Vaandrager, F. W., editors, *FORMATS*, volume 5813 of *Lecture Notes in Computer Science*, pages 195–211. Springer.

Larsen, K. G. (1985). A context dependent equivalence between processes. In Brauer, W., editor, *ICALP*, volume 194 of *Lecture Notes in Computer Science*, pages 373–382. Springer.

Larsen, K. G. (1987). A context dependent equivalence between processes. *Theoretical Computer Science*, 49:184–215.

Larsen, K. G. (1989). Modal specifications. In (Sifakis, 1990), pages 232–246.

Larsen, K. G., Larsen, U., and Wasowski, A. (2005). Color-blind specifications for transformations of reactive synchronous programs. In Cerioli, M., editor, *FASE*, volume 3442 of *Lecture Notes in Computer Science*, pages 160–174. Springer.

Larsen, K. G., Mikucionis, M., and Nielsen, B. (2004). Online testing of real-time systems using uppaal. In Grabowski, J. and Nielsen, B., editors, *FATES*, volume 3395 of *Lecture Notes in Computer Science*, pages 79–94. Springer.

Larsen, K. G. and Milner, R. (1987). Verifying a protocol using relativized bisimulation. In Ottmann, T., editor, *ICALP*, volume 267 of *Lecture Notes in Computer Science*, pages 126–135. Springer.

Larsen, K. G. and Milner, R. (1992). A compositional protocol verification using relativized bisimulation. *Inf. Comput.*, 99(1):80–108.

Larsen, K. G., Nyman, U., and Wasowski, A. (2007). Modal i/o automata for interface and product line theories. In Nicola, R. D., editor, *ESOP*, volume 4421 of *Lecture Notes in Computer Science*, pages 64–79. Springer.

Larsen, K. G. and Thomsen, B. (1988a). Compositional proofs by partial specification of processes. In Chytil, M., Janiga, L., and Koubek, V., editors, *MFCS*, volume 324 of *Lecture Notes in Computer Science*, pages 414–423. Springer.

Larsen, K. G. and Thomsen, B. (1988b). A modal process logic. In *LICS*, pages 203–210. IEEE Computer Society.

Milner, R. (1980). *A Calculus of Communicating Systems*, volume 92 of *Lecture Notes in Computer Science*. Springer.

Milner, R. (1983). Calculi for synchrony and asynchrony. *Theoretical Computer Science*, 25:267–310.

Park, D. M. R. (1981). Concurrency and automata on infinite sequences. In Deussen, P., editor, *Theoretical Computer Science*, volume 104 of *Lecture Notes in Computer Science*, pages 167–183. Springer.

Pierce, B. C. and Sangiorgi, D. (2000). Behavioral equivalence in the polymorphic pi-calculus. *J. ACM*, 47:531–584.

Plotkin, G. (1981). A structural approach to operational semantics. FN 19, DAIMI. Computer Science Department, Aarhus University, Denmark.

Raclet, J.-B. (2008). Residual for component specifications. In *Proc. of the 4th International Workshop on Formal Aspects of Component Software (FACS'07)*, volume 215 of *Electronic Notes in Theoretical Computer Science*, pages 93–110.

Sifakis, J., editor (1990). *Automatic Verification Methods for Finite State Systems, International Workshop, Grenoble, France, June 12-14, 1989, Proceedings*, volume 407 of *Lecture Notes in Computer Science*. Springer.

SPEEDS (2006–2010). Speeds. http://www.speeds.eu.com.

Stirling, C. (1987). Modal logics for communicating systems. *Theoretical Computer Science*, 49:311–347.

Stirling, C. and Walker, D. (1989). Ccs, liveness, and local model checking in the linear time mu-calculus. In (Sifakis, 1990), pages 166–178.

Stoy, J. (1977). *Denotational Semantics: The Scott-Strachey Approach to Progamming Language Theory*. The MIT Press.

van Glabbeek, R. J. (1990). The linear time-branching time spectrum (extended abstract). In Baeten, J. C. M. and Klop, J. W., editors, *CONCUR*, volume 458 of *Lecture Notes in Computer Science*, pages 278–297. Springer.