On the Metric-based Approximate Minimization of Markov Chains*

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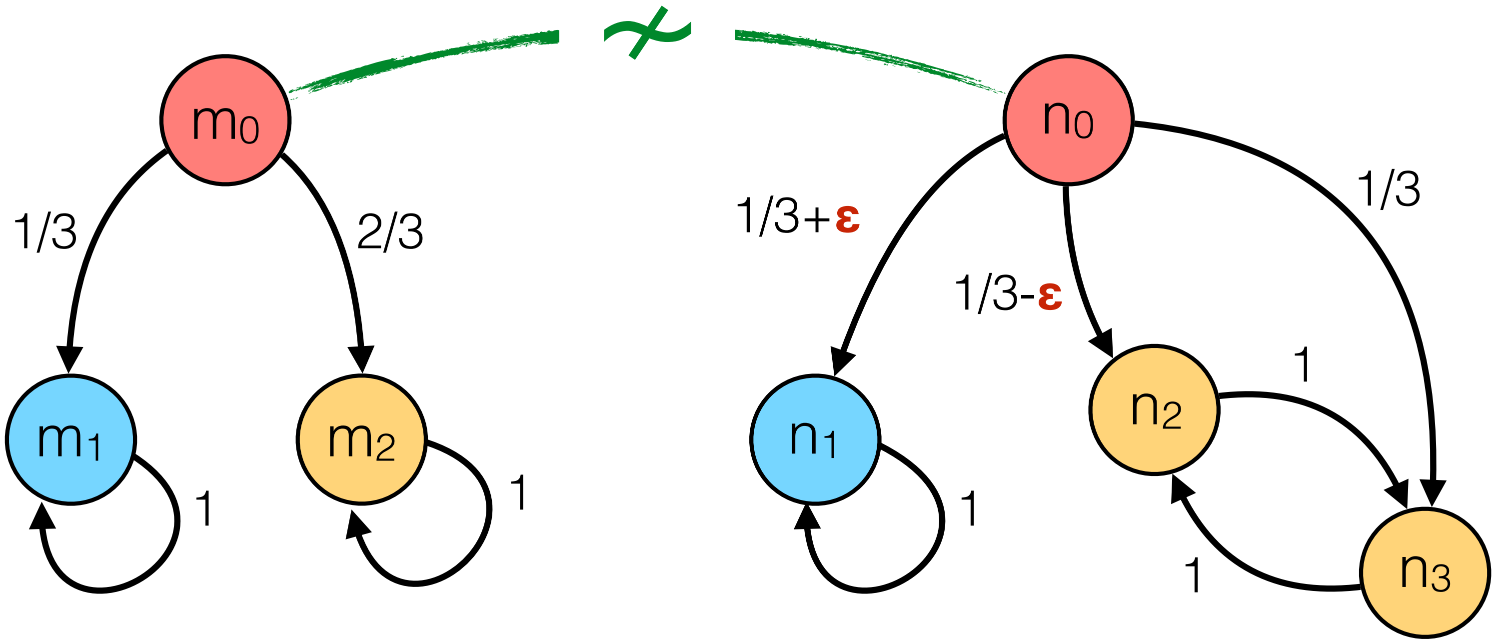
Introduction

- **Moore'56, Hopcroft'71:** Minimization algorithm for DFA (*partition refinement wrt Myhill-Nerode equiv.*)
- Minimization via partition refinement:
  - **Kanellakis-Smolka'83:** minimization of LTSs wrt Milner's strong bisimulation
  - **Baier'96:** minimization of MCs wrt Larsen-Skou probabilistic bisimulation
  - **Alur et al.'92, Yannakakis-Lee'97:** minimization of timed & real-time transition systems.
  - and many more…

A fundamental problem

Jou-Smolka'90 observed that behavioral equivalences are not robust for systems with real-valued data



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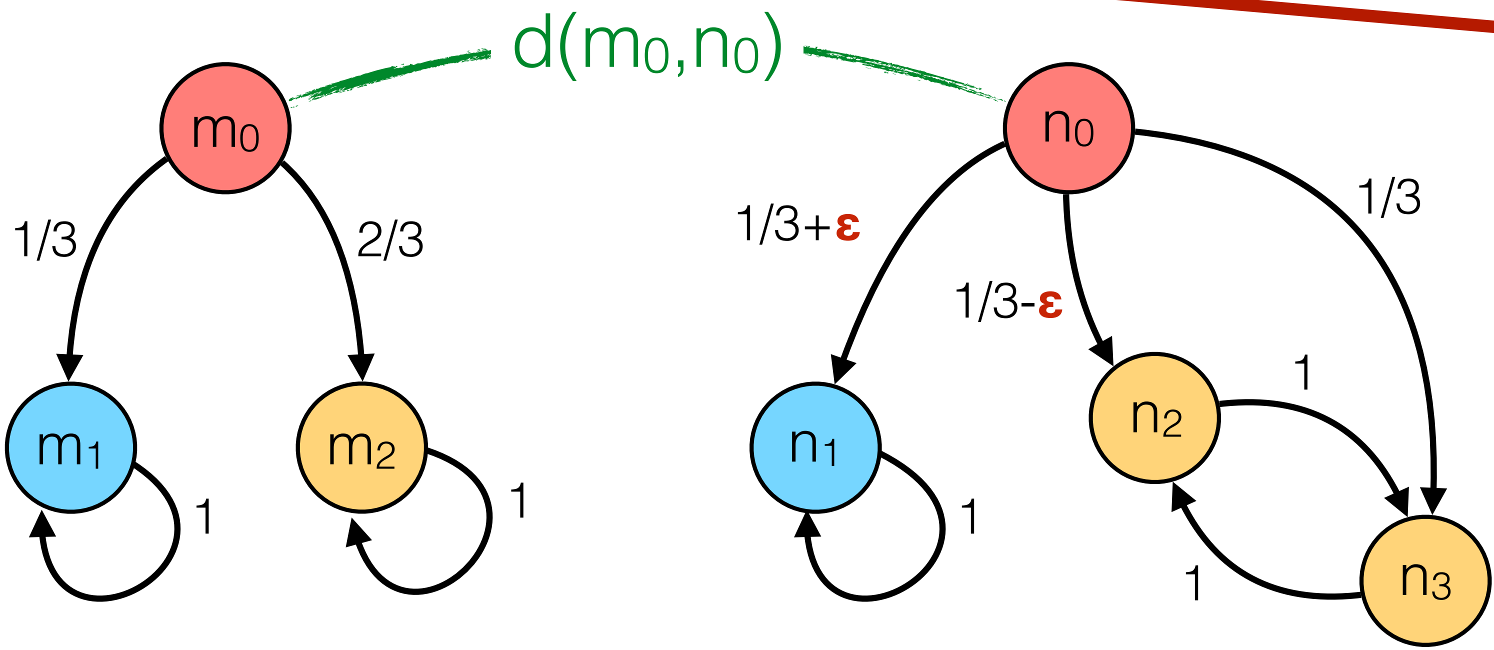
Solution!
equiv. ⇨ distance



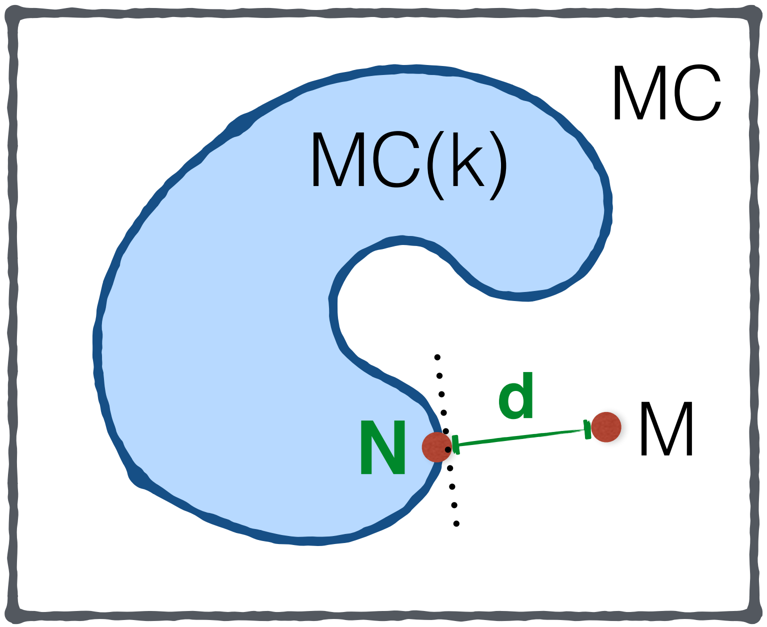
Metric-based Approximate Minimization

Closest Bounded Approximant (CBA)

Minimum Significant Approximant Bound (MSAB)

Metric-based Approximate Minimization

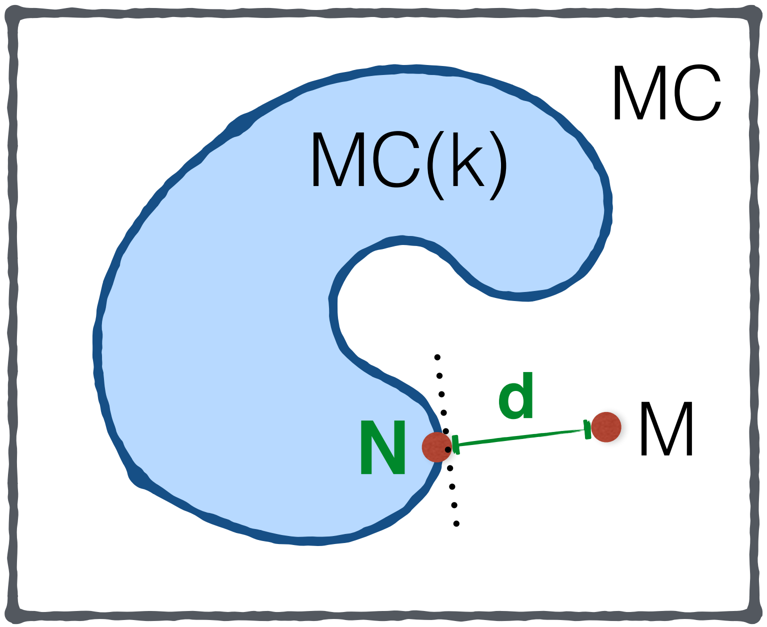
Closest Bounded Approximant (CBA)



Minimum Significant Approximant Bound (MSAB)

Metric-based Approximate Minimization

Closest Bounded Approximant (CBA)



minimize d

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Metric-based Approximate Minimization

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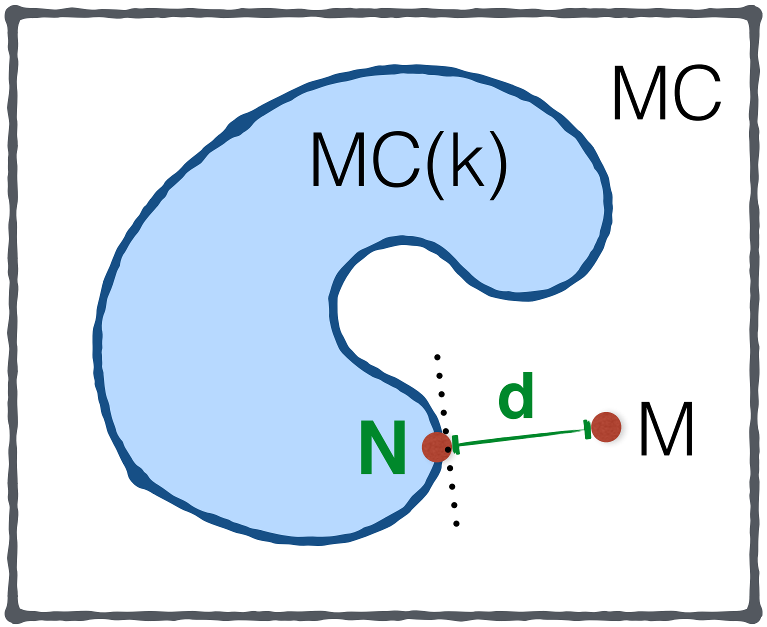

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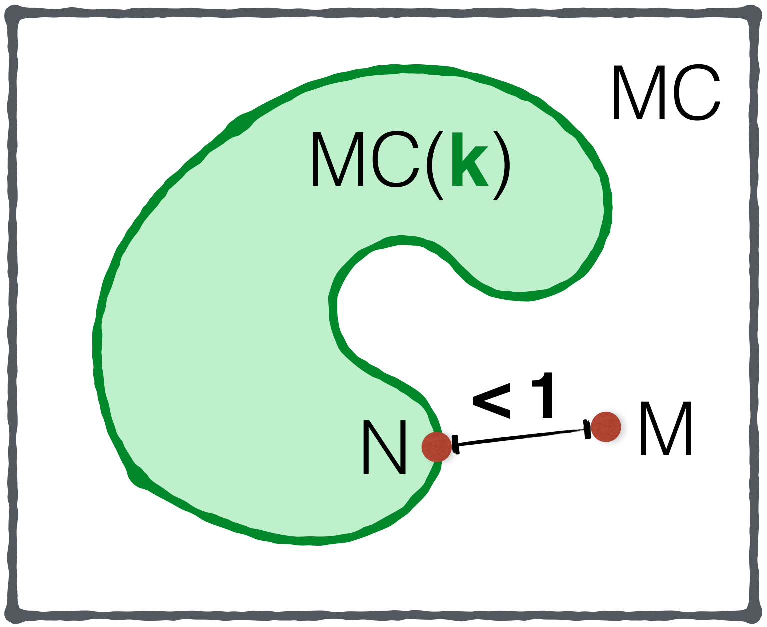

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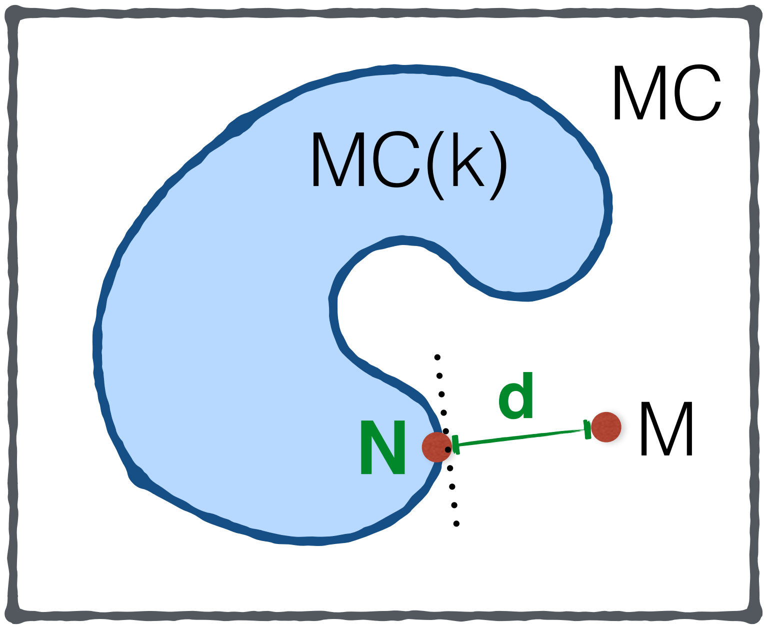

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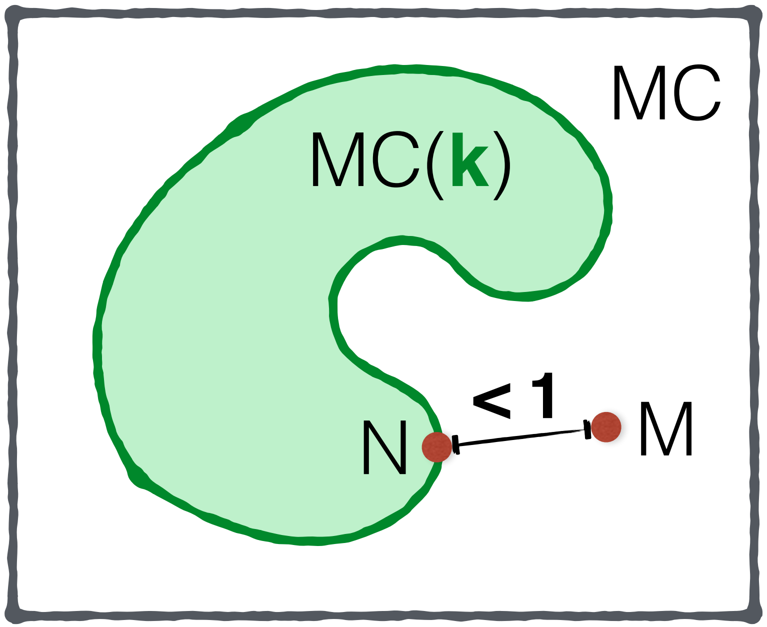
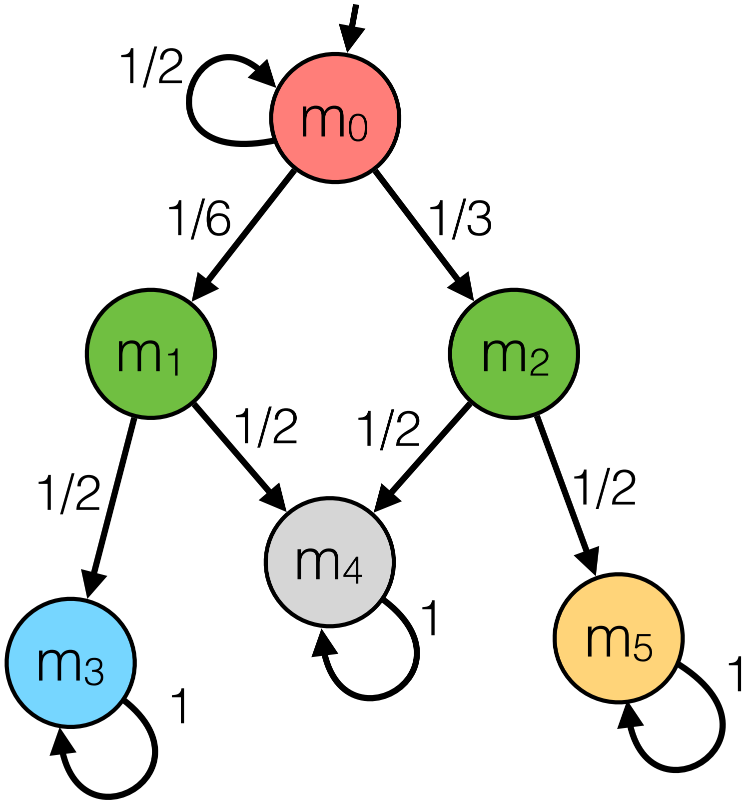
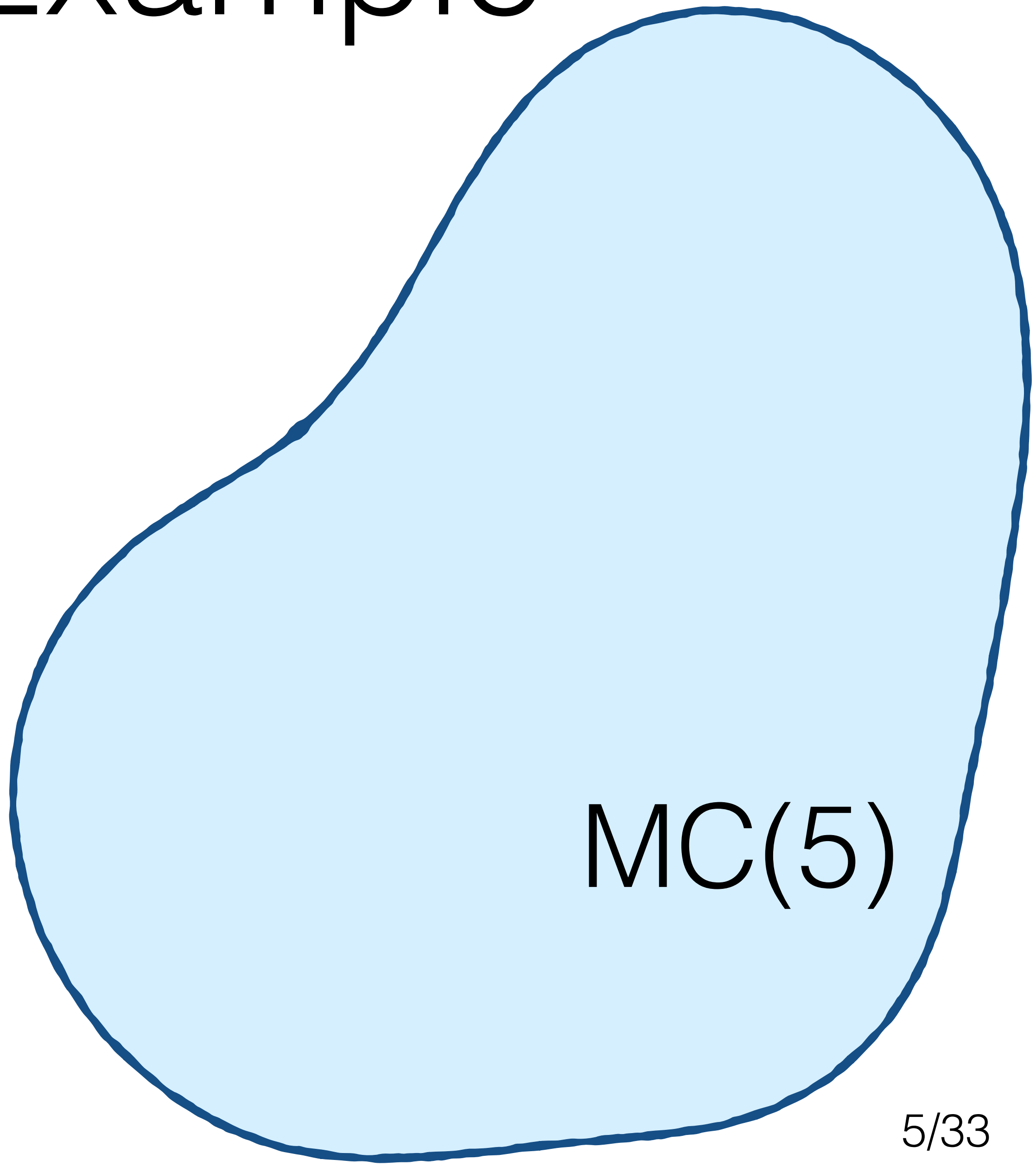


minimize k

CBA: Example*





(*) With respect to the undiscounted probabilistic bisimilarity distance

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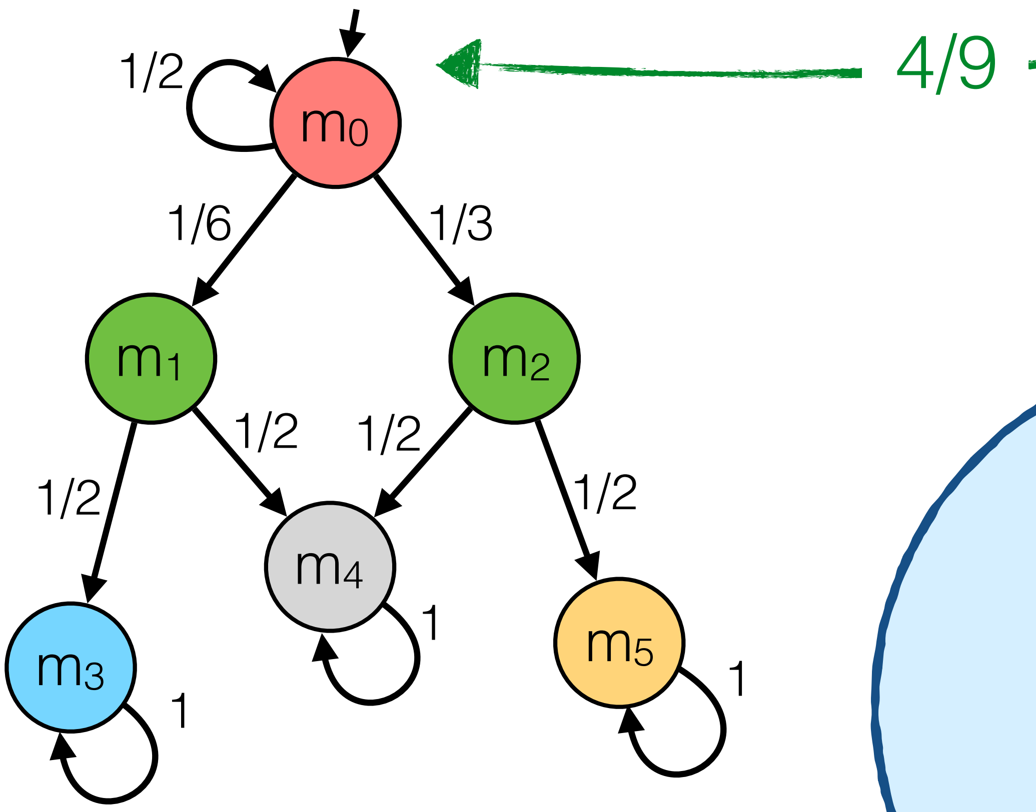

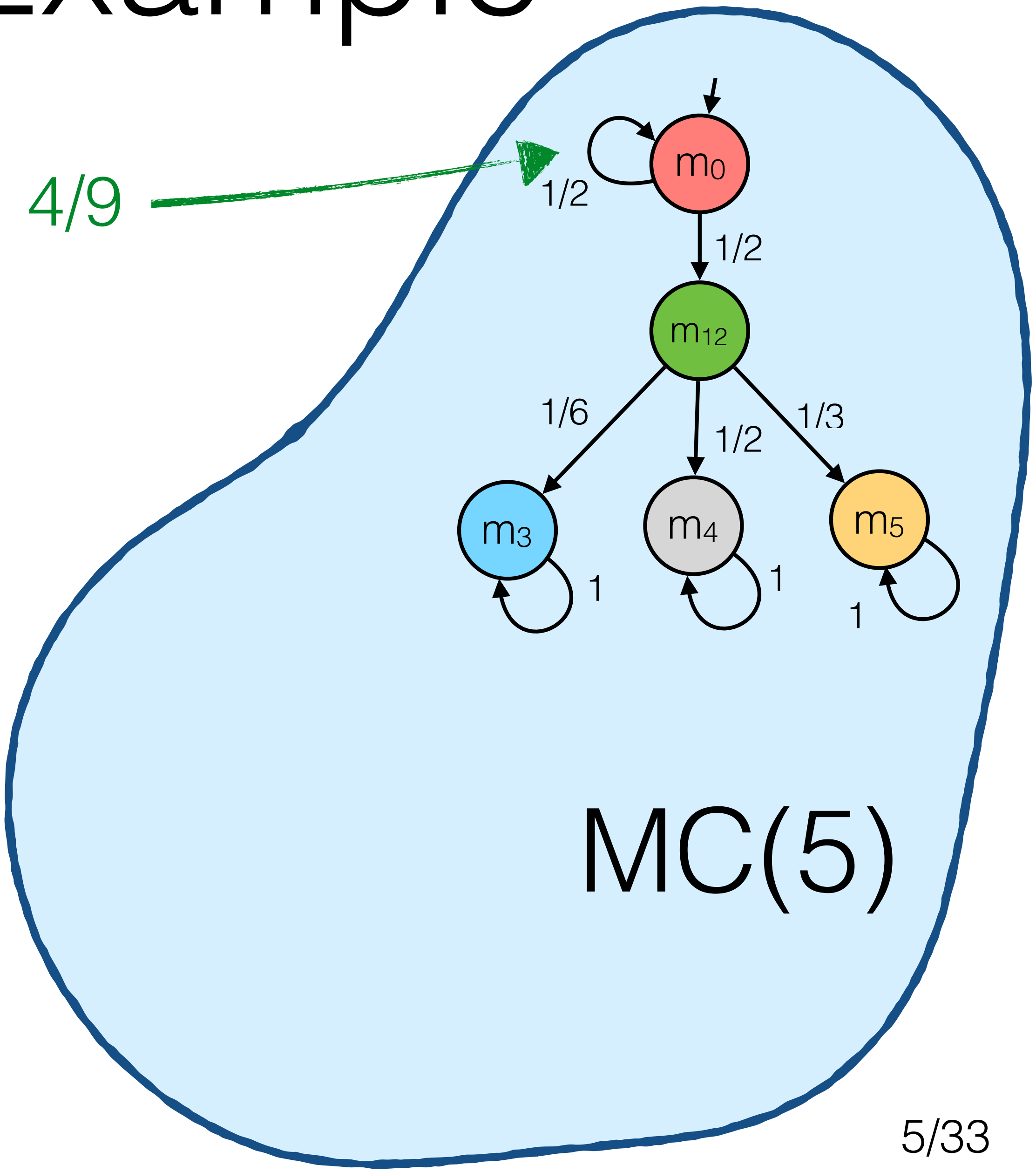

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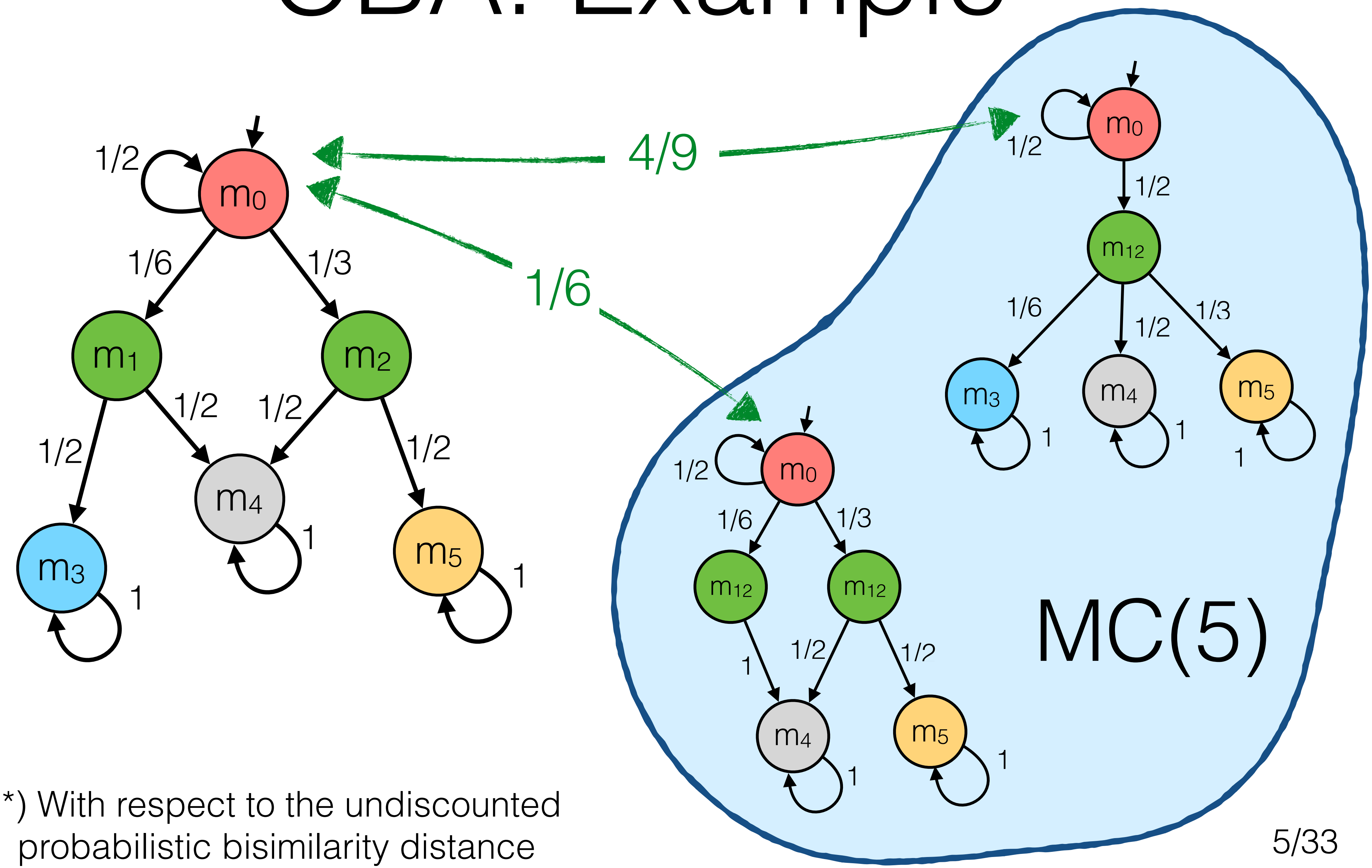

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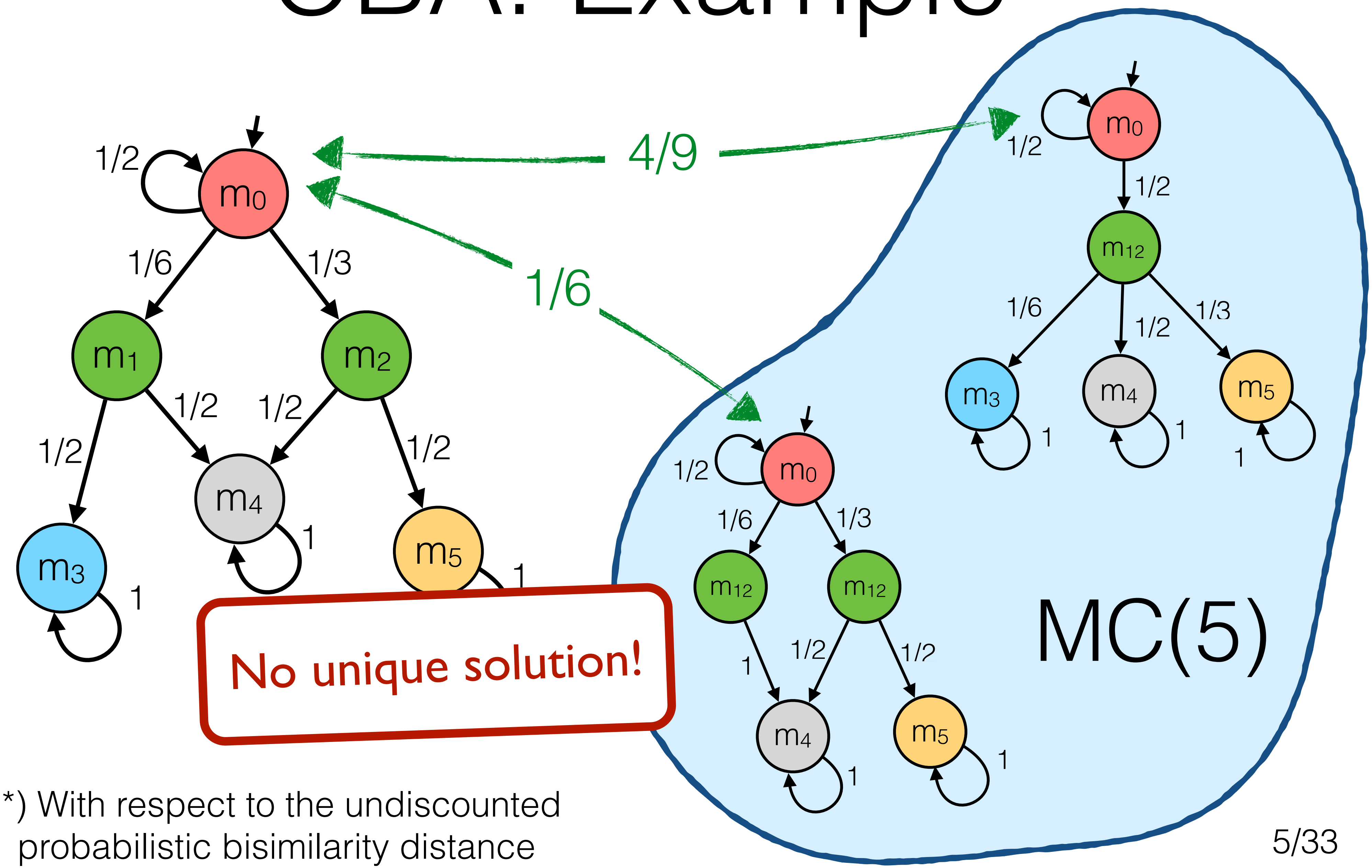

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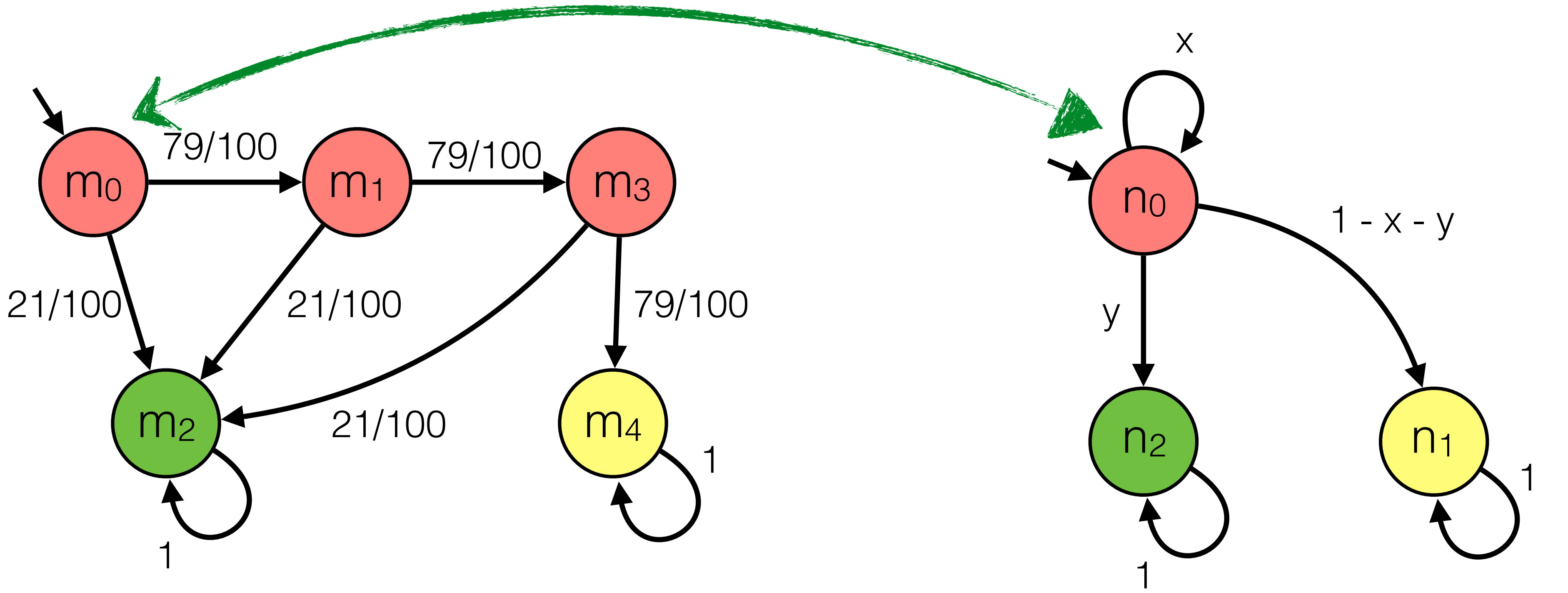

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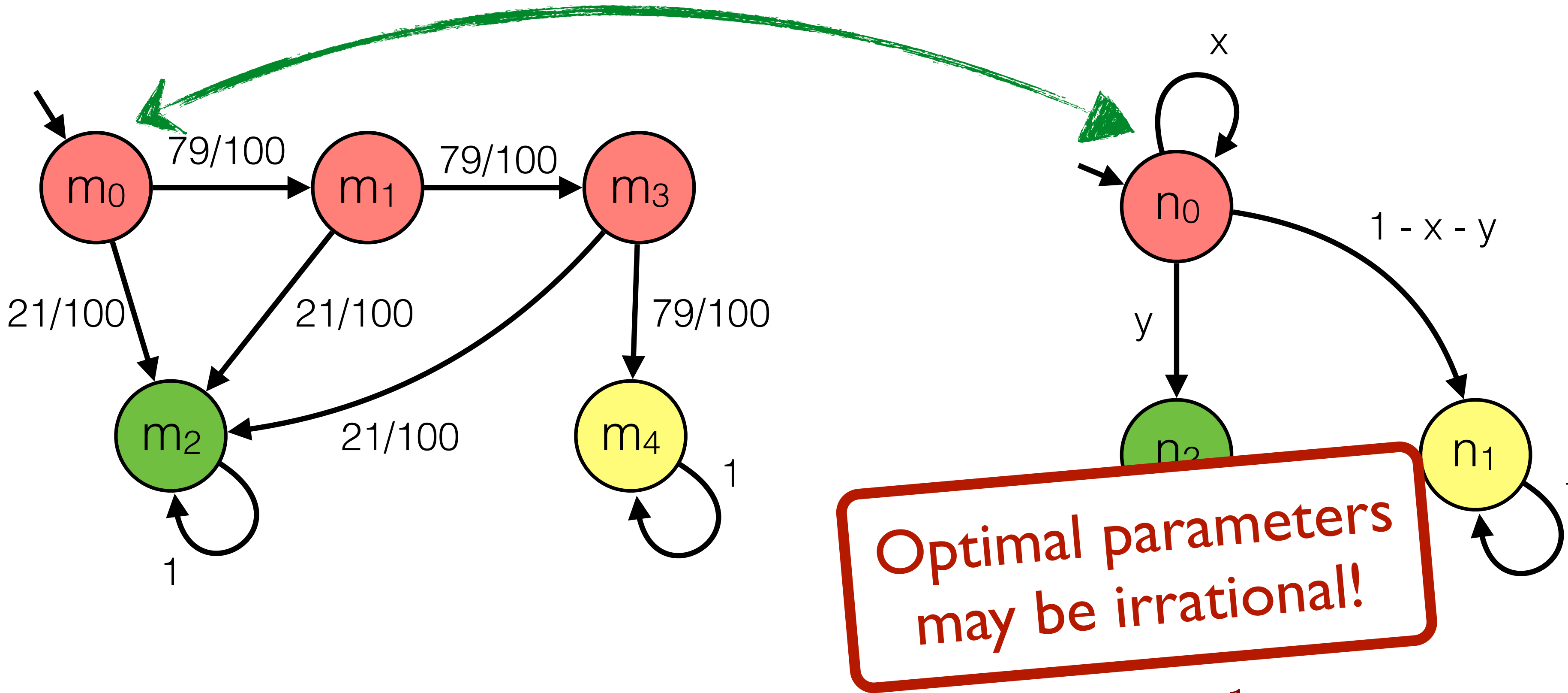
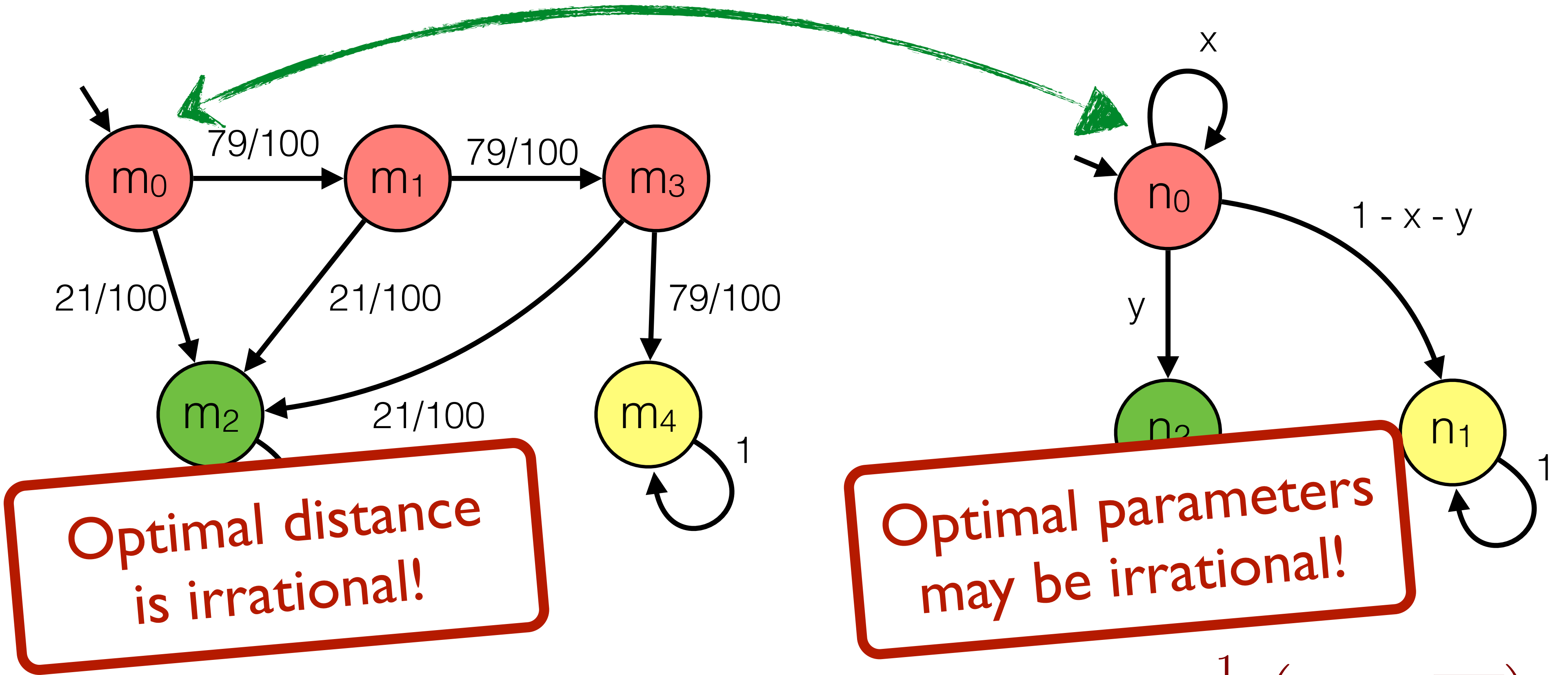


$$x = \frac{1}{30}\left(10 + \sqrt{163}\right)$$
$$y = \frac{21}{200}$$

(*) With respect to the undiscounted probabilistic bisimilarity distance

CBA: Example*



$$\delta(m_0, n_0) = \frac{436}{675} - \frac{163\sqrt{163}}{13500} \approx 0.49$$

$$x = \frac{1}{30}\left(10 + \sqrt{163}\right)$$

$$y = \frac{21}{200}$$

(*) With respect to the undiscounted probabilistic bisimilarity distance

Talk Outline

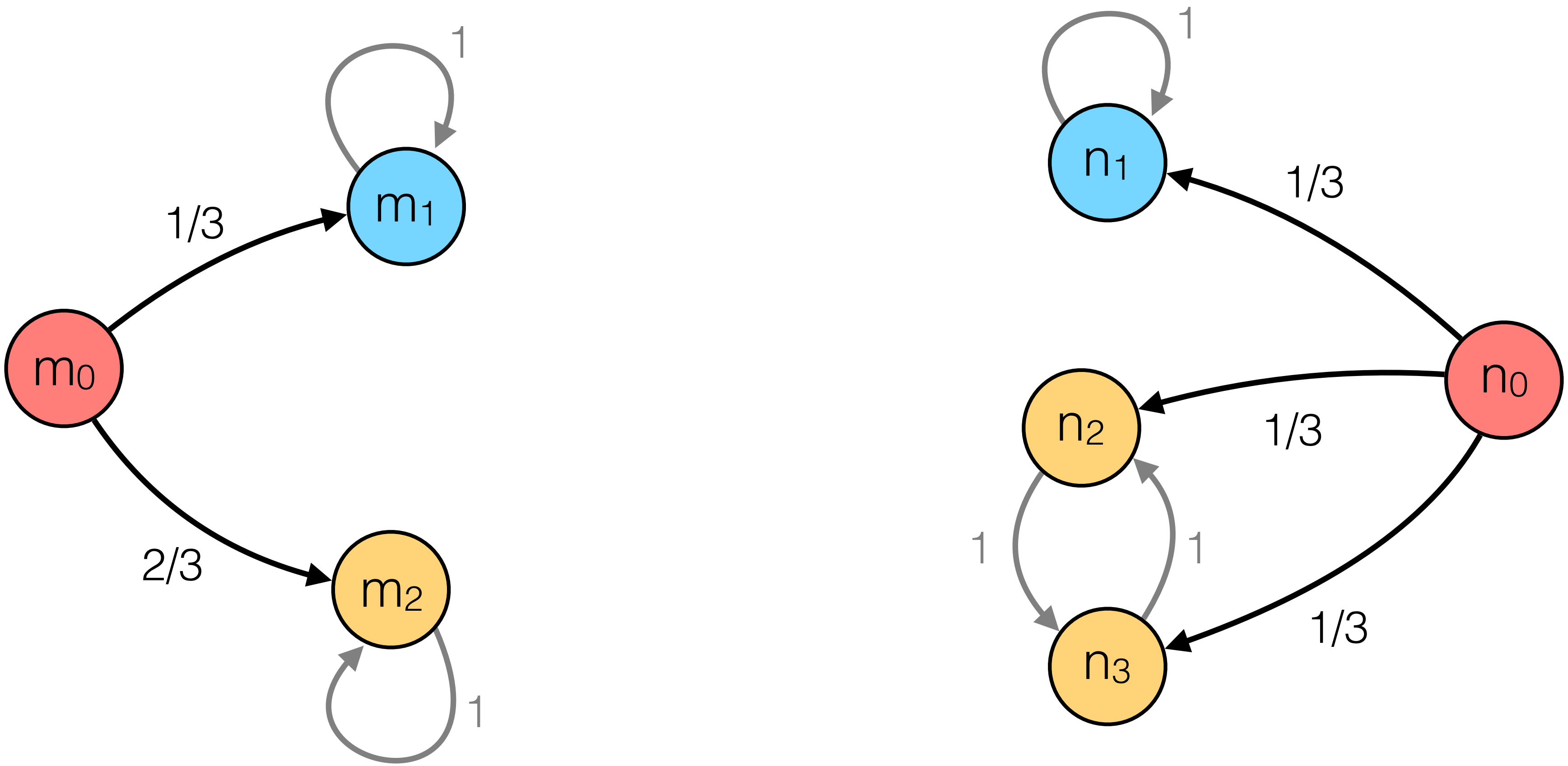
★ **Probabilistic bisimilarity distance**

- fixed point characterization (Kantorovich oper.)
- remarkable properties
- relation with probabilistic model checking

★ **Metric-based Optimal Approximate Minimization**

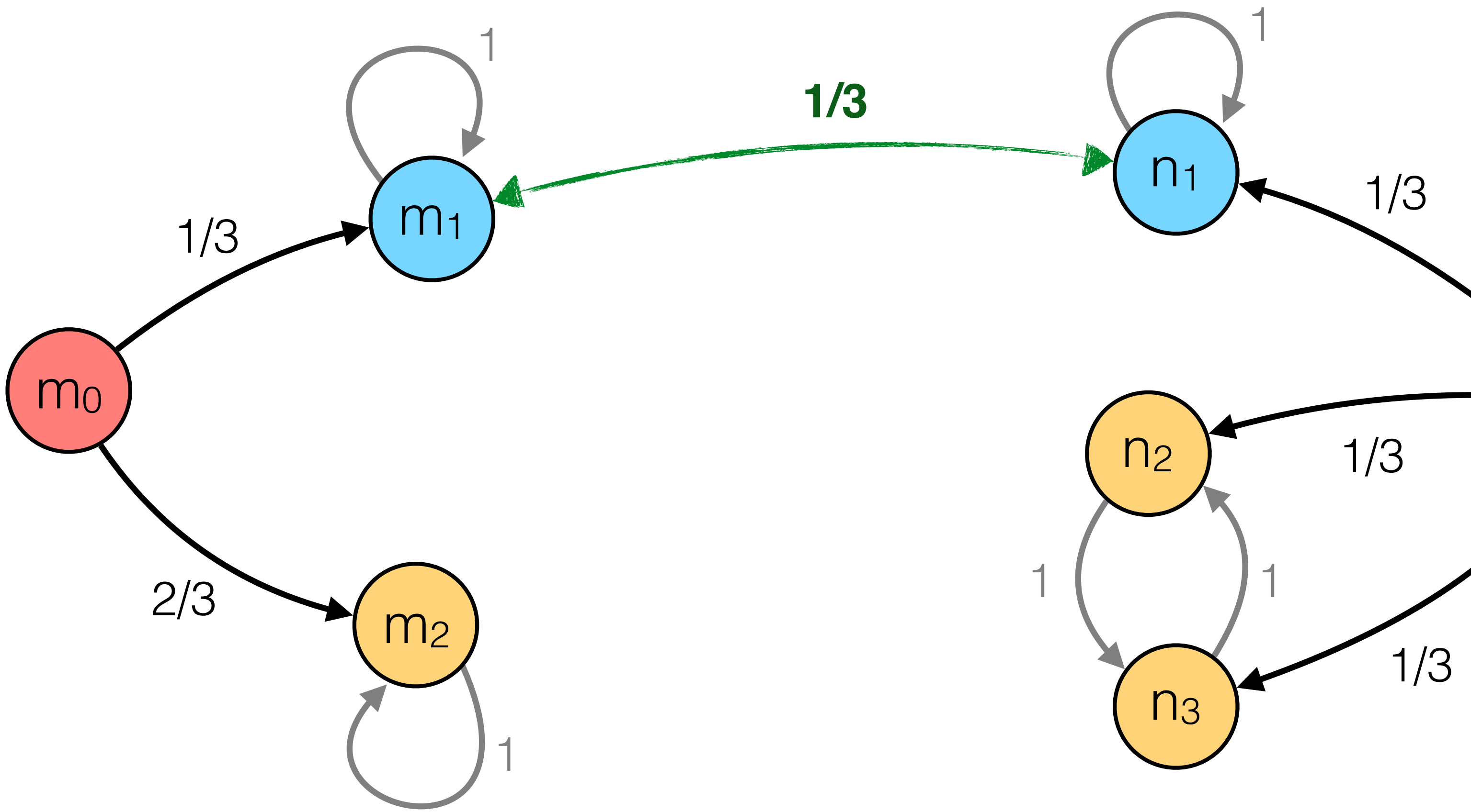
- *Closest Bounded Approximant* (CBA)
  — definition, characterization, complexity
- *Minimum Significant Approximant Bound* (MSAB)
  — definition, characterization, complexity
- *Expectation Maximization-like algorithm*
  — 2 heuristics + experimental results

Probabilistic bisimulation



It tries to match the behaviors "quantitatively"

Probabilistic bisimulation



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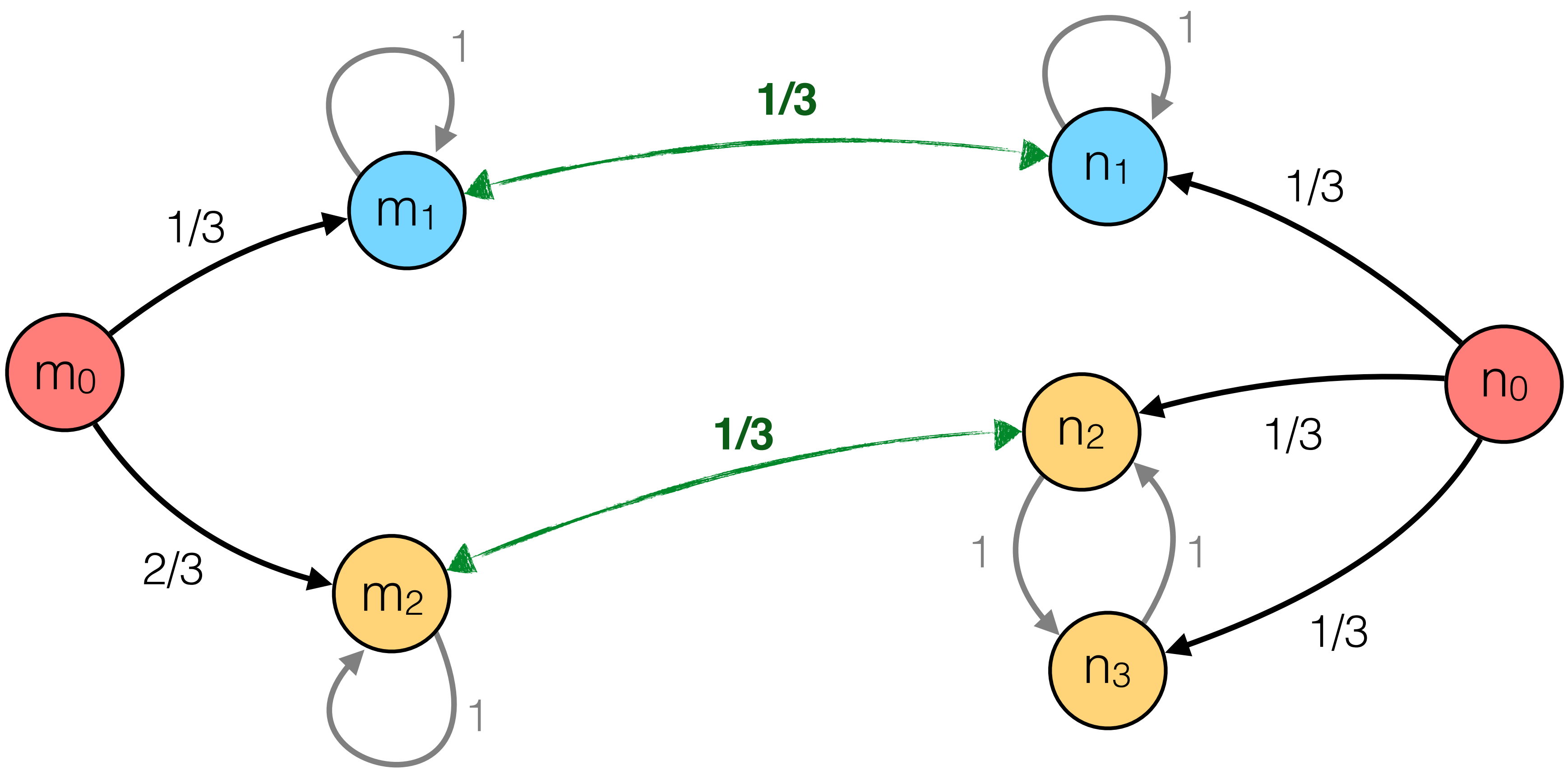

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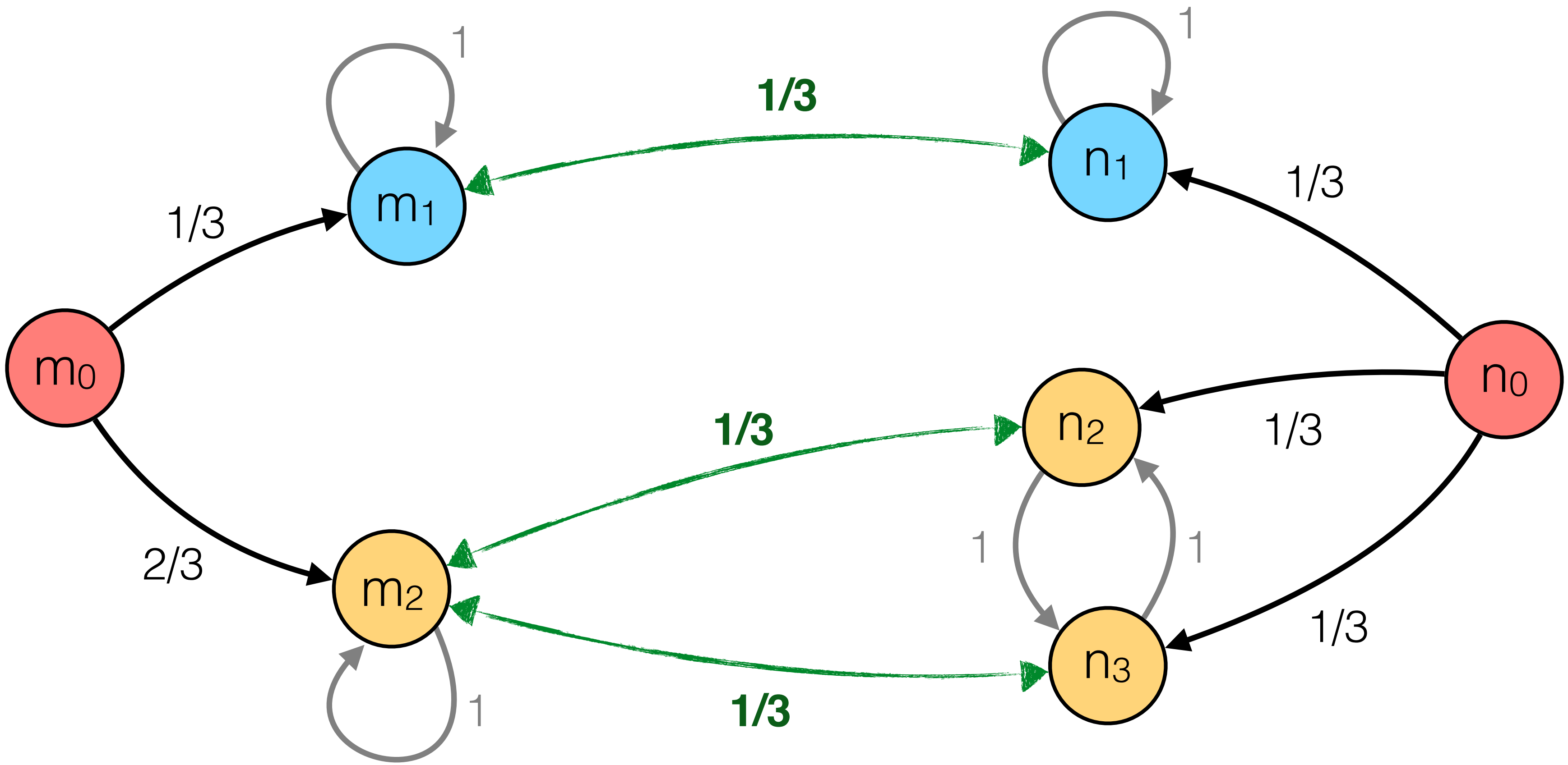

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Coupling

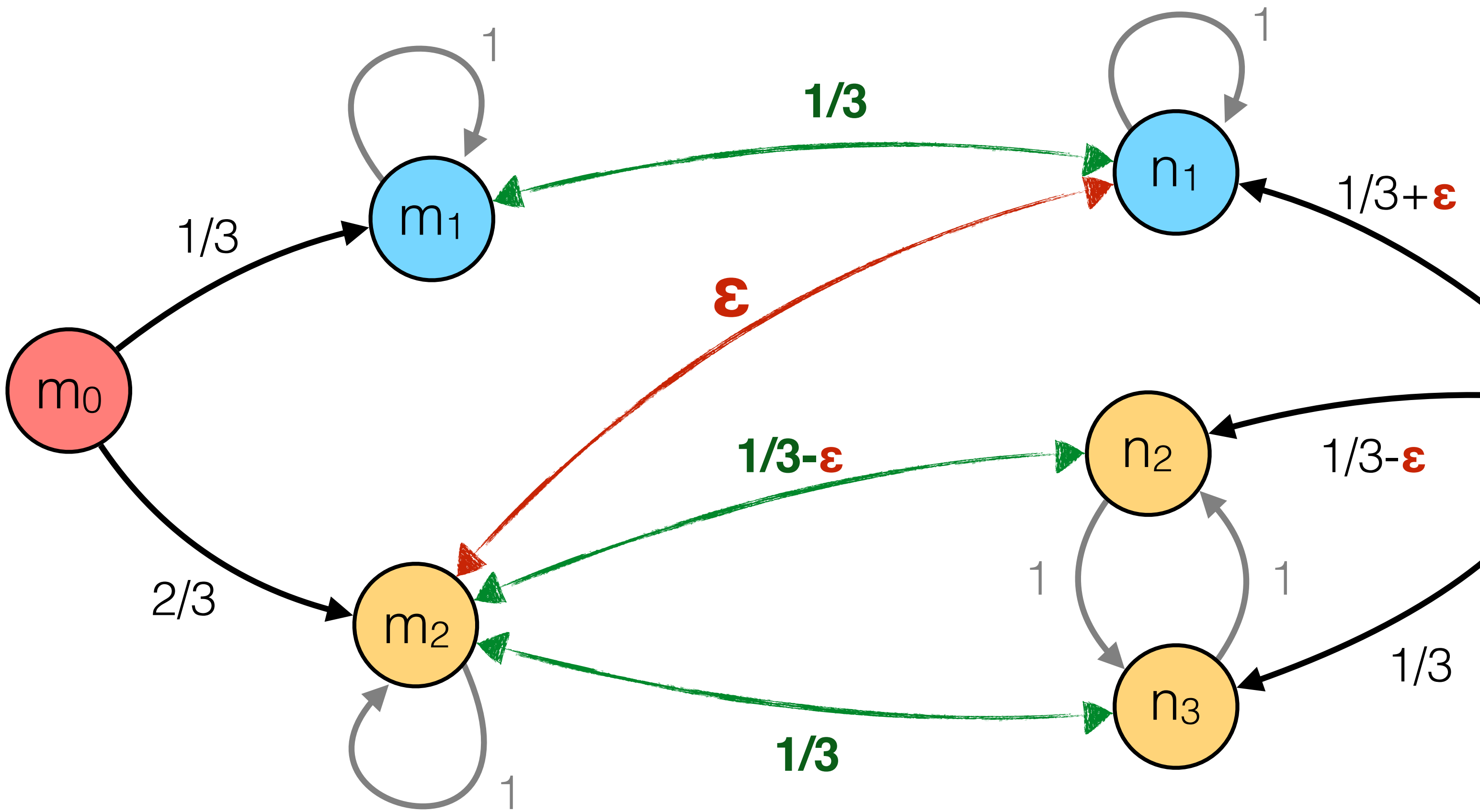
Definition (W. Doeblin 36)

A *coupling* of a pair (μ,ν) of probability distributions on M is a distribution ω on M×M such that

- $\sum_{n \in M} \omega(m,n) = \mu(m)$ *(left marginal)*
- $\sum_{m \in M} \omega(m,n) = \nu(n)$ *(right marginal)*.

One can think of a coupling as a measure-theoretic relation between probability distribution

A quantitative generalization



$$\text{minimize} \sum_{u,v \in M} \omega(u,v)\, d(u,v)$$

A quantitative generalization of probabilistic bisimilarity

The λ-discounted *probabilistic bisimilarity pseudometric* is the smallest $d_\lambda : M \times M \to [0,1]$ such that

$$d_\lambda(m,n) = \begin{cases} 1 & \text{if } \ell(m) \neq \ell(n) \\ \min_{\omega \in \Omega(\tau(m),\tau(n))} \lambda \sum_{u,v \in M} \omega(u,v)\, d_\lambda(u,v) & \text{otherwise} \end{cases}$$

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Kantorovich distance

$$K(d)(\mu,\nu) = \min_{\omega \in \Omega(\mu,\nu)} \sum_{u,v \in M} \omega(u,v)\, d(u,v)$$

Remarkable properties

Theorem (Desharnais et. al 99)

$$m \sim n \quad \text{iff} \quad d_\lambda(m,n) = 0$$

Theorem (Chen, van Breugel, Worrell 12)

The probabilistic bisimilarity distance
can be computed in polynomial time

Relation with Model Checking

Theorem (Chen, van Breugel, Worrell 12)

For all $\phi \in LTL$ $\quad |\Pr(m \vDash \phi) - \Pr(n \vDash \phi)| \leq d_1(m,n)$

Relation with Model Checking

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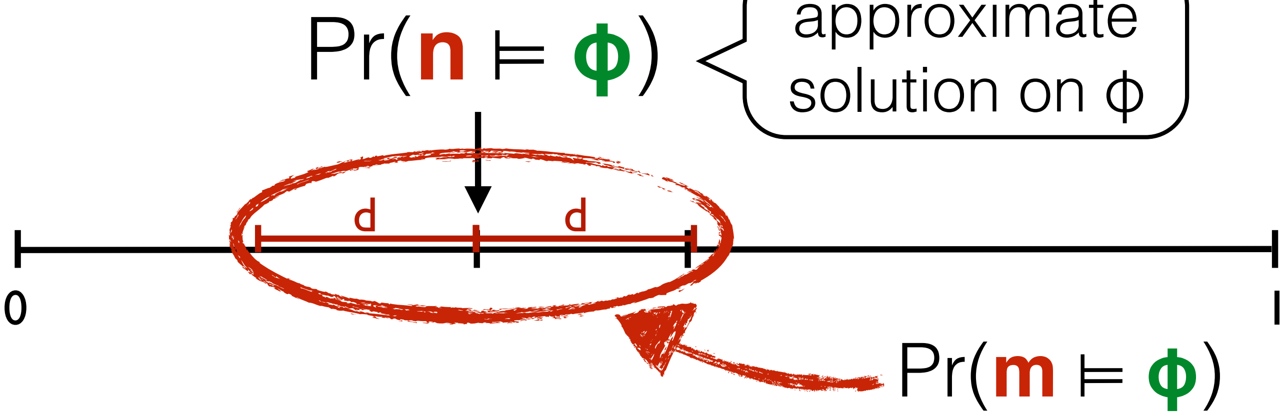
For all $\phi \in LTL$ $\quad | Pr(m \vDash \phi) - Pr(n \vDash \phi) | \leq d_1(m,n)$

…imagine that $|M| \gg |N|$, we can use N in place of M



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The CBA-λ problem

The Closest Bounded Approximant wrt d_λ

Instance: An MC M, and a positive integer k

Ouput: An MC Ñ, with at most k states minimizing $d_\lambda(m_0,\tilde{n}_0)$

$$d_\lambda(m_0,\tilde{n}_0) = \inf \{ d_\lambda(m_0,n_0) \mid N \in MC(k) \}$$

we get a solution iff the infimum is a minimum

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generalization of bisimilarity quotient

CBA-λ as a Bilinear Program

$$d_\lambda(m_0,\tilde{n}_0) = \inf \{\, d_\lambda(m_0,n_0) \mid N \in MC(k) \,\}$$
$$= \inf \{\, d(m_0,n_0) \mid \Gamma_\lambda(d) \leq d,\ N \in MC(k) \,\}$$

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mimimize	d_{m_0,n_0}	
such that	$d_{m,n} = 1$	$\ell(m) \neq \alpha(n)$
	$\lambda \sum_{(u,v) \in M \times N} c^{m,n}_{u,v} \cdot d_{u,v} \leq d_{m,n}$	$\ell(m) = \alpha(n)$
	$\sum_{v \in N} c^{m,n}_{u,v} = \tau(m)(u)$	$m, u \in M,\ n \in N$
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\end{array}
$$

what labels should the MC N have?

CBA-λ as a Bilinear Program

Lemma (Meaningful labels)

For any $N \in MC(k)$, there exists $N' \in MC(k)$ with labels taken from M, such that $d_\lambda(M,N) \geq d_\lambda(M,N')$

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& 1 - \alpha_{n,l} \leq d_{m,n} \leq 1 & n \in N,\ l \in L(\mathcal{M}),\ \ell(m) \neq l \\
& \alpha_{n,l} \cdot \alpha_{n,l'} = 0 & n \in N,\ l, l' \in L(\mathcal{M}),\ l \neq l' \\
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CBA-λ as a Bilinear Program

this characterization has two main consequences…

1. CBA-λ admits always a solution
   (finite intersection of closed subsets)

2. CBA-λ can be approximated up to any precision

Complexity of CBA-λ

"To study the complexity of an optimization problem one has to look at its decision variant"

(C. Papadimitriou)

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Bounded Approximant threshold wrt d_λ

Instance: An MC M, a positive integer k, and a **rational ε>0**

Output: **yes** iff there exists N with at most k states such that $d_\lambda(m_0,n_0)$ **≤ ε**

Complexity upper bound

Theorem

BA-λ is in **PSPACE**

Proof sketch: *we can encode the question ⟨M,k,ε⟩∈BA-λ to that of checking the feasibility of a set of bilinear inequalities. This can be encoded as a decision problem for the existential theory of the reals, thus it can be solved in PSPACE [Canny—STOC88].*

Complexity lower bound

Theorem

BA-λ is **NP-hard**

Proof idea: *we provide a reduction from VERTEX COVER.*
(see the appendix for a sketch of the reduction)

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BA-λ is **NP-hard**

unlikely to solve CBA as simple linear program

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For **λ<1**, the MSAB-λ problem is trivial, because the solution is always k=1

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For **λ=1**, the same problem is surprisingly difficult…

Complexity of MSAB-1

…as before we should look at its decision variant

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Significant Bounded Approximant wrt d_1

Instance: An MC M and a **positive k**

Ouput: **yes** iff there exists N with **at most k** states such that $d_1(m_0, n_0) < 1$.

Complexity of MSAB-1

…as before we should look at its decision variant

Significant Bounded Approximant wrt d_1

Instance: An MC M and a **positive k**

Ouput: **yes** iff there exists N with **at most k** states such that $d_1(m_0,n_0)<1$.

Theorem

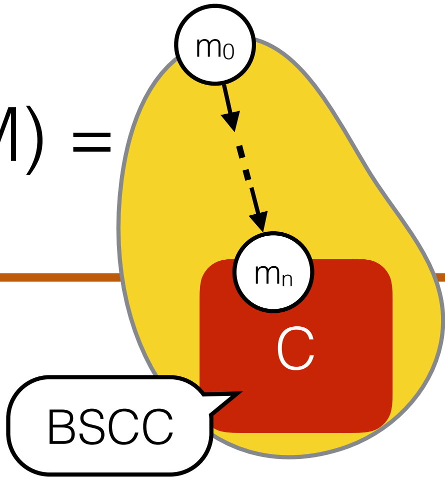
SBA-1 is **NP-complete**

SBA-1 $\subseteq$ **NP**

Lemma

Assume M be maximally collapsed. Then,

$\langle M,k \rangle \in$ SBA-1 iff $\mathcal{G}(M) =$



and $h+|C| \leq k$

number of labels in $m_0 \ldots m_{n-1}$

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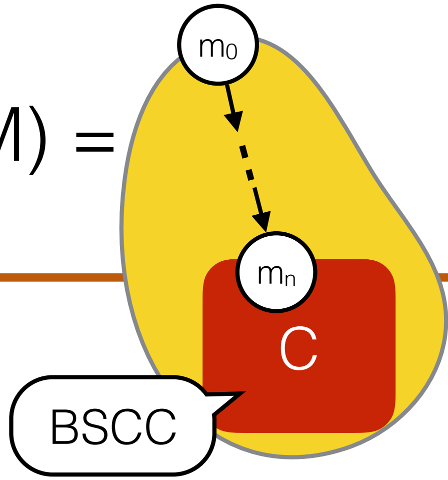
$\langle M,k\rangle \in$ SBA-1 iff $\mathcal{G}(M)$ = 

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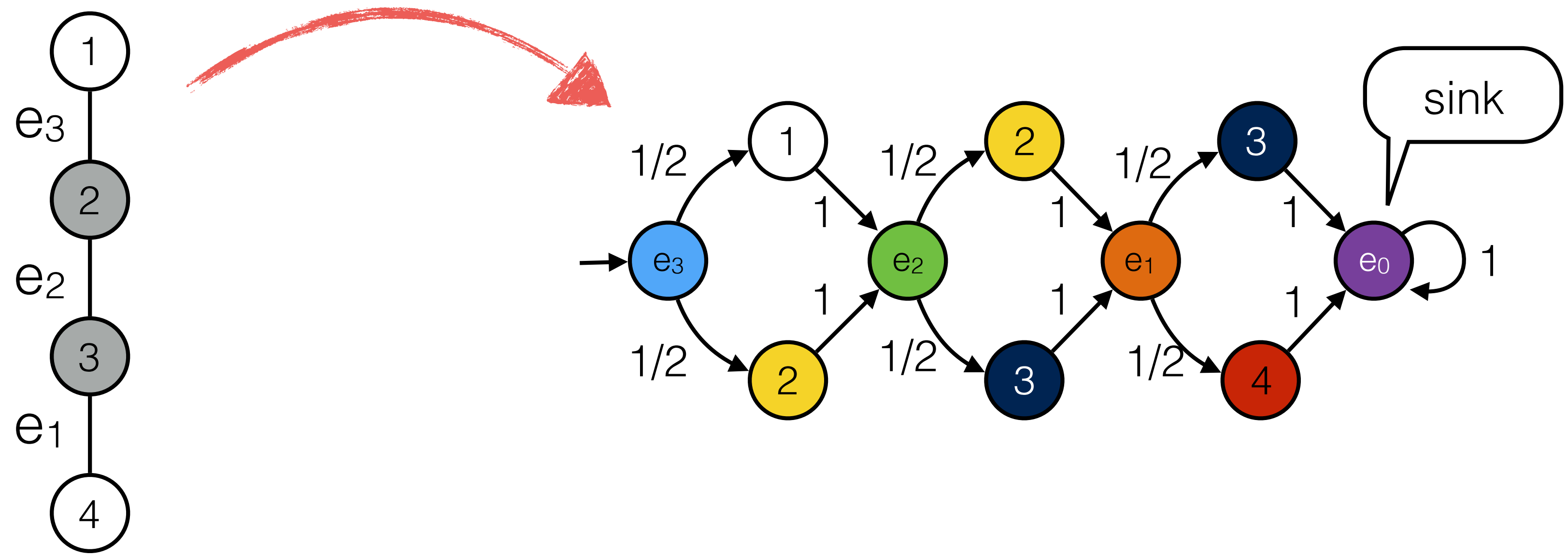
Proof sketch: *compute with Tarjan's algorithm all the SCCs of $\mathcal{G}(M)$. Then non deterministically choose a BSCC and a path to it. In poly-time we can count the number of labels in the path and the size of the BSCC.*

SBA-1 is **NP-hard**



Proof sketch: *by reduction to VERTEX COVER:*

$\langle G,h \rangle \in$ *VERTEX COVER* *iff* $\langle M_G, h+m+1 \rangle \in$ *SBA-1*

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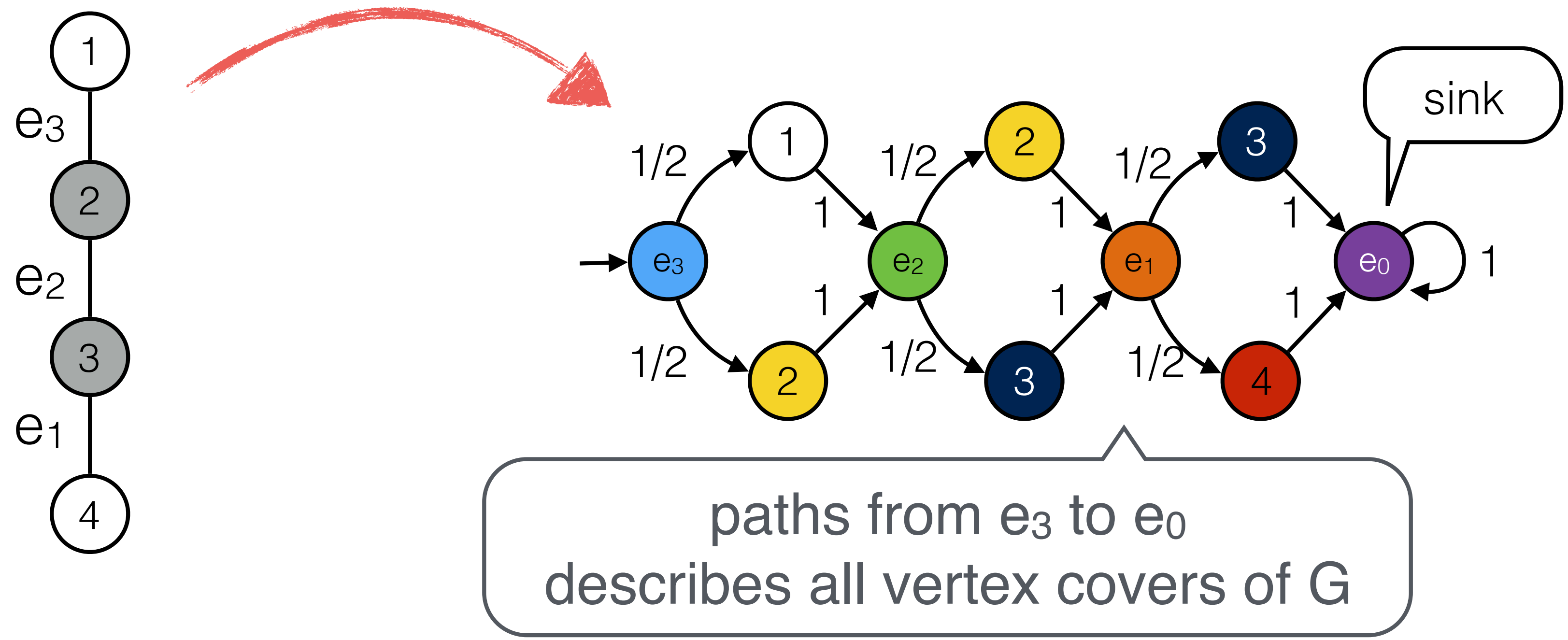

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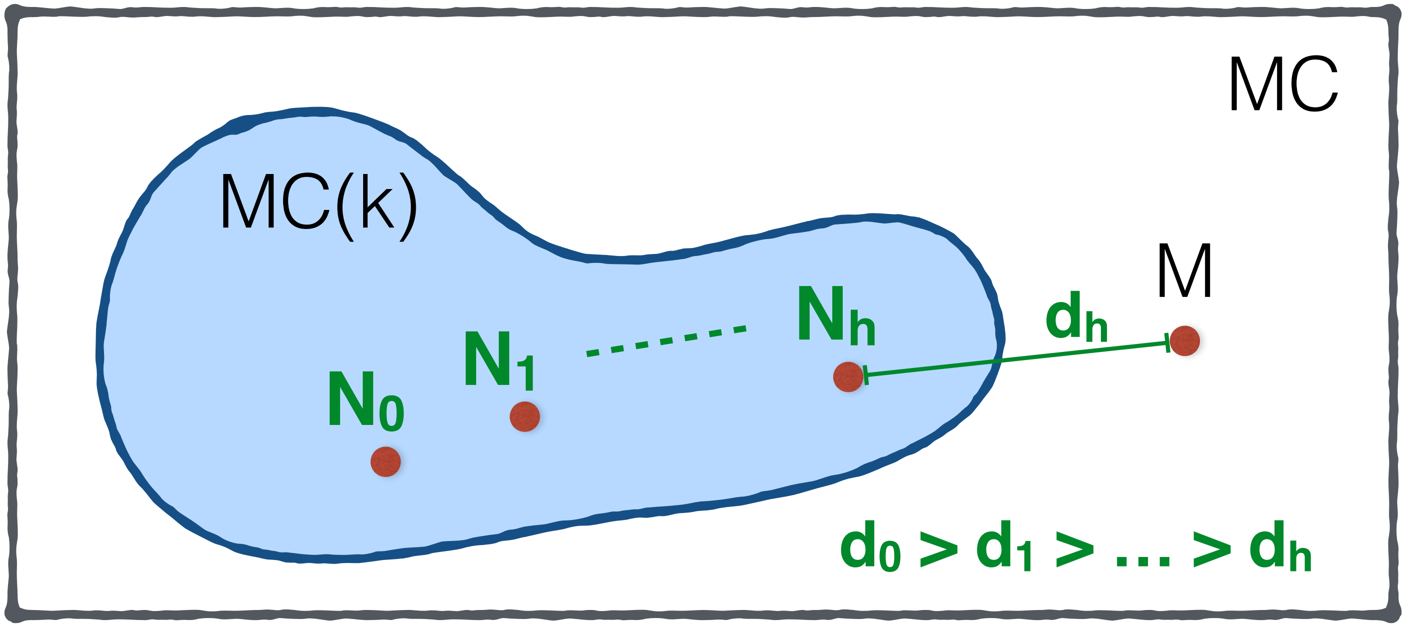
- The CBA can be solved as a bilinear program. Theoretically nice, but practically unfeasible!
  (our implementation in PENBMI can handle MCs with at most 5 states…)

- We are happy with **sub-optimal solutions** if they can be obtained by a practical algorithm.

EM-like Algorithm

- Given the MC M and an initial approximant N_0
- it produces a sequence $N_0, \ldots, N_h$ of approximants having strictly decreasing distance from M
- N_h may be a sub-optimal solution of CBA-λ



EM-like Algorithm

Algorithm 1

Input: $\mathcal{M} = (M, \tau, \ell)$, $\mathcal{N}_0 = (N, \theta_0, \alpha)$, and $h \in \mathbb{N}$.

1. $i \leftarrow 0$
2. **repeat**
3. $\quad i \leftarrow i + 1$
4. $\quad$ compute $\mathcal{C} \in \Omega(\mathcal{M}, \mathcal{N}_{i-1})$ such that $\delta_\lambda(\mathcal{M}, \mathcal{N}_{i-1}) = \gamma_\lambda^{\mathcal{C}}(\mathcal{M}, \mathcal{N}_{i-1})$
5. $\quad \theta_i \leftarrow \textsc{UpdateTransition}(\theta_{i-1}, \mathcal{C})$
6. $\quad \mathcal{N}_i \leftarrow (N, \theta_i, \alpha)$
7. **until** $\delta_\lambda(\mathcal{M}, \mathcal{N}_i) > \delta_\lambda(\mathcal{M}, \mathcal{N}_{i-1})$ or $i \geq h$
8. **return** $\mathcal{N}_{i-1}$

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Intuitive Idea

UpdateTransition assigns greater probability to transitions that are most representative of the behavior of M

Two update heuristics

- **Averaged Marginal (AM)**: given N_k we construct N_{k+1} by averaging the marginal of certain "coupling variables" obtained by optimizing the number of occurrences of the edges that are most likely to be seen in M.

- **Averaged Expectations (AE)**: similar to the above, but now the N_{k+1} looks only the expectation of the number of occurrences of the edges likely to be found in M.

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UpdateTransition in polynomial time for both heuristics!

Case	$\|M\|$	k	$\lambda = 1$				$\lambda = 0.8$			
			δ_λ-init	δ_λ-final	#	time	δ_λ-init	δ_λ-final	#	time
IPv4 (AM)	23	5	0.775	0.054	3	4.8	0.576	0.025	3	4.8
	53	5	0.856	0.062	3	25.7	0.667	0.029	3	25.9
	103	5	0.923	0.067	3	116.3	0.734	0.035	3	116.5
	53	6	0.757	0.030	3	39.4	0.544	0.011	3	39.4
	103	6	0.837	0.032	3	183.7	0.624	0.017	3	182.7
	203	6	–	–	–	TO	–	–	–	TO
IPv4 (AE)	23	5	0.775	0.109	2	2.7	0.576	0.049	3	4.2
	53	5	0.856	0.110	2	14.2	0.667	0.049	3	21.8
	103	5	0.923	0.110	2	67.1	0.734	0.049	3	100.4
	53	6	0.757	0.072	2	21.8	0.544	0.019	3	33.0
	103	6	0.837	0.072	2	105.9	0.624	0.019	3	159.5
	203	6	–	–	–	TO	–	–	–	TO
DrkW (AM)	39	7	0.565	0.466	14	259.3	0.432	0.323	14	252.8
	49	7	0.568	0.460	14	453.7	0.433	0.322	14	420.5
	59	8	0.646	–	–	TO	0.423	–	–	TO
DrkW (AE)	39	7	0.565	0.435	11	156.6	0.432	0.321	2	28.6
	49	7	0.568	0.434	10	247.7	0.433	0.316	2	46.2
	59	8	0.646	0.435	10	588.9	0.423	0.309	2	115.7

Table 1. Comparison of the performance of EM algorithm on the IPv4 zeroconf protocol and the classic Drunkard's Walk w.r.t. the heuristics AM and AE.

What we have seen

Theoretical

Metric-based state space reduction for MCs

1. Closest Bounded Approximant (CBA)
   encoded as a bilinear program
2. Bounded Approximant (BA)
   PSPACE & NP-hard for all $\lambda \in (0,1]$
3. Significant Bounded Approximant (SBA)
   NP-complete for $\lambda = 1$

Practical

We proposed an EM-like method to obtain a sub-optimal approximants

Future Work

- Is BA-λ SUM-OF-SQUARE-ROOTS-hard?
  (conjecture: for $\lambda<1$, BA-λ is in NP)

- Can we obtain a real/better EM-heuristics?

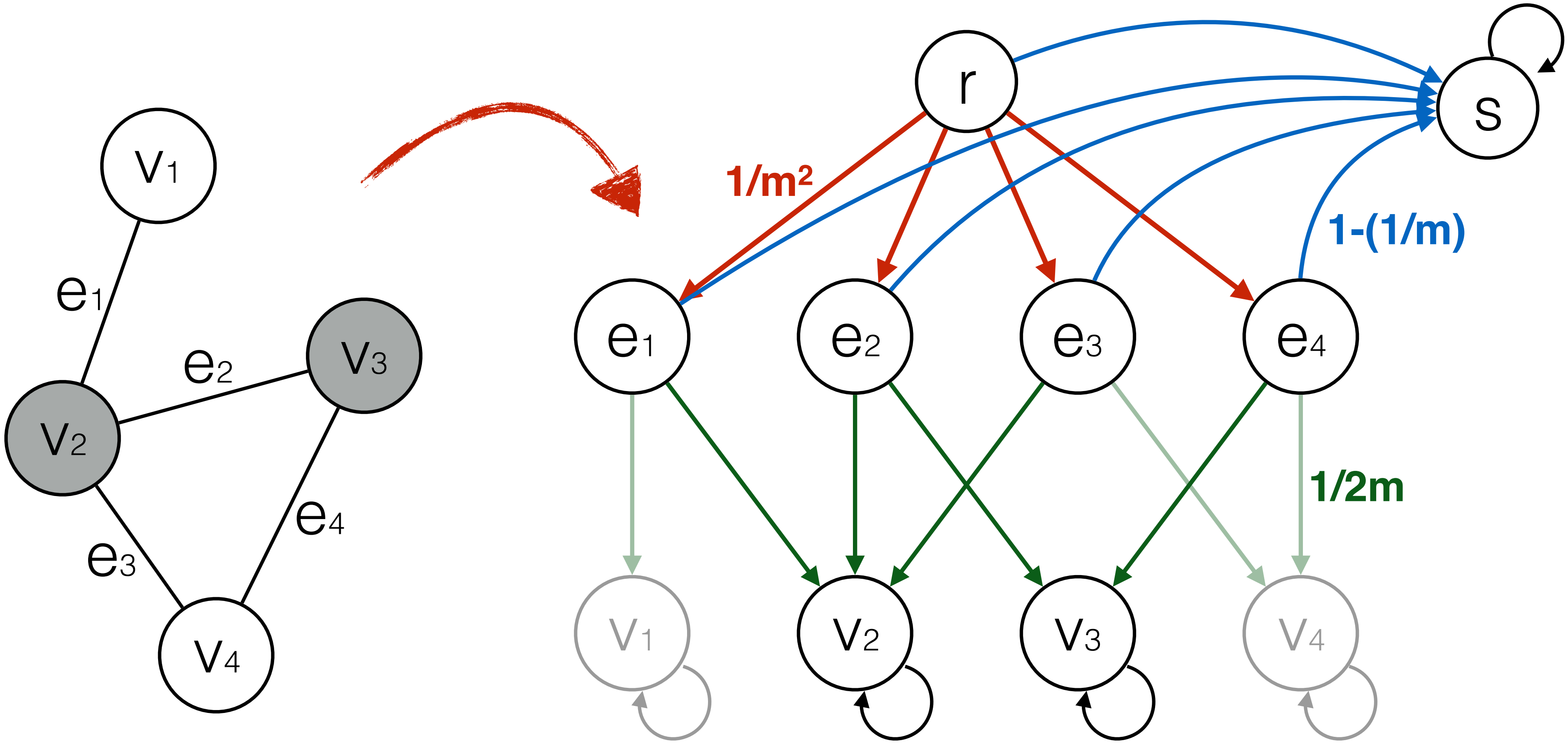
- What about different models/distances?

- What about different constraints?
  —beyond minimization!

Thank you
for your attention

Appendix

BA-λ is NP-hard



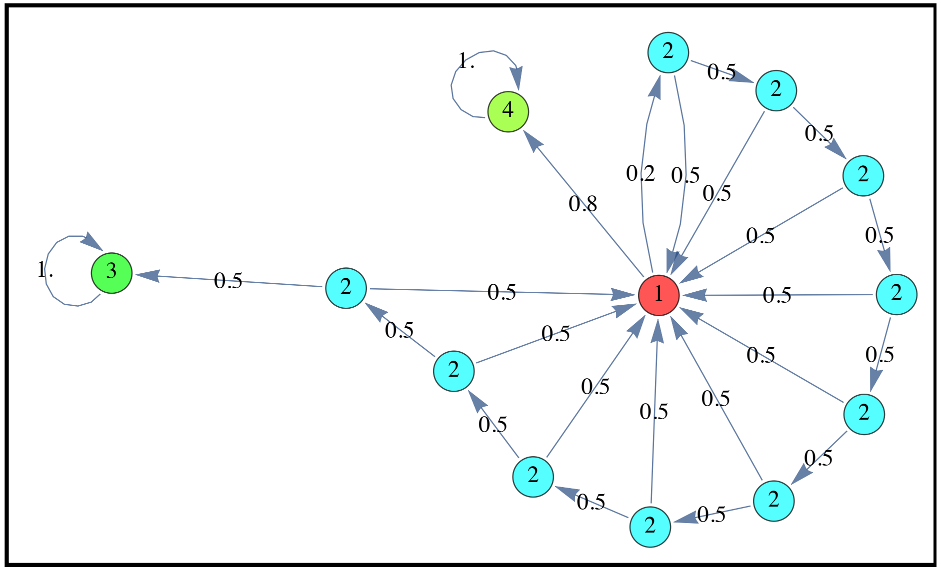
$\langle G,h\rangle \in VERTEX\ COVER$ iff $\langle M_G,\ m+h+2,\ \lambda^2/2m^2\rangle \in BA\text{-}\lambda$

EM-like algorithm (experimental results)

IPv4 Zero Conf Protocol

Averaged Marginal (AM)

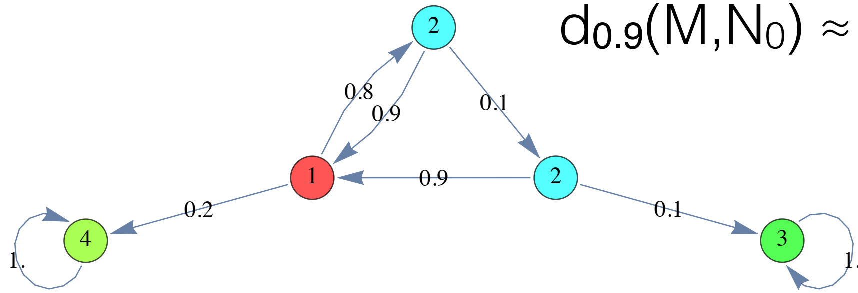
Input model



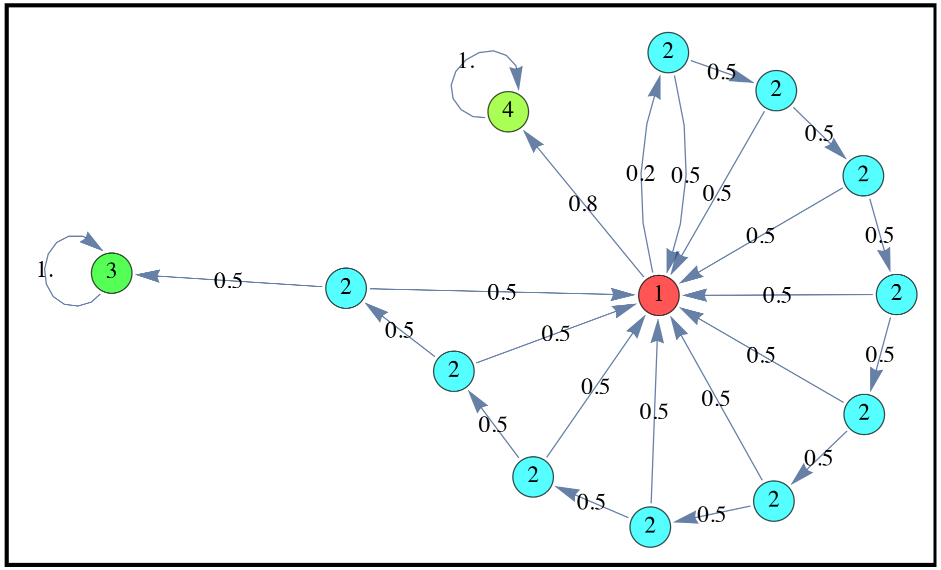
IPv4 Zero Conf Protocol

Averaged Marginal (AM)

$d_{0.9}(M,N_0) \approx 0.67$



Input model



IPv4 Zero Conf Protocol

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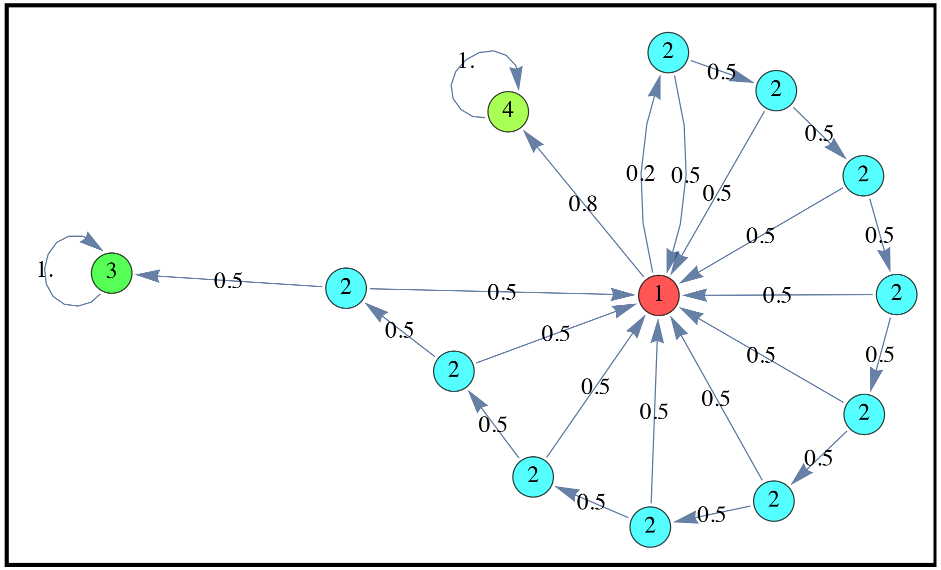
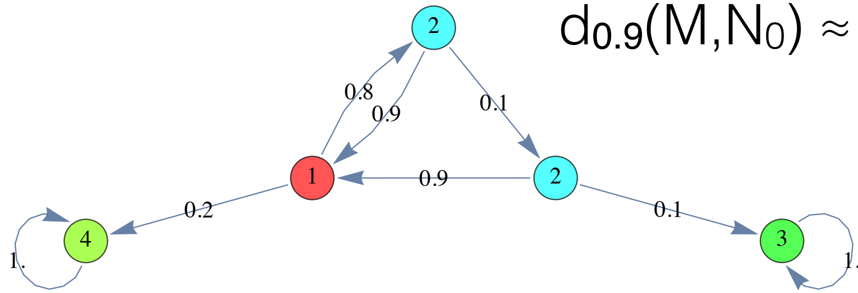
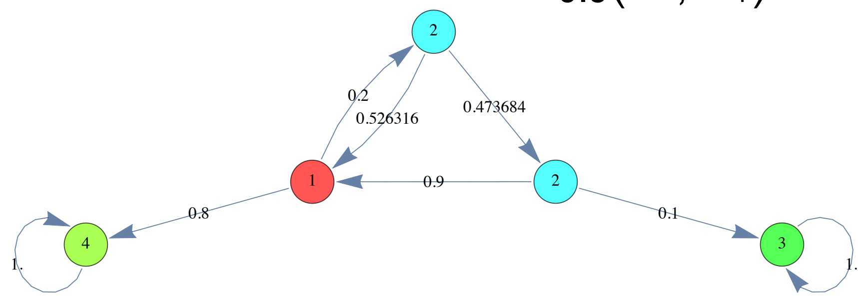


$d_{0.9}(M,N_0) \approx 0.67$



$d_{0.9}(M,N_1) \approx 0.043$



IPv4 Zero Conf Protocol

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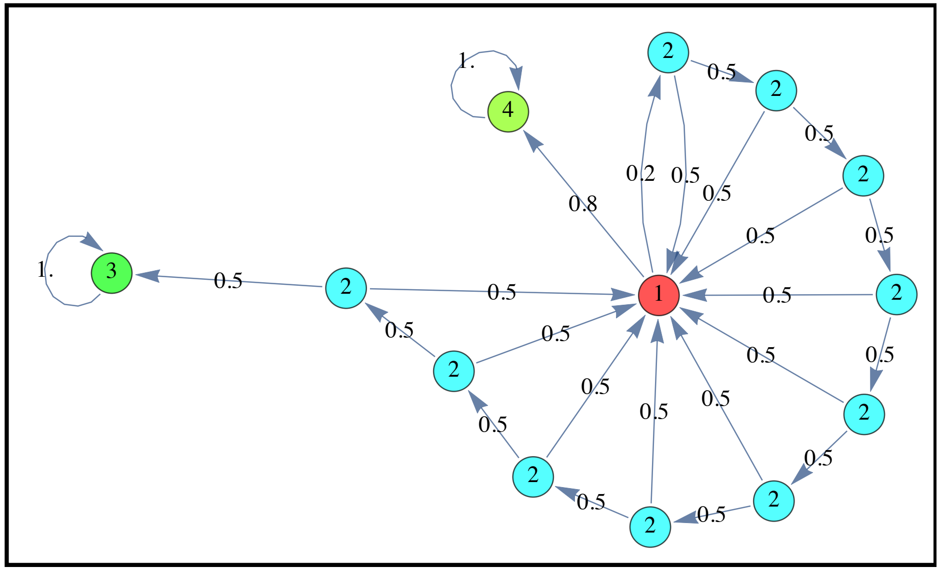
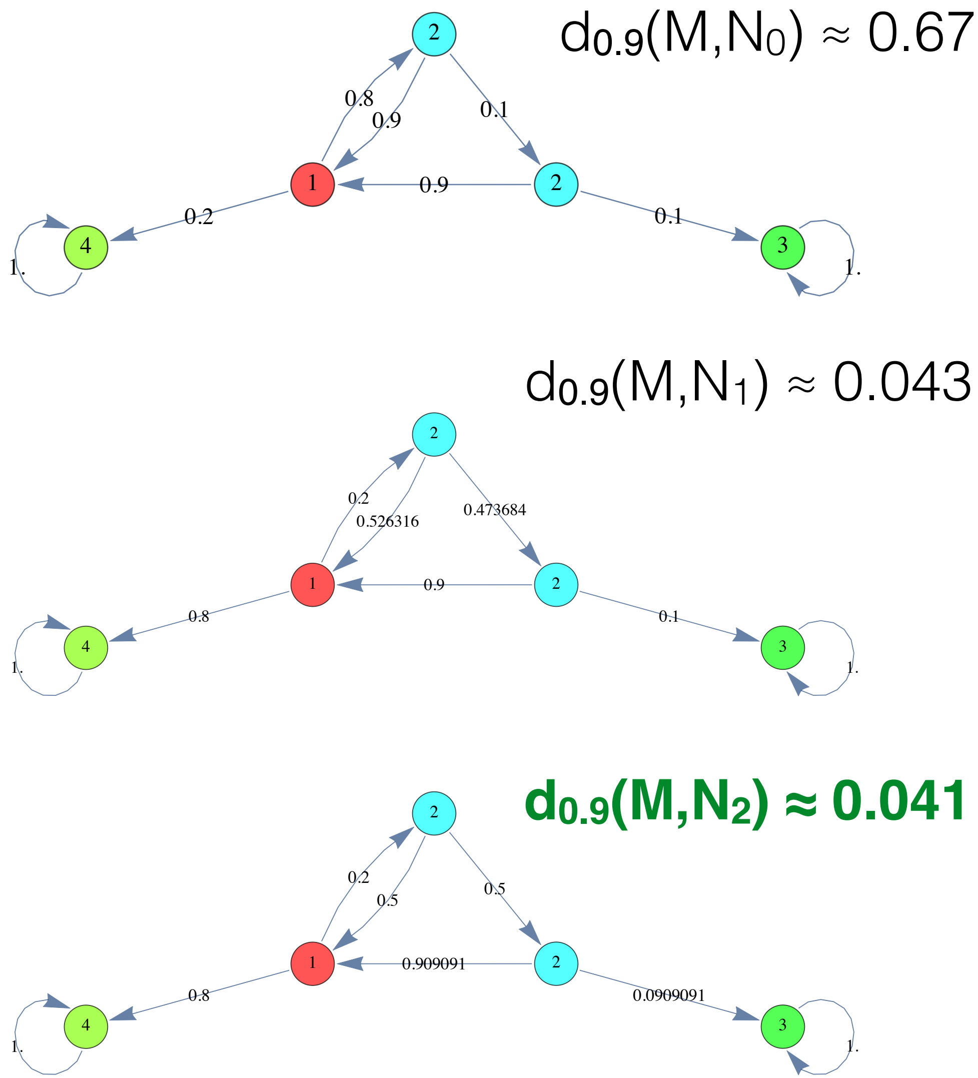
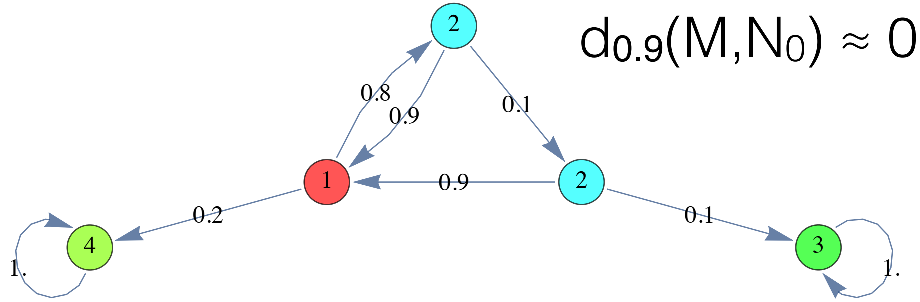




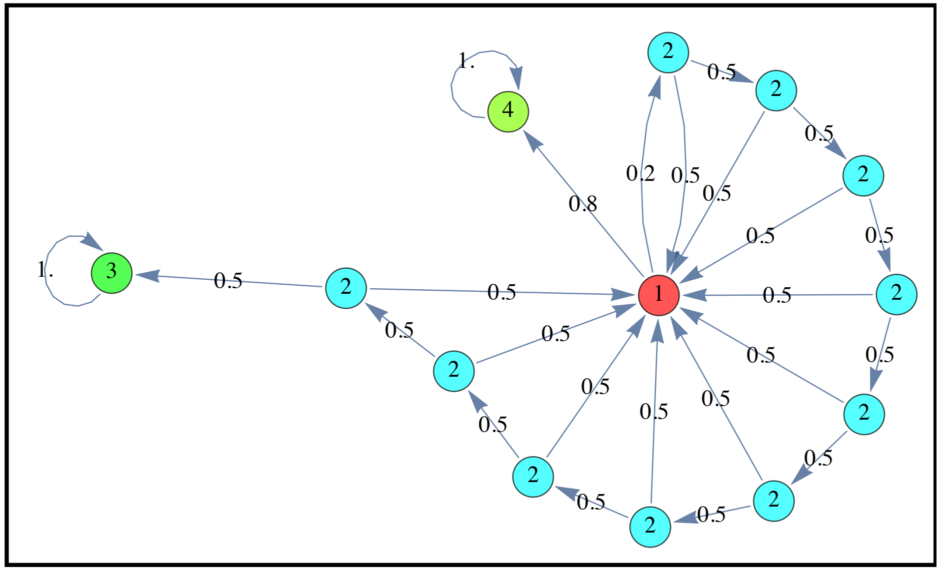
IPv4 Zero Conf Protocol

Averaged Expectations (AE)

$d_{0.9}(M,N_0) \approx 0.67$



Input model



IPv4 Zero Conf Protocol

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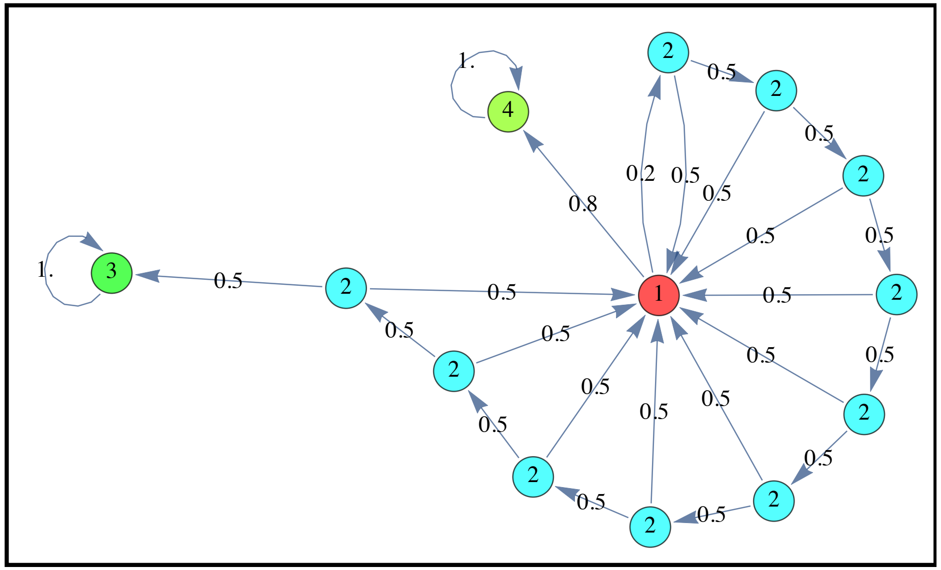

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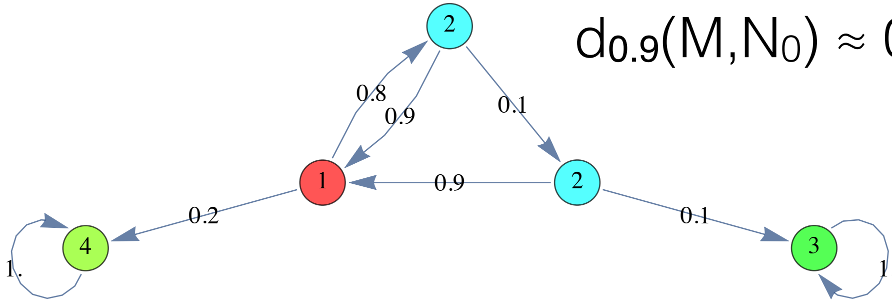
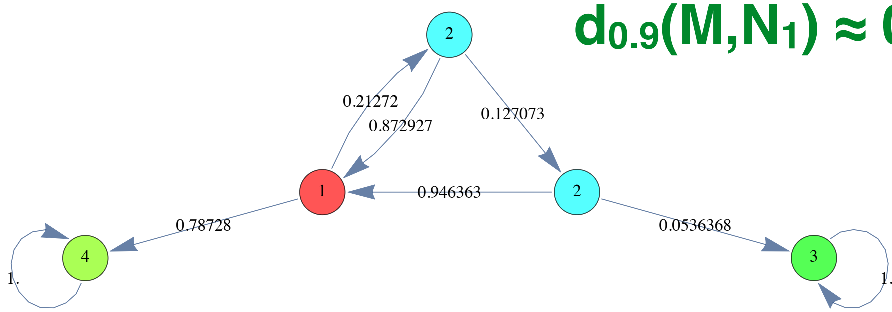


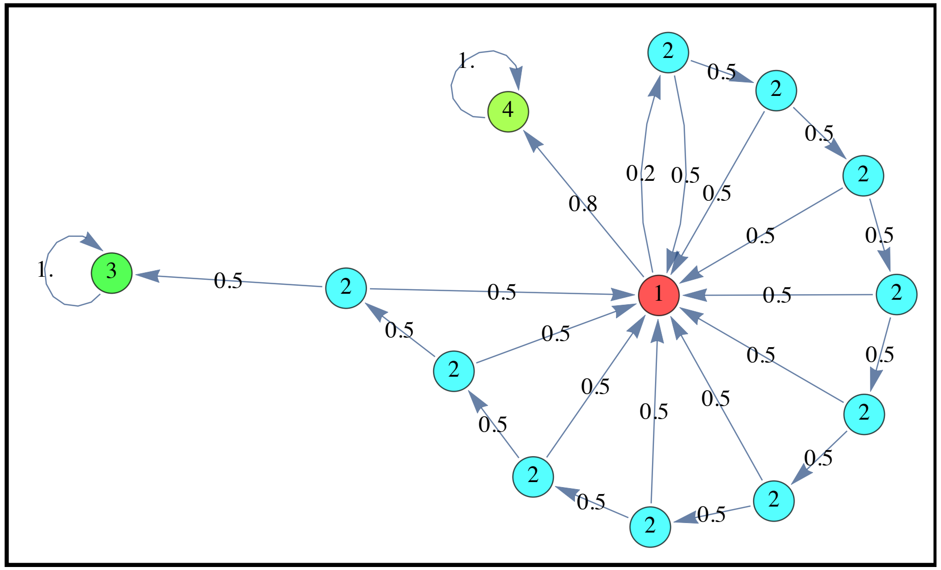
$d_{0.9}(M,N_1) \approx 0.08$



IPv4 Zero Conf Protocol

Averaged Expectations (AE)

Input model



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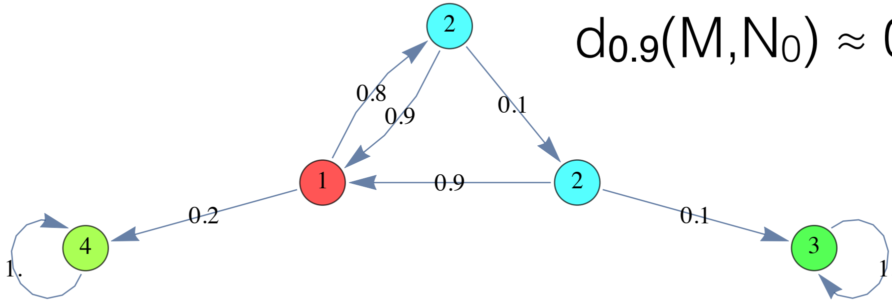

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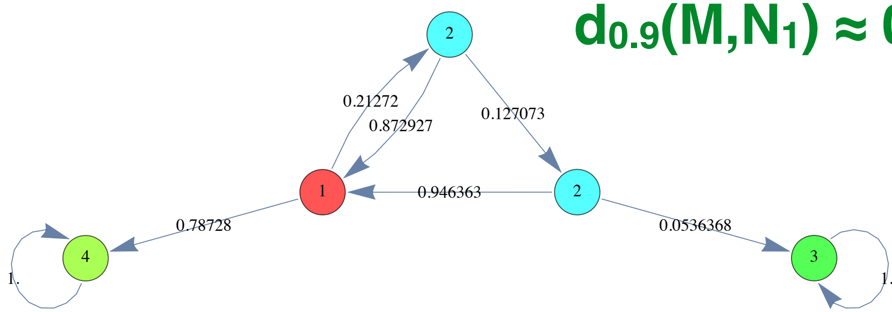
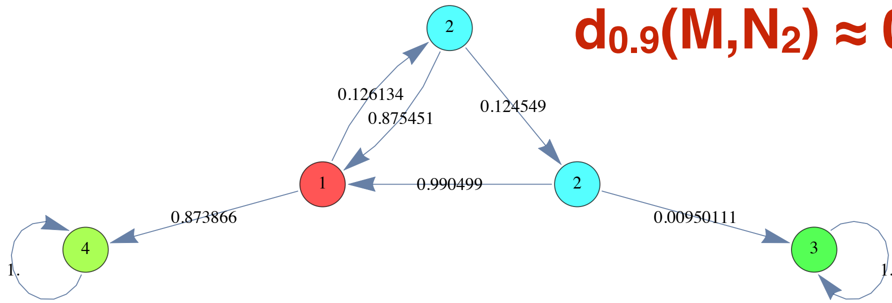


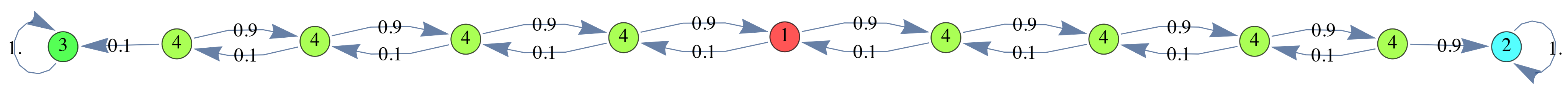
$d_{0.9}(M,N_2) \approx 0.11$



Drunkard's Walk

Averaged Marginal (AM)

Input model



Drunkard's Walk

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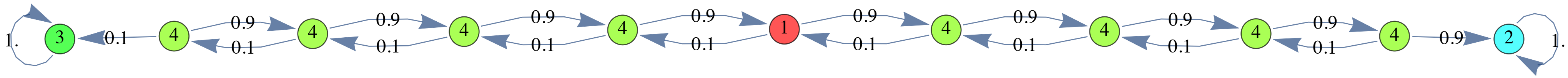
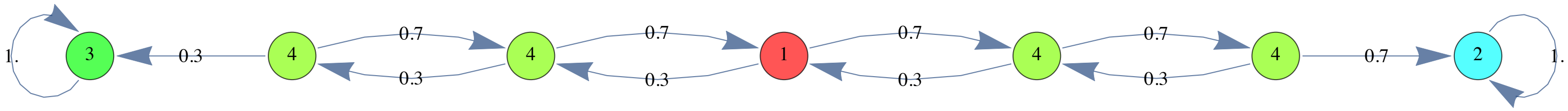
Input model



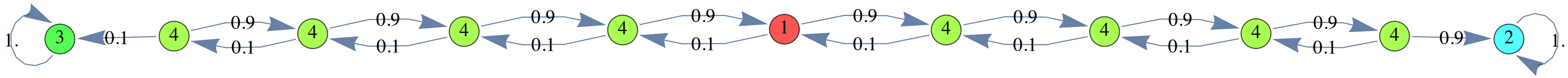


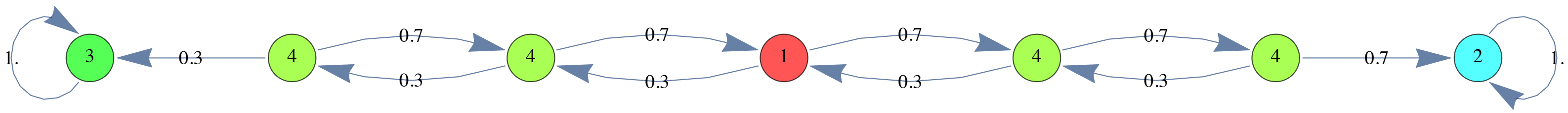
$d_{0.9}(M,N_0) \approx 0.64$

Drunkard's Walk

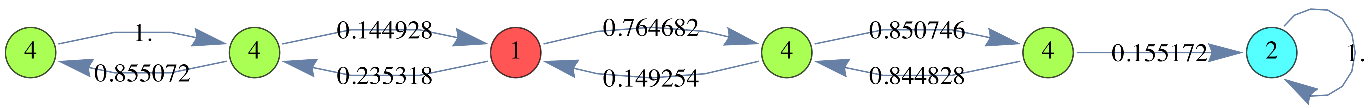
Averaged Marginal (AM)

Input model





$d_{0.9}(M,N_0) \approx 0.64$

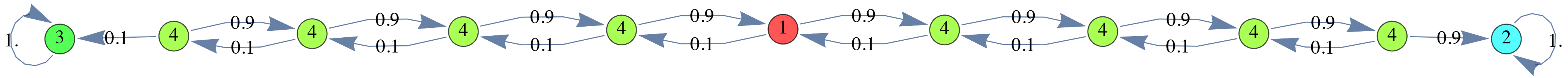


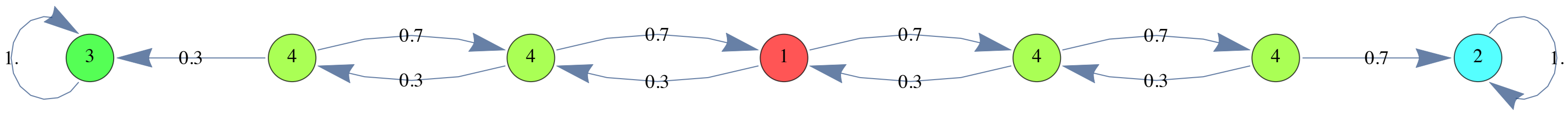
$\mathbf{d_{0.9}(M,N_1) \approx 0.56}$

Drunkard's Walk

Averaged Marginal (AM)

Input model





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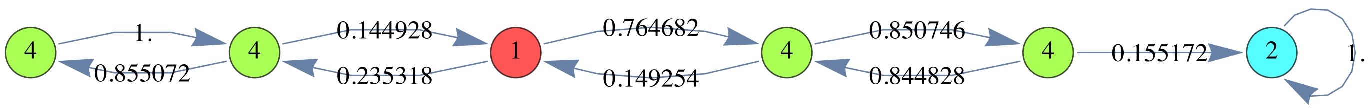
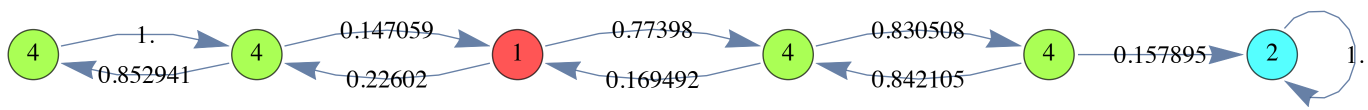


$\mathbf{d_{0.9}(M,N_1) \approx 0.56}$

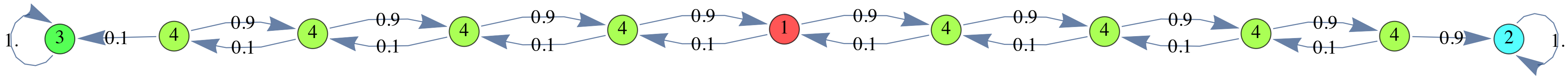


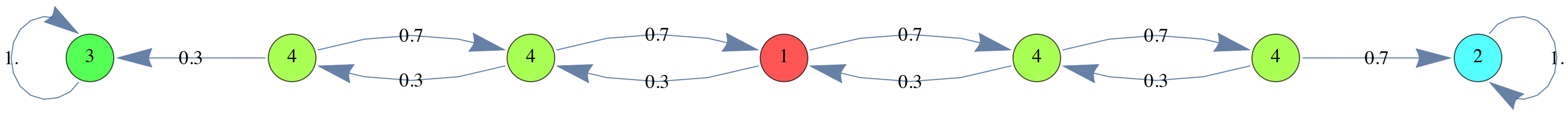
$\mathbf{d_{0.9}(M,N_2) \approx 0.567}$

Drunkard's Walk

Averaged Expectations (AE)

Input model



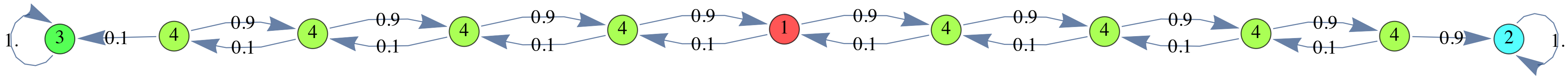


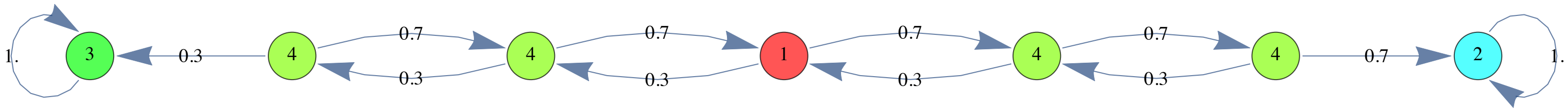
$\delta_{0.9}(M,N_0) \approx 0.64$

Drunkard's Walk

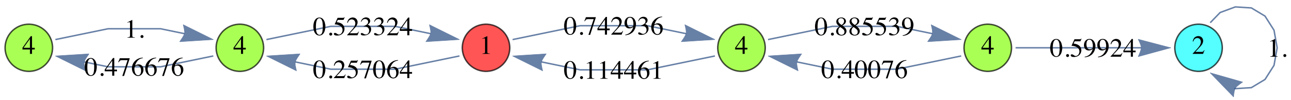
Averaged Expectations (AE)

Input model





$\delta_{0.9}(M, N_0) \approx 0.64$



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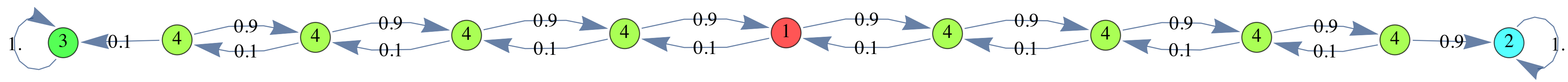
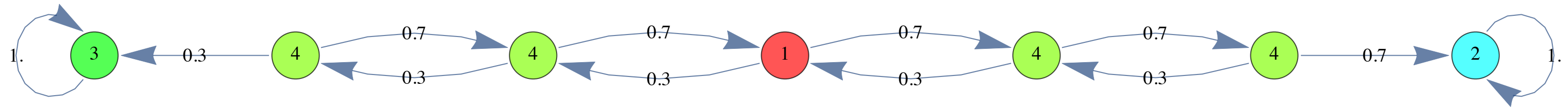
Drunkard's Walk

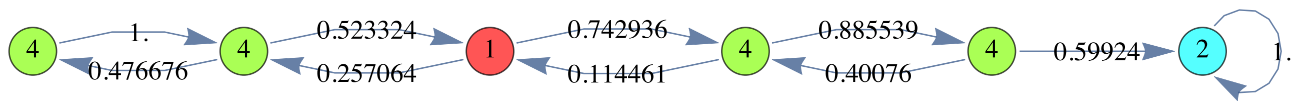
Averaged Expectations (AE)

Input model





$\delta_{0.9}(M,N_0) \approx 0.64$



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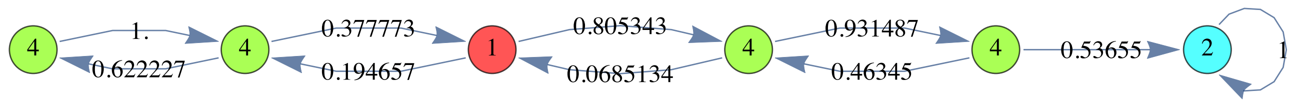
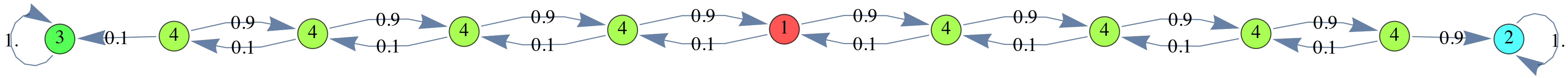


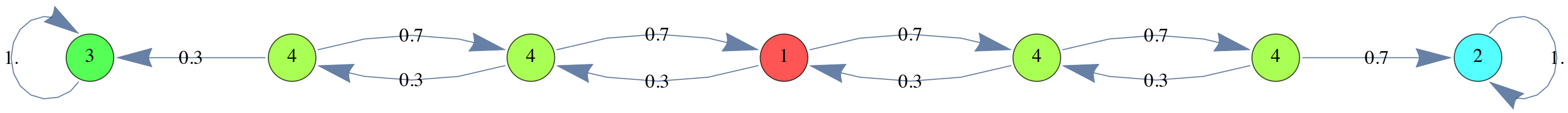
$\delta_{0.9}(M,N_2) \approx 0.543$

Drunkard's Walk

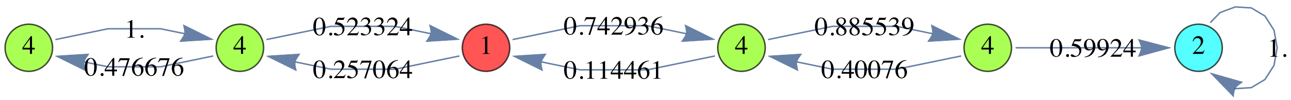
Averaged Expectations (AE)

Input model





$\delta_{0.9}(M,N_0) \approx 0.64$



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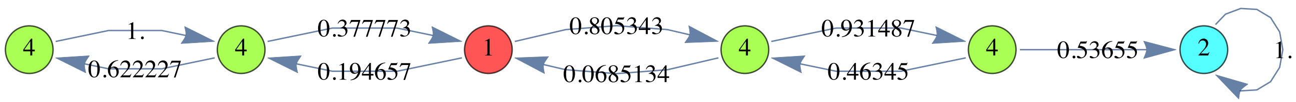
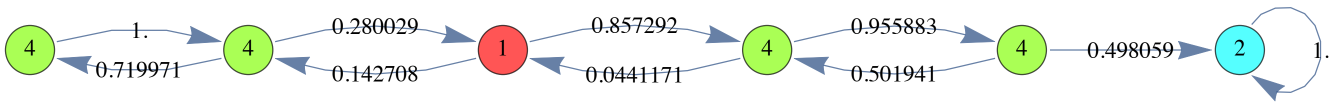


$\delta_{0.9}(M,N_2) \approx 0.543$



$\delta_{0.9}(M,N_3) \approx 0.540$