Quantitative Algebraic Effects
*(Sum & Tensor)**

Giorgio Bacci
Dept. Computer Science, Aalborg University, DK

MFPS 2022
12th July 2022, Ithaca+Paris+online

(*) joint work with R. Mardare, P. Panangaden and G. Plotkin

Historical Perspective

- **Moggi'88:** *How to incorporate effects into denotational semantics?* -**Monads** as notions of computations
- **Plotkin & Power'01:** *(most of the) Monads are given by operations and equations* -**Algebraic Effects**
- **Hyland, Plotkin, Power'06:** *sum and tensor of theories* -**Combining Algebraic Effects**

Historical Perspective

- Probabilistic programming languages (eg. Church, Anglican, Pyro, WebPPL, etc.) use *probabilistic nondeterminism as built-in effect*

- **Giacalone, Jou, Smolka'90:** (pseudo)metrics for reasoning about "how close are two programs"

- **Mardare, Panangaden, Plotkin (LICS'16)**
  - *Algebraic reasoning in a metric setting:* operations & *quantitative equations* (+ complete deduction system)
  - **Quantitative Algebraic Effects** (monads on **metric spaces**)

- **Bacci, Mardare, Panangaden, Plotkin (LICS'18, CALCO'21)**
  sum & tensor of theories -**Combining Quantitative Effects**

The Basic Idea

a quantitative analogue of equational reasoning

not an equivalence!

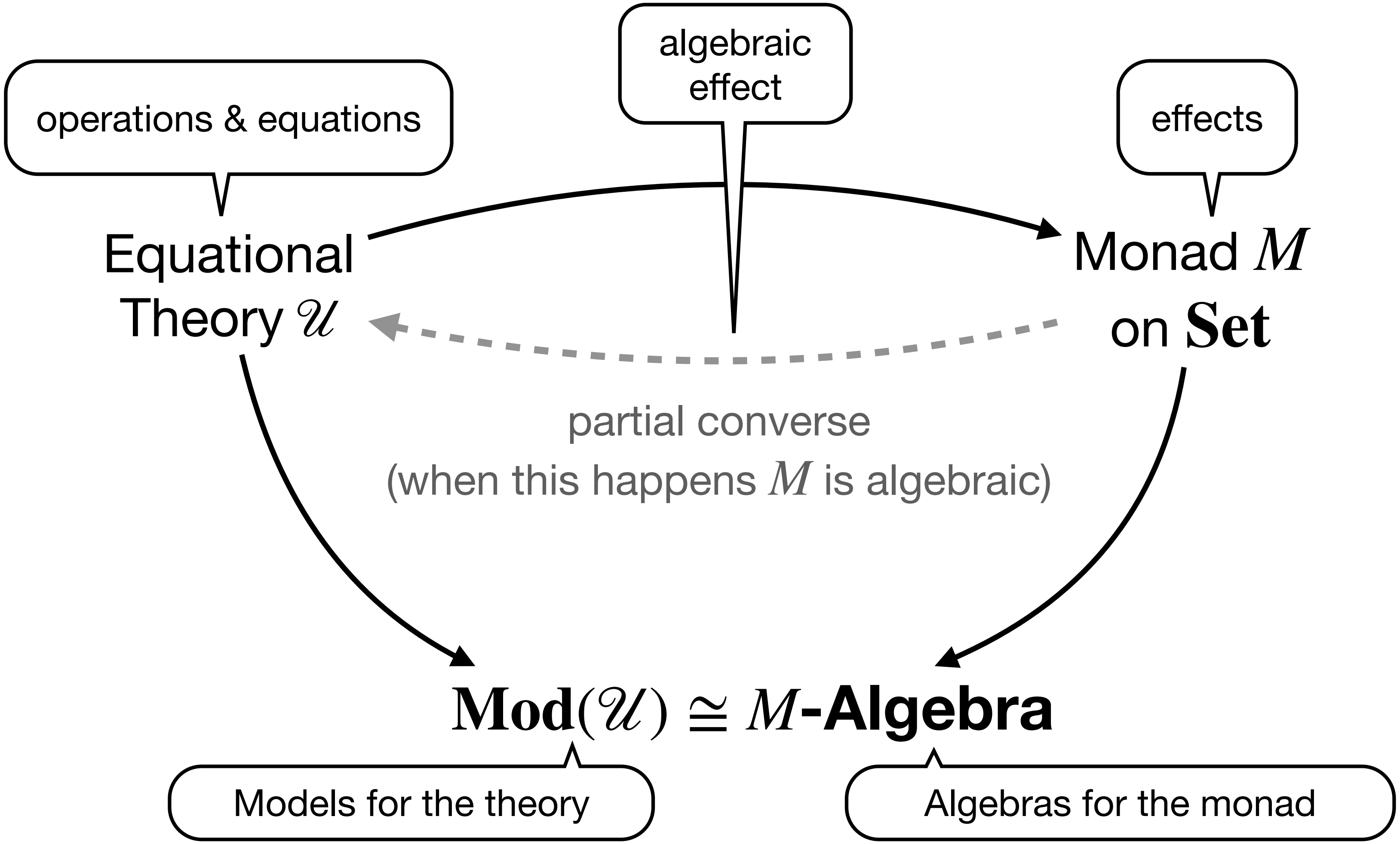
quantitative equation

$$s =_{\varepsilon} t$$

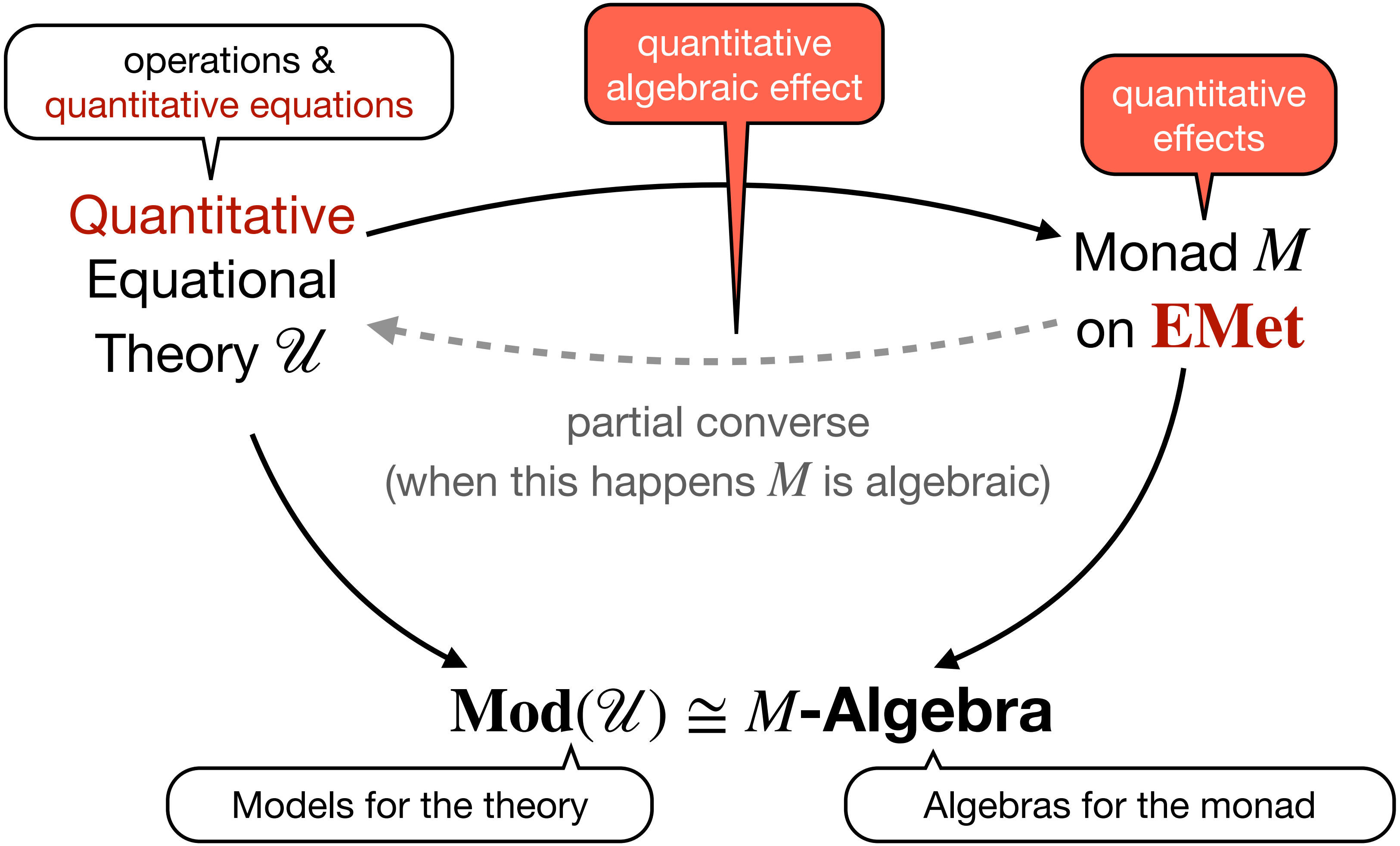
"s is within ε of t"

completeness results, universality of free algebras,
Birkhoff-like variety theorem, and **algebraic effects**...

The Standard Picture



The *Quantitative* Picture



Quantitative Equational Theories

- **Signature of operations**: $\Sigma = \{(f_0 \colon n_0), \ldots, (f_k \colon n_k), \ldots\}$
- **Terms**: $s, t ::= x \mid f(t_1, \ldots, t_n)$ ($\mathbb{T}_\Sigma X$ set of terms over X)
- **Quantitative equations**: $s =_\varepsilon t$ where $\varepsilon \in \mathbb{Q}_{\geq 0}$
- **Quantitative inferences**: $\{s_1 =_{\varepsilon_1} t_1, \ldots, s_n =_{\varepsilon_n} t_n\} \vdash s =_\varepsilon t$
- **Quantitative equational theories**: sets $\mathscr{U}$ of quantitative inferences satisfying certain closure properties telling us what can be deduced...

Closure Properties

(Refl) $\emptyset \vdash t =_0 t \in \mathcal{U}$

(Symm) $\{s =_\varepsilon t\} \vdash t =_\varepsilon s \in \mathcal{U}$

(Triang) $\{s =_\varepsilon t, t =_\delta u\} \vdash s =_{\varepsilon+\delta} u \in \mathcal{U}$

(Max) $\{s =_\varepsilon t\} \vdash s =_{\varepsilon+\delta} t \in \mathcal{U}$ (for $\delta > 0$)

(NExp) for $f \colon n \in \Sigma$,
$\{s_1 =_\varepsilon t_1, \ldots, s_n =_\varepsilon t_n\} \vdash f(s_1, \ldots, s_n) =_\varepsilon f(t_1, \ldots, t_n) \in \mathcal{U}$

typically, we describe a quantitative theory using a set of **axioms**

(Cont) $\{\Gamma \vdash s =_\delta t \mid \delta > \varepsilon\} \subseteq \mathcal{U}$ implies $\Gamma \vdash s =_\varepsilon t \in \mathcal{U}$

(Subst) $\Gamma \vdash s =_\varepsilon t \in \mathcal{U}$ implies $\Gamma[u/x] \vdash s[u/x] =_\varepsilon t[u/x] \in \mathcal{U}$

(Cut) $\Gamma \vdash \Theta \subseteq \mathcal{U}$ and $\Theta \vdash s =_\varepsilon t \in \mathcal{U}$ implies $\Gamma \vdash s =_\varepsilon t \in \mathcal{U}$

(Assum) $s =_\varepsilon t \in \Gamma$ implies $\Gamma \vdash s =_\varepsilon t \in \mathcal{U}$

Quantitative Algebras

$$\mathscr{A} = \left(A, d_A, \{f_{\mathscr{A}} \colon A^n \to A \mid f \colon n \in \Sigma\}\right)$$

- (A, d_A) is an *extended* metric space (carrier)
- $f_{\mathscr{A}} \colon A^n \to A$ interpretations of operations are non-expansive

$$\max_i d_A(a_i, b_i) \geq d(f_{\mathscr{A}}(a_1, \ldots, a_n), f_{\mathscr{A}}(b_1, \ldots, b_n))$$

Morphisms: $h \colon \mathscr{A} \to \mathscr{B}$

- Σ-homomorphisms
  $h(f_{\mathscr{A}}(a_1, \ldots, a_n)) = f_{\mathscr{B}}(h(a_1), \ldots, h(a_n))$ for all $f \colon n \in \Sigma$
- non-expansive $d_A(a, a') \geq d_B(h(a), h(a'))$

Models of a theory $\mathscr{U}$

A quantitative algebra $\mathscr{A}$ *is a model for a quantitative theory* $\mathscr{U}$
if it satisfies all quantitative inferences in it

Satisfiability

$$\mathscr{A} \models \left(\{ t_i =_{\varepsilon_i} s_i \mid i = 1,\ldots,n \} \vdash t =_{\varepsilon} s \right)$$

iff

for any Σ-homomorphism $\iota \colon \mathbb{T}_\Sigma X \to A$

$d_A(\iota(t_i), \iota(s_i)) \leq \varepsilon_i$, for $i = 1,\ldots,n$ implies $d_A(\iota(t), \iota(s)) \leq \varepsilon$

Barycentric Algebras

(Stone 1949)

Signature: $\{ +_e : 2 \mid e \in [0,1] \}$

convex sum

(B1) $\vdash s +_1 t = s$

(B2) $\vdash t +_e t = t$

(SC) $\vdash s +_e t = t +_{1-e} s$

(SA) $\vdash (s +_e t) +_d u = s +_{ed} (t +_{\frac{(1-e)d}{1-ed}} u)$, **for** $e, d \in (0,1)$

Barycentric Algebras

Mardare, Panangaden, Plotkin (LICS'16)

Signature: $\{ +_e : 2 \mid e \in [0,1] \}$

convex sum

(B1) $\vdash s +_1 t =_0 s$

(B2) $\vdash t +_e t =_0 t$

(SC) $\vdash s +_e t =_0 t +_{1-e} s$

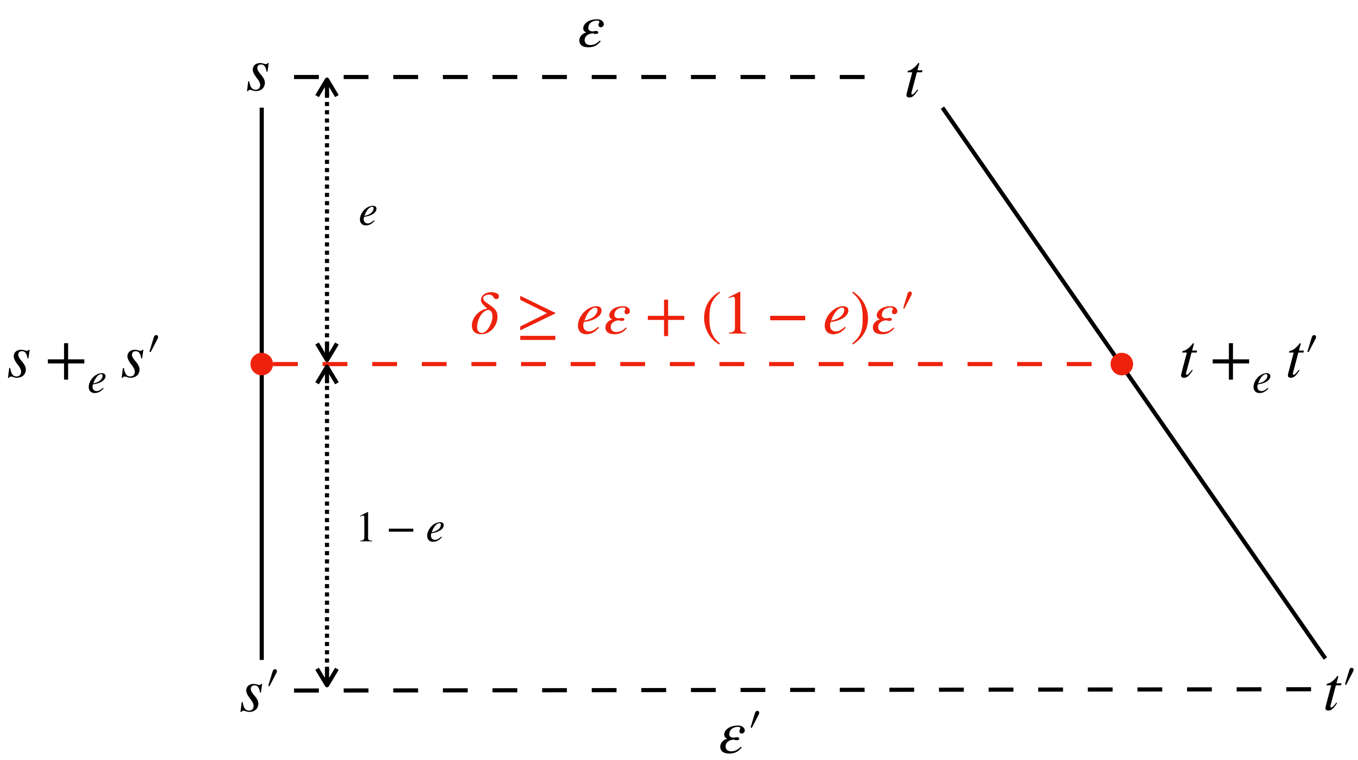
(SA) $\vdash (s +_e t) +_d u =_0 s +_{ed} (t +_{\frac{(1-e)d}{1-ed}} u)$, **for** $e, d \in (0,1)$

(IB) $\{s =_\varepsilon t, s' =_{\varepsilon'} t'\} \vdash s +_e s' =_\delta t +_e t'$,

where $\delta \geq e\varepsilon + (1-e)\varepsilon'$

A geometric intuition

(IB) $\{s =_\varepsilon t, s' =_{\varepsilon'} t'\} \vdash s +_e s' =_\delta t +_e t'$, **where** $\delta \geq e\varepsilon + (1-e)\varepsilon'$



..some of models

Unit interval with Euclidian distance and convex combinators

$$([0,1], d_{[0,1]}) \qquad (\,+_e\,)^{[0,1]}(a,b) = ea + (1-e)b$$

Finitely supported distributions with **Kantorovich distance**

$$(\mathscr{D}(M), \mathscr{K}(d_M)) \qquad (\,+_e\,)^{\mathscr{D}}(\mu,\nu) = e\mu + (1-e)\nu$$

Borel probability measures with **Kantorovich distance**

$$(\Delta(M), \mathscr{K}(d_M)) \qquad (\,+_e\,)^{\Delta}(\mu,\nu) = e\mu + (1-e)\nu$$

Kantorovich Distance

Let $\mu, \nu \in \mathscr{D}(M)$ probability distributions on (M, d_M)

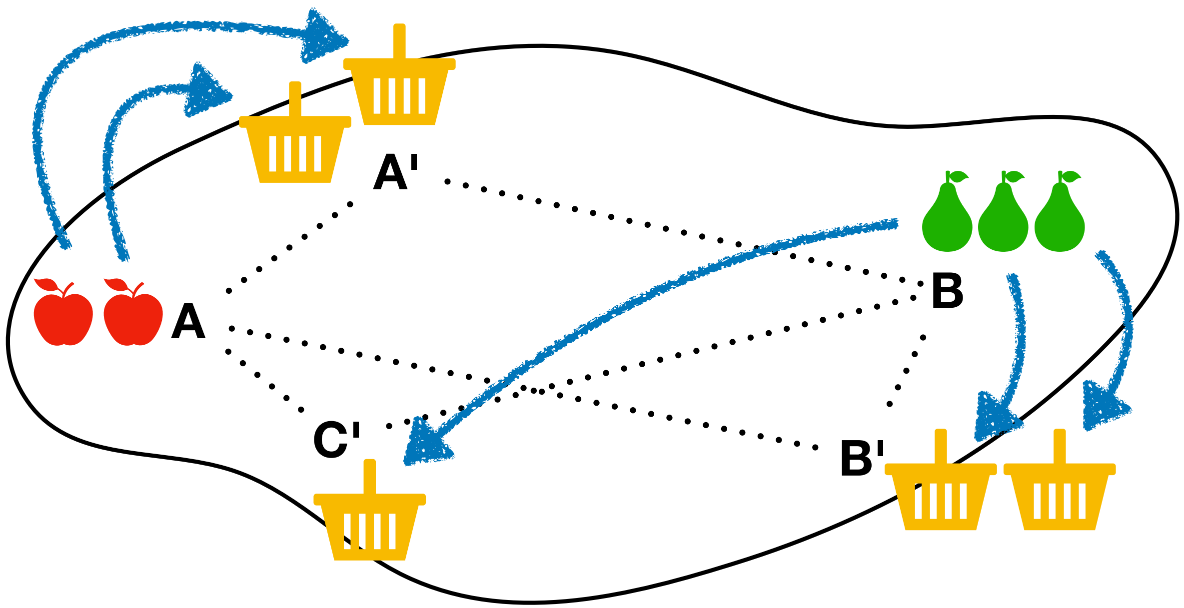
$$\mathscr{K}(d_M)(\mu, \nu) = \min \left\{ \sum_{m,n} \omega(m,n) d(m,n) \mid \omega \in \mathscr{C}(\mu, \nu) \right\}$$

distribution of "fruit"

distribution of "baskets"

amount of fruit moved from m to n

coupling



Main General Results

- **Completeness theorem:**
  $\forall \mathscr{A} \in \mathbf{Mod}(\mathscr{U}) \,.\, \mathscr{A} \models (\Gamma \vdash s =_\varepsilon t)$ iff $(\Gamma \vdash s =_\varepsilon t) \in \mathscr{U}$

- **Birkhoff-like (quasi-)variety theorem:**
  A collection of quantitative algebras is a variety iff it closed under sub-algebras, direct products, <u>reflexive</u> homomorphic images.

- **Free-models:**

$$\textbf{EMet} \qquad\qquad\qquad \mathbf{Mod}(\mathscr{U})$$

$$\begin{tikzcd} M \arrow[r, "\eta_M"] \arrow[dr, "\alpha"'] & T_{\mathscr{U}}M \arrow[d, "h"] \\ & A \end{tikzcd} \qquad\qquad \begin{tikzcd} \mathbf{T}_{\mathscr{U}}M \arrow[d, dashed, "h"] \\ \mathscr{A} \end{tikzcd}$$

Free model of $\mathscr{U}$ over M

Free Models of $\mathscr{U}$

- Given $\mathscr{U}$ quantitative theory for the signature Σ
- and (M, d_M) an *extended* metric space,
- we define $\mathscr{U}_M$ as the quantitative theory for the signature $\Sigma + M$ with $\mathscr{U}$ and $\{ \vdash m =_\varepsilon n \mid d_M(m,n) \leq \varepsilon \}$ as set of axioms

An extended (pseudo)metric on $\mathbb{T}_\Sigma M$

$$d_{\mathscr{U}}(s,t) = \inf\{\varepsilon \mid \emptyset \vdash t =_\varepsilon s \in \mathscr{U}_M\}$$

Free model of $\mathscr{U}$ over (M, d_M)

$$\mathbf{T}_{\mathscr{U}} M = \big((\mathbb{T}_\Sigma M)/_{=_0}, d_{\mathscr{U}}, \{ f_{\mathscr{U}} \mid f \colon n \in \Sigma \} \big)$$

Algebraic Effects on $\mathbf{EMet}$

$$\begin{tikzcd} \mathbf{Mod}(\mathscr{U}) \arrow[d, bend left] \arrow[d, phantom, "\dashv"] \\ \mathbf{EMet} \arrow[u, bend left] \arrow[d, Rightarrow] \\ T_{\mathscr{U}} \end{tikzcd}$$

Models of $\mathscr{U}$

Free Model functor
(maps a extended metric space to the free model of $\mathscr{U}$)

Forgetful functor
(maps a quantitative algebra to its carrier)

This is how we get algebraic **quantitative effects**!

Monad induced by $\mathscr{U}$ on $\mathbf{EMet}$

Quantitative Algebraic Effects

(Examples)

Ex1: Barycentric Algebras

Mardare, Panangaden, Plotkin (LICS'16)

Signature: $\{\, +_e : 2 \mid e \in [0,1] \,\}$

(B1) $\vdash s +_1 t =_0 s$

(B2) $\vdash t +_e t =_0 t$

(SC) $\vdash s +_e t =_0 t +_{1-e} s$

(SA) $\vdash (s +_e t) +_d u =_0 s +_{ed} (t +_{\frac{(1-e)d}{1-ed}} u)$, **for** $e, d \in (0,1)$

(IB) $\{s =_\varepsilon t, s' =_{\varepsilon'} t'\} \vdash s +_e s' =_\delta t +_e t'$,

where $\delta \geq e\varepsilon + (1-e)\varepsilon'$

$$\begin{tikzcd} \mathbf{Mod}(\mathscr{B}) \arrow[r, bend right] \arrow[r, phantom, "\perp"] & \mathbf{EMet} \arrow[l, bend right] \end{tikzcd} \Rightarrow T_{\mathscr{B}} \cong \mathscr{D}$$

Finitely supported distributions with **Kantorovic distance**

Ex2: Quantitative Semilattices

Mardare, Panangaden, Plotkin (LICS'16)

Signature: $\{\mathbf{0} : 0, \oplus : 2\}$

bottom

join

(S0) $\vdash 0 \oplus t =_0 t$

(S1) $\vdash t \oplus t =_0 t$

(S2) $\vdash s \oplus t =_0 t \oplus s$

(S3) $\vdash (s \oplus t) \oplus u =_0 s \oplus (t \oplus u)$

(S4) $\{s =_\varepsilon t, s' =_{\varepsilon'} t'\} \vdash s \oplus s' =_\delta t \oplus t'$, **where** $\delta \geq \max\{\varepsilon, \varepsilon'\}$

$$\begin{tikzcd} \mathbf{Mod}(\mathcal{S}) \arrow[r, bend right] \arrow[r, phantom, "\perp"] & \mathbf{EMet} \arrow[l, bend right] \end{tikzcd} \Rightarrow T_{\mathcal{S}} \cong \mathscr{H}$$

Finite powerset monad with **Hausdorff distance**

Ex3: Quantitative Exceptions

Bacci, Mardare, Panangaden, Plotkin (LICS'18)

Signature: $\{e : 0 \mid e \in E\}$

A metric space (E, d_E) of exceptions

(E0) $\vdash e_1 =_\varepsilon e_2\,,$ **where** $\varepsilon \geq d_E(e_1, e_2)$

$$\begin{tikzcd} \mathbf{Mod}(\mathscr{E}_E) \arrow[r, bend right] \arrow[r, phantom, "\perp"] & \mathbf{EMet} \arrow[l, bend right] \end{tikzcd} \quad \Rightarrow \quad T_{\mathscr{E}_E} \cong (\, - + E)$$

Quantitative Exception Monad

Ex4: Contractive Transitions

Bacci, Mardare, Panangaden, Plotkin (LICS'18)

Signature: $\{ \diamond : 1 \}$

transition step

contractive factor $c \in (0,1)$

(c-Lip) $\{s =_\varepsilon t\} \vdash \diamond(s) =_\delta \diamond(t)\,,$ **where** $\delta \geq c\varepsilon$

$$\begin{tikzcd} \mathbf{Mod}(\mathscr{T}) \arrow[r, bend right] \arrow[r, phantom, "\perp"] & \mathbf{EMet} \arrow[l, bend right] \end{tikzcd} \Rightarrow T_{\mathscr{T}} \cong (c \cdot -)^*$$

free monad on the rescaling functor $(c \cdot -)$

Ex5: Quantitative Reader

Bacci, Mardare, Panangaden, Plotkin (CALCO'21)

Signature: $\{\mathbf{r} : |A|\}$

reads from a finite set of input actions $A = \{a_1, \ldots, a_n\}$ and proceeds

(Idem) $\vdash x =_0 \mathbf{r}(x, \ldots, x)$

(Diag) $\vdash \mathbf{r}(x_{1,1}, \ldots, x_{n,n}) =_0 \mathbf{r}(\mathbf{r}(x_{1,1}, \ldots, x_{1,n}), \ldots, \mathbf{r}(x_{n,1}, \ldots, x_{n,n}))$

Monad in $\mathbf{EMet}$ only for *discrete* spaces of inputs!

$$\begin{tikzcd} \mathbf{Mod}(\mathscr{R}) \arrow[r, bend right] \arrow[r, phantom, "\perp"] & \mathbf{EMet} \arrow[l, bend right] \end{tikzcd} \Rightarrow T_{\mathscr{R}} \cong (\,-\,)^{\underline{A}}$$

Reader monad for the discrete space $\underline{A}$

Ex6: Quantitative Writer

Bacci, Mardare, Panangaden, Plotkin (CALCO'21)

metric space

Let $(\Lambda, \star, 0)$ be a monoid with non-expansive multiplication

Signature: $\{\mathbf{w}_a : 1 \mid a \in \Lambda\}$

writes the output symbol a and proceeds

(Zero) $\vdash x =_0 \mathbf{w}_0(x)$

(Mult) $\vdash \mathbf{w}_a(\mathbf{w}_b(x)) =_0 \mathbf{w}_{a\star b}(x)$

(Diff) $\{x =_\varepsilon x'\} \vdash \mathbf{w}_a(x) =_\delta \mathbf{w}_b(x')\,,$ **for** $\delta \geq d_\Lambda(a,b) + \varepsilon$

$$\begin{tikzcd} \mathbf{Mod}(\mathscr{W}) \arrow[r, bend right] \arrow[r, phantom, "\perp"] & \mathbf{EMet} \arrow[l, bend right] \end{tikzcd} \Rightarrow T_{\mathscr{W}} \cong (\Lambda \,\square\, -)$$

Writer monad for the metric space Λ

Combining theories (sum & tensor)

Sum of theories

Let $\mathscr{U}, \mathscr{U}'$ be *quantitative theories* with disjoint signatures Σ, Σ'.

The *sum* $\mathscr{U} + \mathscr{U}'$ is the smallest theory containing $\mathscr{U}, \mathscr{U}'$

combines two theories by taking the disjoint union of their axioms

Ex: pointed Barycentric Algebras

$$\begin{tikzcd} \mathbf{Mod}(\mathscr{B} + \mathscr{E}_1) \arrow[r, bend right] & \mathbf{EMet} \arrow[l, bend right] \arrow[l, phantom, "\perp"] \end{tikzcd} \Rightarrow \mathscr{D}(- + 1)$$

Sub-probability distributions with **Kantorovic distance**

Tensor of theories

Let $\mathscr{U}, \mathscr{U}'$ be *quantitative theories* with disjoint signatures Σ, Σ'.
The *tensor* $\mathscr{U} \otimes \mathscr{U}'$ is the smallest theory containing $\mathscr{U}, \mathscr{U}'$ and

$$\vdash f(g(x_{1,1}, \ldots, x_{1,m}), \ldots, g(x_{n,1}, \ldots, x_{n,m}))$$

$$=_0$$

$$g(f(x_{1,1}, \ldots, x_{n,1}), \ldots, f(x_{1,m}, \ldots, x_{n,m}))$$

for all $f\colon n \in \Sigma$ and $g : m \in \Sigma'$

combines two theories by imposing the *commutation* of the operations of the theories over each other

Theorem (Sum)

The sum of quantitative theories corresponds to the categorical sum of their quantitative effects as monads

$$
\begin{tikzcd}
\mathbf{Mod}(\mathscr{U}) \arrow[d, bend left] \arrow[d, phantom, "\dashv"] & \mathbf{Mod}(\mathscr{U}') \arrow[d, bend left] \arrow[d, phantom, "\dashv"] & & \mathbf{Mod}(\mathscr{U}+\mathscr{U}') \arrow[d, bend left] \arrow[d, phantom, "\dashv"] \\
\mathbf{EMet} \arrow[u, bend left] \arrow[d, Rightarrow] & \mathbf{EMet} \arrow[u, bend left] \arrow[d, Rightarrow] & & \mathbf{EMet} \arrow[u, bend left] \arrow[d, Rightarrow] \\
T_{\mathscr{U}} \quad + & T_{\mathscr{U}'} & \cong & T_{\mathscr{U}+\mathscr{U}'}
\end{tikzcd}
$$

Sum of monads

Theorem (Tensor)

The tensor of quantitative theories corresponds to the categorical tensor of their quantitative effects as monads

$$\begin{tikzcd}
\mathbf{Mod}(\mathscr{U}) \arrow[d, bend left] \arrow[d, phantom, "\dashv"] & & \mathbf{Mod}(\mathscr{U}') \arrow[d, bend left] \arrow[d, phantom, "\dashv"] & & \mathbf{Mod}(\mathscr{U} \otimes \mathscr{U}') \arrow[d, bend left] \arrow[d, phantom, "\dashv"] \\
\mathbf{EMet} \arrow[u, bend left] \arrow[d, Rightarrow] & & \mathbf{EMet} \arrow[u, bend left] \arrow[d, Rightarrow] & & \mathbf{EMet} \arrow[u, bend left] \arrow[d, Rightarrow] \\
T_{\mathscr{U}} & \otimes & T_{\mathscr{U}'} & \cong & T_{\mathscr{U} \otimes \mathscr{U}'}
\end{tikzcd}$$

Tensor product of monads

Quantitative Theory Transformers

We can obtain quantitative analogues of Cenciarelli and Moggi's monad transformers at the level of theories via sum & tensor

Exception transformer

$$\mathscr{U} \mapsto \mathscr{U} + \mathscr{E}_E \quad\Longrightarrow\quad T_{\mathscr{U}} + \mathscr{E}_E \cong T_{\mathscr{U}}(-+E)$$

Transition transformer

$$\mathscr{U} \mapsto \mathscr{U} + \mathscr{T} \quad\Longrightarrow\quad T_{\mathscr{U}} + \mathscr{T} \cong \mu y \,.\, T_{\mathscr{U}}(c \cdot y + -)$$

Reader transformer

$$\mathscr{U} \mapsto \mathscr{U} \otimes \mathscr{R} \quad\Longrightarrow\quad T_{\mathscr{U}} \otimes \mathscr{R} \cong (T_{\mathscr{U}}-)^{\underline{A}}$$

Writer transformer

$$\mathscr{U} \mapsto \mathscr{U} \otimes \mathscr{W} \quad\Longrightarrow\quad T_{\mathscr{U}} \otimes \mathscr{W} \cong (\Lambda \,\Box\, T_{\mathscr{U}}-)$$

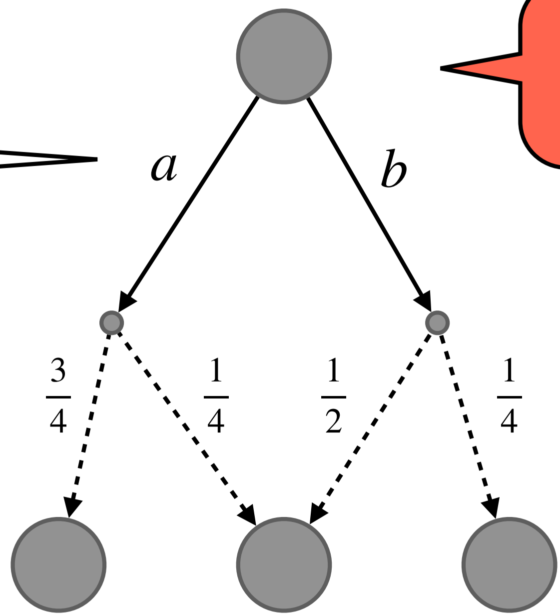
as the combination of simpler theories, via sum & tensor

Labelled Markov Chains

(with discounted bisimilarity metrics)

Labelled Markov Chains

Action labels from
a finite set A



are coalgebras
$M \xrightarrow{\pi} (\mathcal{D}(M+1))^{A}$

sub-probability distribution
to jump to next state

contractive factor
$c \in (0,1)$

c-discounted bisimilarity metric

$$d(m,n) = c\Big(\max_{a\in A} \mathcal{K}(d)(\pi(m)(a), \pi(n)(a))\Big)$$

The quantitative theory $\mathscr{U}_{LMC}$

Signature: $\{\, +_e \mid e \in [0,1] \,\} \cup \{\, \star : 0,\ \mathbf{r} : |A|,\ \diamond : 1 \,\}$

probabilistic choice

termination

reaction
input actions

transition step

$$\mathscr{U}_{LMC} = \big((\mathscr{B} + \mathscr{E}_1) \otimes \mathscr{R} \big) + \mathscr{T}$$

(B1) $\vdash x +_1 y =_0 x$

(B2) $\vdash x +_e x =_0 x$

(B3) $\vdash x + y =_0 y + x$

(SC) $\vdash x +_e y =_0 y +_{1-e} x$

(SA) $\vdash (x +_e y) +_{e'} z =_0 x +_{ee'} (y +_{\frac{(1-e)e'}{1-ee'}} z)$, **for** $e, e' \in (0,1)$

(IB) $x =_\epsilon y, x' =_{\epsilon'} y' \vdash x +_e x' =_\delta y +_e y'$, **where** $\delta = e\epsilon + (1-e)\epsilon'$

(Idem) $\vdash x \equiv_0 \mathbf{r}(x, x)$

(Diag) $\vdash \mathbf{r}(x, y) \equiv_0 \mathbf{r}(\mathbf{r}(x, z), \mathbf{r}(w, y))$

(Comm) $\vdash \mathbf{r}(x +_e y, x' +_e y') \equiv_0 \mathbf{r}(x, x') +_e \mathbf{r}(y, y')$

($\diamond$-Lip) $x =_\epsilon y \vdash \diamond x =_{c\epsilon} \diamond y$

The step-by-step recipe

STEP 1: *We represent sub-probability distributions as the sum of the quantitative barycentric theory and exception theory on* $1 = \{\star\}$

$$\mathscr{U}_1 = \mathscr{B} + \mathscr{E}_1 \Rightarrow T_{\mathscr{U}_1} \cong \mathscr{D}(- + 1)$$

quantitative barycentric theory

termination as exception

sub-probability distributions monad

STEP 2: *apply the quantitative reader theory transformer*

$$\mathscr{U}_2 = \mathscr{U}_1 \otimes \mathscr{R} \Rightarrow T_{\mathscr{U}_2} \cong (\mathscr{D}(- + 1))^{\underline{A}}$$

reader theory

adds reaction to action labels

STEP 3: *add a unary* c*-Lipschitz transition step operator* $\diamond : 1$

$$\mathscr{U}_{LMC} = \mathscr{U}_2 + \mathscr{T} \Rightarrow T_{\mathscr{U}_{LMC}} \cong \mu y . ((\mathscr{D}(c \cdot y + 1 + -))^{\underline{A}}$$

transition theory transformer

LMCs with c-discounted bisimilarity metric

Conclusions

- Quantitative theories are *the right tool* to algebraically describe **quantitative effects** ("effects with a metric twist")
- Plenty of *non-trivial examples*: Kantorovich metric, Hausdorff metric, Total variation, p-Wasserstein metric, etc.
- *Compositional reasoning*: sum & tensor
  Labelled Markov Processes, Markov Decision Processes, etc.
- The list of papers: general theory (LICS'16), *variety theorem* (LICS'17), Sum of theories (LICS'18), *Fixed-points* (LICS'21), Tensor of theories (CALCO'21).