Complete Axiomatization for the Total Variation Distance of MCs

Giorgio Bacci, Giovanni Bacci,
Kim G. Larsen, and Radu Mardare

Aalborg University, DK

MFPS XXXIII
Lubljana, 14th June 2017

Introduction

- **Kleene's Theorem**: fundamental correspondence between regular expressions and DFAs
- **Salomaa'66**, **Kozen'91**: complete axiomatization for proving equivalence of regular expressions
- **Milner'83**: applied the above program on process behaviors and LTSs
- **Rabinovich'83:** distributivity axiom capturing trace equivalence of LTSs
- Many variations of the above schema

Example: Markov chains

Expressions: $t, s := X \mid a.t \mid t +_e s \mid \text{rec } X.t$

Example: Markov chains

names
$X \in \mathbb{X}$

Expressions: $t,s := X \mid \mathbf{a}.t \mid t +_e s \mid \mathrm{rec}\ X.t$

Example: Markov chains

Expressions: t,s := X | **a**.t | t $+_e$ s | rec X.t

names $X \in \mathbb{X}$

action-prefix $a \in \mathbb{A}$

Example: Markov chains

names
$X \in \mathbb{X}$

probabilistic choice
$e \in [0,1]$

Expressions: $t,s := X \mid a.t \mid t +_e s \mid \text{rec } X.t$

action-prefix
$a \in \mathbb{A}$

Example: Markov chains

names
$X \in \mathbb{X}$

probabilistic choice
$e \in [0,1]$

Expressions: $t,s := X \mid \mathbf{a}.t \mid t +_e s \mid \text{rec } X.t$

action-prefix
$a \in \mathbb{A}$

recursion

Example: Markov chains

names $X \in \mathbb{X}$

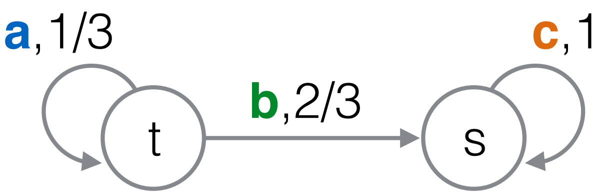
probabilistic choice $e \in [0,1]$

Expressions: $t, s := X \mid a.t \mid t +_e s \mid \text{rec } X.t$

action-prefix $a \in \mathbb{A}$

recursion

Kleene's theorem for MCs



$\Longleftrightarrow$

$$t = \text{rec } X.(a.X +_{1/3} b.s)$$
$$s = \text{rec } Y.(c.Y)$$

Example: Markov chains

names $X \in \mathbb{X}$

probabilistic choice $e \in [0,1]$

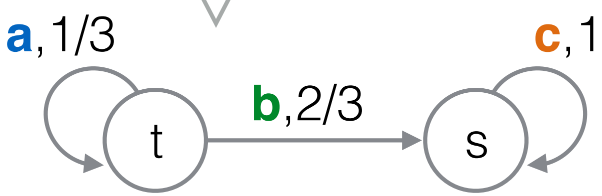
Expressions: $t,s := X \mid a.t \mid t +_e s \mid \text{rec } X.t$

action-prefix $a \in \mathbb{A}$

recursion

Kleene's theorem for MCs

finite MCs



$\Longleftrightarrow$

$t = \text{rec } X.(a.X +_{1/3} b.s)$

$s = \text{rec } Y.(c.Y)$

Example: Markov chains

(B1) $\vdash t +_1 s = t$

(B2) $\vdash t +_e t = t$

(SC) $\vdash t +_e s = s +_{1-e} t$

(SA) $\vdash (t +_e s) +_{e'} u = t +_{ee'} (s +_{\frac{e'-ee'}{1-ee'}} u)$ — for $e,e' \in [0,1)$

(Unfold) $\vdash \text{rec } X.t = t[\text{rec } X.t / X]$

(Fix) $\{t = s[t / X]\} \vdash t = \text{rec } X.s$ — for X guarded in t

(Unguard) $\vdash \text{rec } X.(t +_e X) = \text{rec } X.t$

(Dist-pref) $\vdash a.(t +_e s) = a.t +_e a.s$

Example: Markov chains

Silva-Sokolova axiomatization probabilistic trace equivalence (MFPS'11)

(B1) $\vdash t +_1 s = t$

(B2) $\vdash t +_e t = t$

(SC) $\vdash t +_e s = s +_{1-e} t$

(SA) $\vdash (t +_e s) +_{e'} u = t +_{ee'} (s +_{\frac{e'-ee'}{1-ee'}} u)$ — for $e, e' \in [0,1)$

(Unfold) $\vdash \mathrm{rec}\ X.t = t[\mathrm{rec}\ X.t / X]$

(Fix) $\{t = s[t / X]\} \vdash t = \mathrm{rec}\ X.s$ — for X guarded in t

(Unguard) $\vdash \mathrm{rec}\ X.(t +_e X) = \mathrm{rec}\ X.t$

(Dist-pref) $\vdash a.(t +_e s) = a.t +_e a.s$

Example: Markov chains

Silva-Sokolova axiomatization probabilistic trace equivalence (MFPS'11)

Stone's barycentric axioms

(B1) $\vdash t +_1 s = t$

(B2) $\vdash t +_e t = t$

(SC) $\vdash t +_e s = s +_{1-e} t$

(SA) $\vdash (t +_e s) +_{e'} u = t +_{ee'} (s +_{\frac{e'-ee'}{1-ee'}} u)$ — for $e,e' \in [0,1)$

(Unfold) $\vdash \mathrm{rec}\ X.t = t[\mathrm{rec}\ X.t / X]$

(Fix) $\{t = s[t / X]\} \vdash t = \mathrm{rec}\ X.s$ — for X guarded in t

(Unguard) $\vdash \mathrm{rec}\ X.(t +_e X) = \mathrm{rec}\ X.t$

(Dist-pref) $\vdash a.(t +_e s) = a.t +_e a.s$

Example: Markov chains

Silva-Sokolova axiomatization
probabilistic trace equivalence (MFPS'11)

Stone's barycentric axioms

(B1) $\vdash t +_1 s = t$

(B2) $\vdash t +_e t = t$

(SC) $\vdash t +_e s = s +_{1-e} t$

(SA) $\vdash (t +_e s) +_{e'} u = t +_{ee'} (s +_{\frac{e'-ee'}{1-ee'}} u)$ — for $e,e' \in [0,1)$

Milner's recursion axioms

(Unfold) $\vdash \text{rec } X.t = t[\text{rec } X.t \,/\, X]$

(Fix) $\{t = s[t \,/\, X]\} \vdash t = \text{rec } X.s$ — for X guarded in t

(Unguard) $\vdash \text{rec } X.(t +_e X) = \text{rec } X.t$

(Dist-pref) $\vdash a.(t +_e s) = a.t +_e a.s$

Example: Markov chains

Silva-Sokolova axiomatization probabilistic trace equivalence (MFPS'11)

Stone's barycentric axioms

(B1) $\vdash t +_1 s = t$

(B2) $\vdash t +_e t = t$

(SC) $\vdash t +_e s = s +_{1-e} t$

(SA) $\vdash (t +_e s) +_{e'} u = t +_{ee'} (s +_{\frac{e'-ee'}{1-ee'}} u)$ — for $e,e' \in [0,1)$

Milner's recursion axioms

(Unfold) $\vdash \text{rec } X.t = t[\text{rec } X.t \,/\, X]$

(Fix) $\{t = s[t \,/\, X]\} \vdash t = \text{rec } X.s$ — for X guarded in t

(Unguard) $\vdash \text{rec } X.(t +_e X) = \text{rec } X.t$

Rabinovich's distributivity axiom

(Dist-pref) $\vdash a.(t +_e s) = a.t +_e a.s$

…for probabilistic systems

- **Generative Markov chains:**
  Baeten-Bergstra-Smolka'95 & Stark-Smolka'00
- **Simple Probabilistic Automata:**
  Bandini-Segala'01
- **(fully) Probabilistic Automata:**
  Mislove-Ouaknine-Worrell'04 (strong-bisimulation)
  Deng-Palamidessi'07 (weak-bisimulation & behavioral eq.)
- **Quantitative Kleene Coalgebras:**
  Silva-Bonchi-Bonsangue-Rutten'11 (coagebraic bisim.)
- etc…

In this talk…

In this talk…

- From equivalences to distances: we present

In this talk…

- From equivalences to distances: we present
  - a ***sound & complete axiomatization*** *of the* ***total variation distance*** of Markov chains

In this talk…

- From equivalences to distances: we present
  - a ***sound & complete axiomatization*** *of the* ***total variation distance*** of Markov chains
  - and a **quantitative Kleene's Theorem** for Markov chains (generalize Silva-Sokolova'11)

In this talk…

- From equivalences to distances: we present
  - a ***sound & complete axiomatization*** *of the* ***total variation distance*** of Markov chains
  - and a **quantitative Kleene's Theorem** for Markov chains (generalize Silva-Sokolova'11)
- How do we do it?
  By using **Quantitative Equational Theories*** of **Mardare-Panangaden-Plotkin** (LICS'16)

In this talk…

- From equivalences to distances: we present
  - a ***sound & complete axiomatization*** *of the* ***total variation distance*** of Markov chains
  - and a **quantitative Kleene's Theorem** for Markov chains (generalize Silva-Sokolova'11)

- How do we do it?
  By using **Quantitative Equational Theories*** of **Mardare-Panangaden-Plotkin** (LICS'16)

$$s = t \quad \Longrightarrow \quad s =_{\varepsilon} t$$

Equational Theories

$$\{t_i = s_i \mid i \in I\} \vdash t = s$$

inference

Equational Theories

$$\{t_i = s_i \mid i \in I\} \vdash t = s$$

inference

(Refl) $\vdash t = t$

(Symm) $\{t = s\} \vdash s = t$

(Trans) $\{t = u, u = s\} \vdash t = s$

(Cong) $\{t_1 = s_1, \ldots, t_n = s_n\} \vdash f(t_1, \ldots, t_n) = f(s_1, \ldots s_n)$ — for $f \in \Sigma$

Quantitative Theories

Mardare-Panangaden-Plotkin (LICS'16)

$$\{t_i =_{\varepsilon_i} s_i \mid i \in I\} \vdash t =_{\varepsilon} s$$

quantitative inference

(Refl) $\vdash t =_0 t$

(Symm) $\{t =_\varepsilon s\} \vdash s =_\varepsilon t$

(Triang) $\{t =_\varepsilon u, u =_\delta s\} \vdash t =_{\varepsilon+\delta} s$

(NExp) $\{t_1 =_\varepsilon s_1,\dots,t_n =_\varepsilon s_n\} \vdash f(t_1,\dots,t_n) =_\varepsilon f(s_1,\dots s_n)$ — for $f \in \Sigma$

(Max) $\{t =_\varepsilon s\} \vdash t =_{\varepsilon+\delta} s$ — for $\delta > 0$

(Arch) $\{t =_\delta s \mid \delta > \varepsilon\} \vdash t =_\varepsilon s$

Quantitative Semantics

Quantitative Algebra

$\mathcal{A} = (A, \Sigma_A, d_A)$

- (A, Σ_A) — Universal algebra
- (A, d_A) — (pseudo)metric space

Satisfiability

$$\mathcal{A} \models \left(\{ t_i =_{\varepsilon_i} s_i \mid i \in I \} \vdash t =_{\varepsilon} s \right)$$

iff

for all $i \in I$. $d_A([\![t_i]\!], [\![s_i]\!]) \leq \varepsilon_i$ implies $d_A([\![t]\!], [\![s]\!]) \leq \varepsilon$

quantitative algebra

quantitative theory

completeness

$$\mathcal{A} \models (\vdash t =_\varepsilon s) \qquad (\vdash t =_\varepsilon s) \in \mathcal{U}$$

soundness

completeness

quantitative algebra

quantitative theory

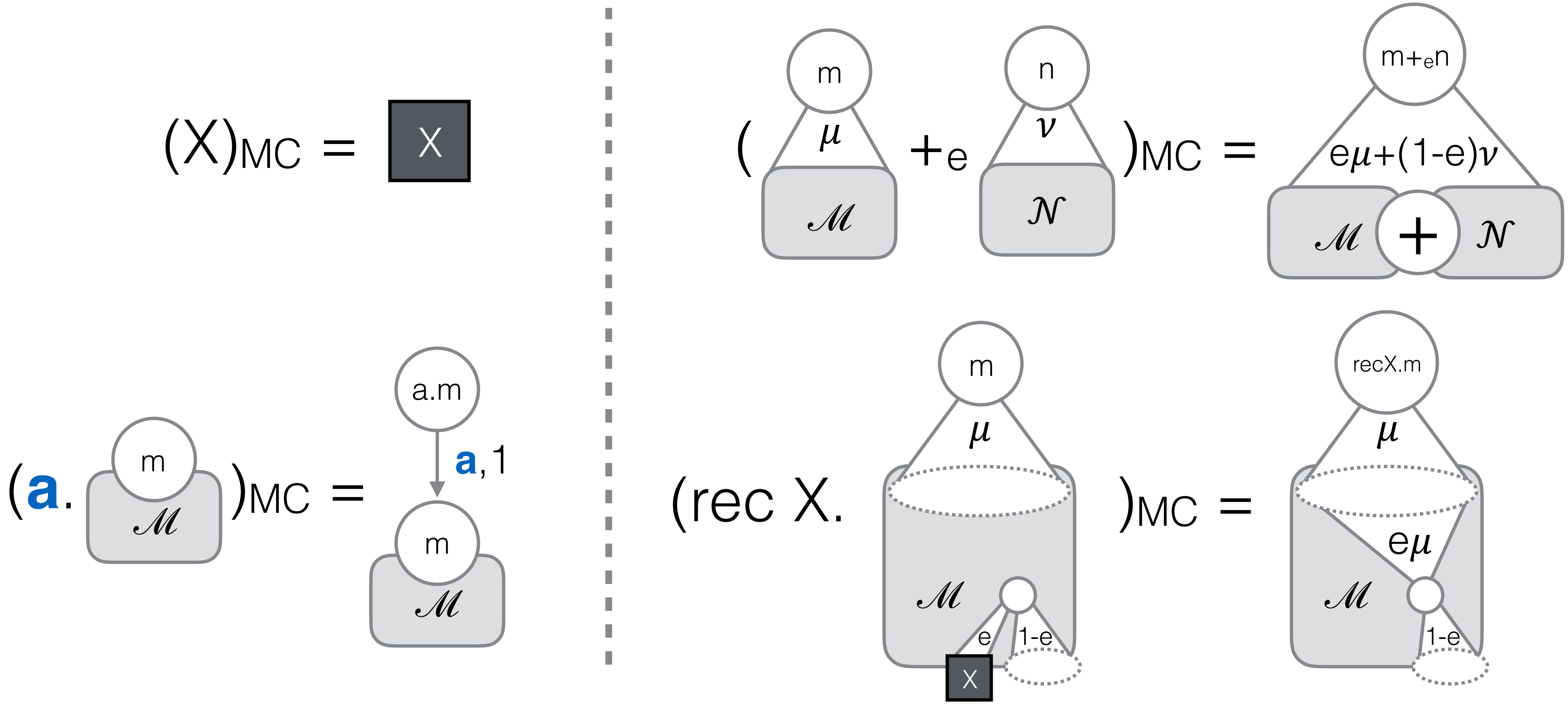
$\mathcal{A}_{MC} \models (\vdash t =_{\varepsilon} s)$ $\quad$ $(\vdash t =_{\varepsilon} s) \in \mathcal{U}_{MC}$

soundness

The Quantitative Universal Algebra

Universal Algebra of MCs

Signature: X : 0 | **a**.- : 1 | $+_e$: 2 | rec X : 1



Total variation distance

$$tv(m,n) = \sup_{E \in \sigma(\Pi)} | P(m)(E) - P(n)(E) |$$

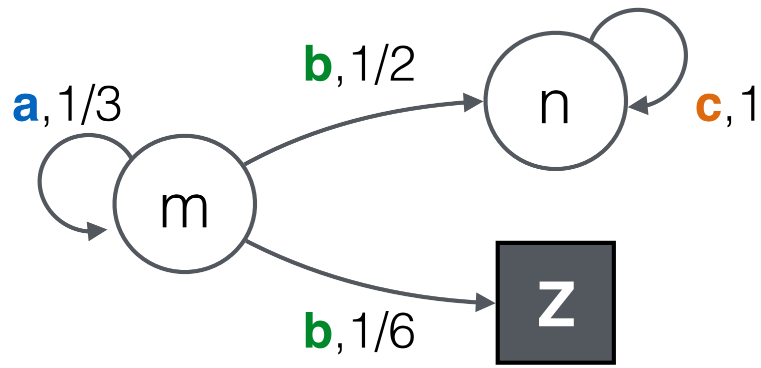
$\Pi = \mathbb{A}^\omega \cup \mathbb{A}^*\mathbb{X}$ *(observable traces)*

ω-traces over $\mathbb{A}$

finite traces over $\mathbb{A}$ terminating in $\mathbb{X}$

$$P(m)(\mathfrak{C}(\mathbf{aabc})) = \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{2} \cdot 1$$

$$P(m)(\mathfrak{C}(\mathbf{aab}Z)) = \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{6}$$



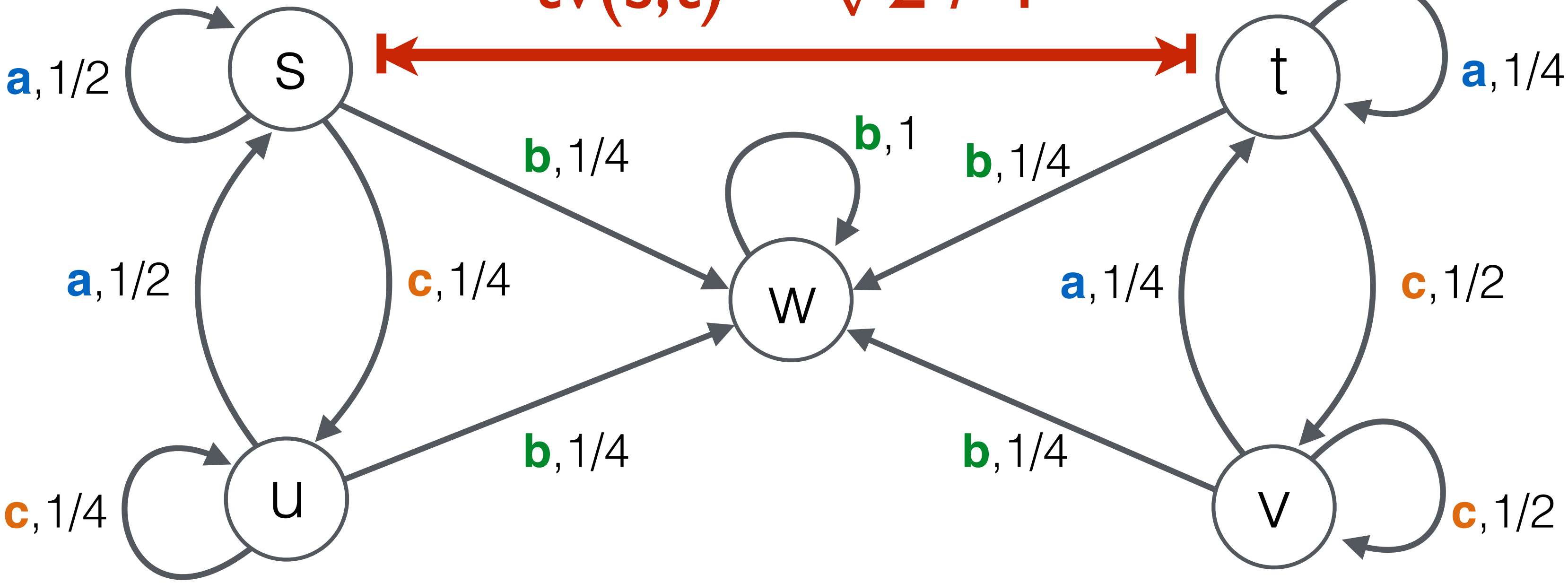
A tiny yet tricky example

(from Chen-Kiefer LICS'14)

maximizing event is not ω-regular!

irrational number

$tv(s,t) = \sqrt{2} / 4$



Converging to Total Variation

(Bacci et al. ICTAC'15)

$\mathbb{K}$ — poset of positive integers ordered by divisibility

Theorem

1. The net $(d_k)_{k \in \mathbb{K}}$ converges point-wise to **tv**
2. for all $k \in \mathbb{K}$, $d_k \geq$ **tv**

Converging to Total Variation

(Bacci et al. ICTAC'15)

$\mathbb{K}$ — poset of positive integers ordered by divisibility

Theorem

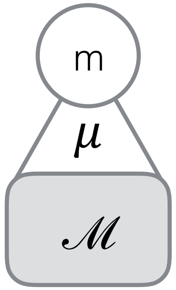
1. The net $(d_k)_{k \in \mathbb{K}}$ converges point-wise to **tv**
2. for all $k \in \mathbb{K}$, $d_k \geq$ **tv**

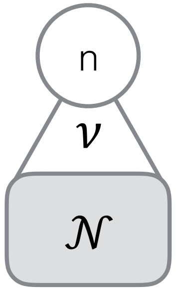
probabilistic k-bisimilarity distance

k-bisimilarity distance

(generalizes Desharnais et al. TCS'04)

it is the least 1-bounded pseudometric satisfying

$d_k($ 

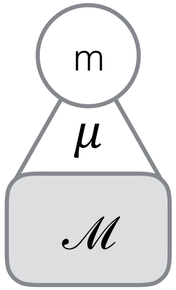
, 

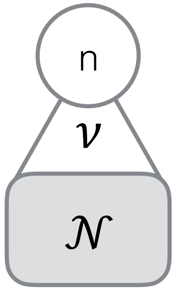
$) = \min \{ \int \Lambda(d_k)\, d\omega \mid \omega \in \Omega(\mu^k, \nu^k) \}$

k-bisimilarity distance

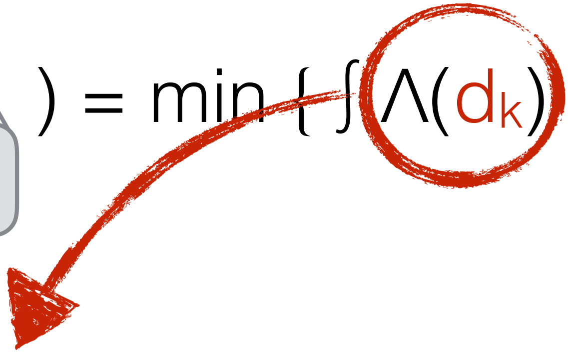
(generalizes Desharnais et al. TCS'04)

it is the least 1-bounded pseudometric satisfying

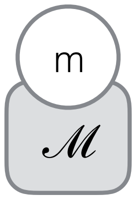
$d_k($ 

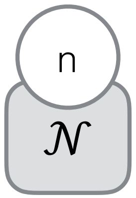
$,$ 

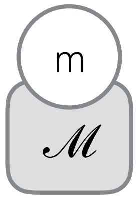
$) = \min \{ \int \Lambda(d_k)\, d\omega \mid \omega \in \Omega(\mu^k, \nu^k) \}$

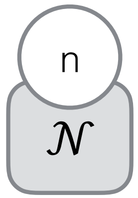


$\Lambda(d_k)$ —greatest 1-bounded pseudometric on $(\mathbb{A}^k \times MC) \cup \mathbb{A}^{<k}\mathbb{X}$

s.t, for all $\mathbf{w} \in \mathbb{A}^k$, $\Lambda(d_k)((\mathbf{w},$ 

$),(\mathbf{w},$ 

$)) = d_k($ 

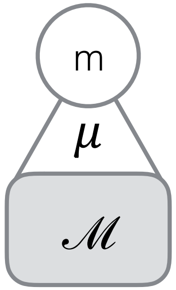
$,$ 

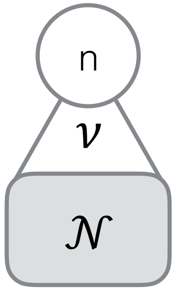
$)$

k-bisimilarity distance

(generalizes Desharnais et al. TCS'04)

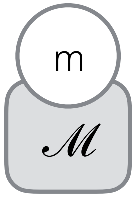
it is the least 1-bounded pseudometric satisfying

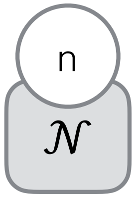
$d_k($ 

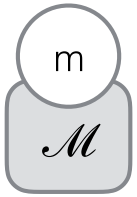
, 

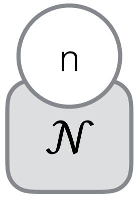
$) = \min \{ \int \Lambda(d_k)\, d\omega \mid \omega \in \Omega(\mu^k, \nu^k) \}$

$\Lambda(d_k)$ —greatest 1-bounded pseudometric on $(\mathbb{A}^k \times MC) \cup \mathbb{A}^{<k}\mathbb{X}$

s.t, for all $\mathbf{w} \in \mathbb{A}^k$, $\Lambda(d_k)((\mathbf{w},$ 

$),(\mathbf{w},$ 

$)) = d_k($ 

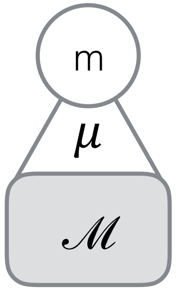
, 

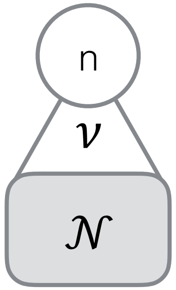
$)$

k-bisimilarity distance

(generalizes Desharnais et al. TCS'04)

it is the least 1-bounded pseudometric satisfying

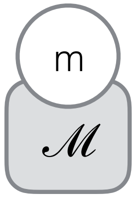
$d_k($ 

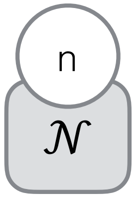
, 

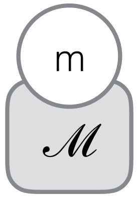
$) = \min \{ \int \Lambda(d_k)\, d\omega \mid \omega \in \Omega(\mu^k, \nu^k) \}$

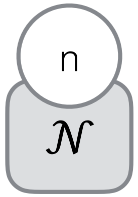
Kantorovich lifting

$\Lambda(d_k)$ —greatest 1-bounded pseudometric on $(\mathbb{A}^k \times MC) \cup \mathbb{A}^{<k}\mathbb{X}$

s.t, for all $\mathbf{w} \in \mathbb{A}^k$, $\Lambda(d_k)((\mathbf{w},$ 

$),(\mathbf{w},$ 

$)) = d_k($ 

, 

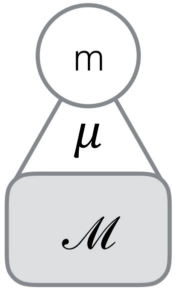
$)$

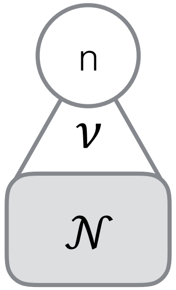
k-bisimilarity distance

(generalizes Desharnais et al. TCS'04)

it is the least 1-bounded pseudomet

couplings
=
probabilistic "relations"

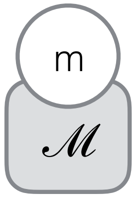
$d_k($ 

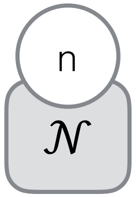
, 

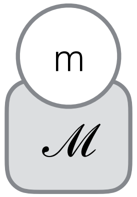
$) = \min \{ \int \Lambda(d_k)\, d\omega \mid \omega \in \Omega(\mu^k, \nu^k) \}$

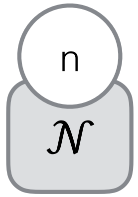
Kantorovich lifting

$\Lambda(d_k)$ —greatest 1-bounded pseudometric on $(\mathbb{A}^k \times MC) \cup \mathbb{A}^{<k}\mathbb{X}$

s.t, for all $\mathbf{w} \in \mathbb{A}^k$, $\Lambda(d_k)((\mathbf{w},$ 

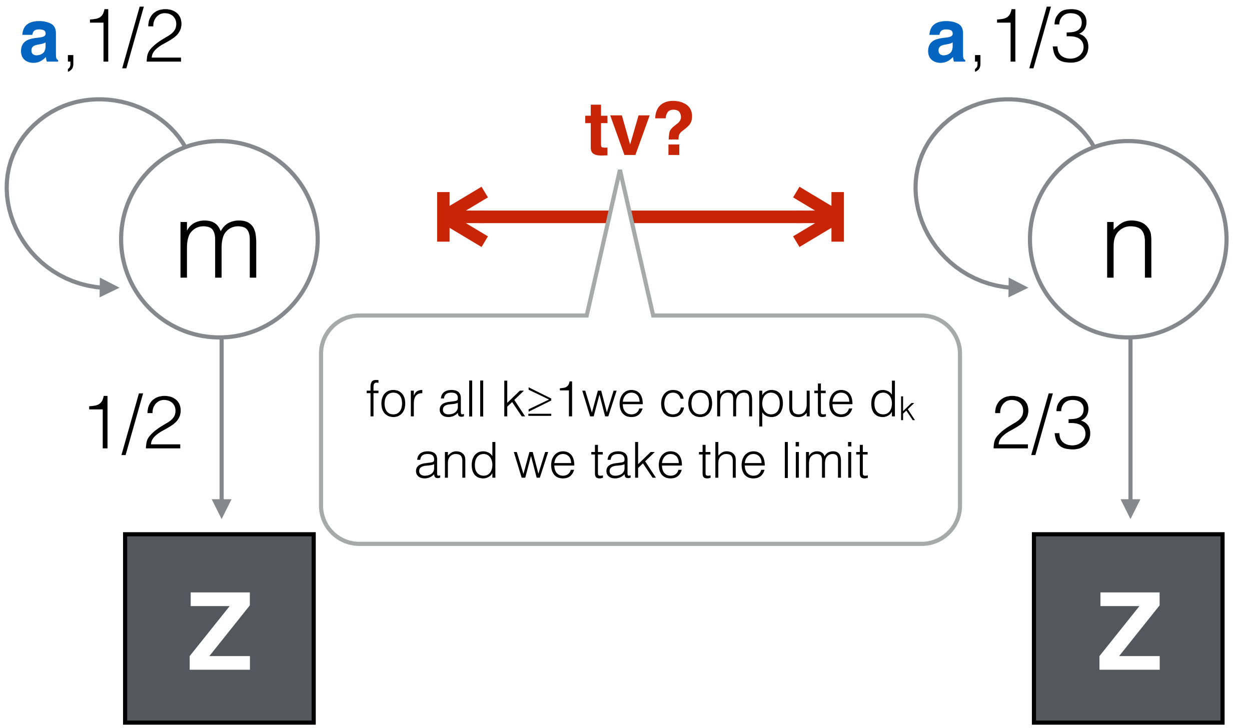
$),(\mathbf{w},$ 

$)) = d_k($ 

, 

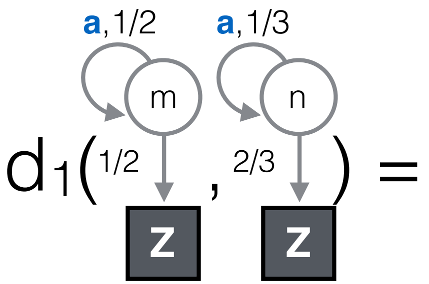
$)$

Running example



$m = \text{rec } X. (\mathbf{a}.X +_{1/2} Z)$ $\quad$ $n = \text{rec } Y. (\mathbf{a}.Y +_{1/3} Z)$

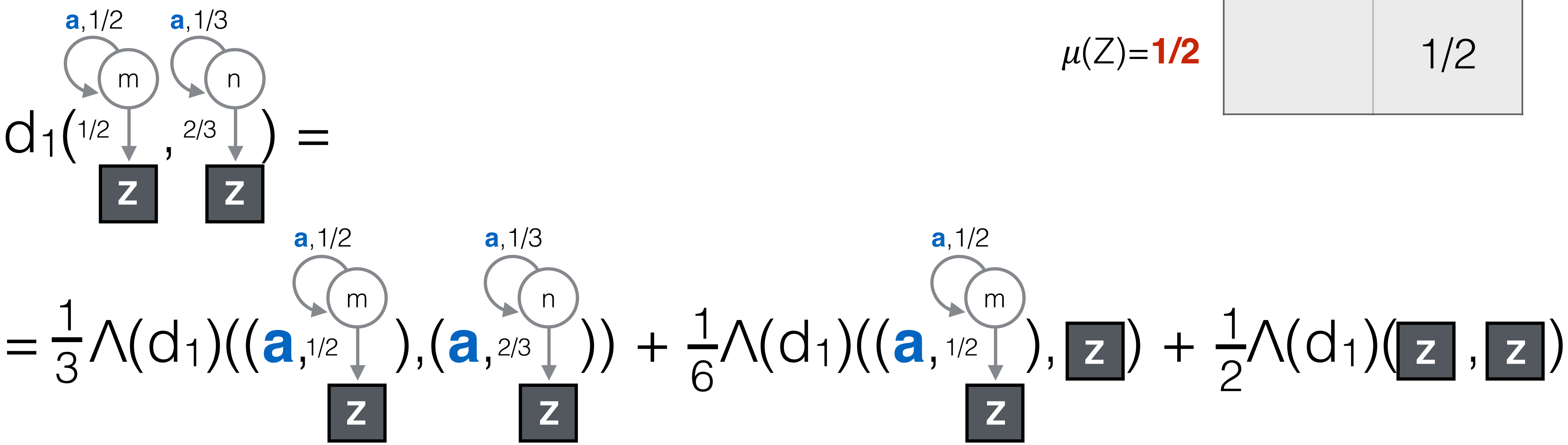
optimal coupling between transition probabilities of m and n



ω^*	$\nu((\mathbf{a},n))$ = **1/3**	$\nu(Z)$ = **2/3**
$\mu((\mathbf{a},m))$=**1/2**	1/3	1/6
$\mu(Z)$=**1/2**		1/2

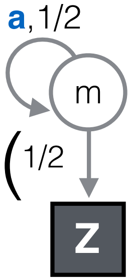
optimal coupling between
transition probabilities
of m and n

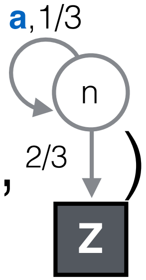
ω^*	$\nu((\mathbf{a},n)) = \mathbf{1/3}$	$\nu(Z) = \mathbf{2/3}$
$\mu((\mathbf{a},m)) = \mathbf{1/2}$	1/3	1/6
$\mu(Z) = \mathbf{1/2}$		1/2



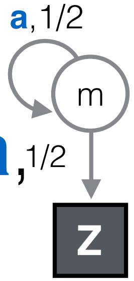
optimal coupling between transition probabilities of m and n

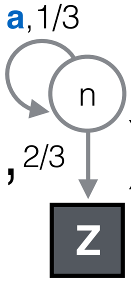
ω^*	$\nu((\mathbf{a},n))$ = **1/3**	$\nu(Z)$ = **2/3**
$\mu((\mathbf{a},m))$=**1/2**	1/3	1/6
$\mu(Z)$=**1/2**		1/2

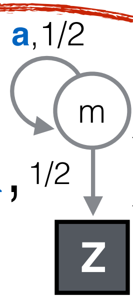
$d_1($ 

$,$ 

$) =$

$= \frac{1}{3}\Lambda(d_1)((\mathbf{a},$ 

$),(\mathbf{a},$ 

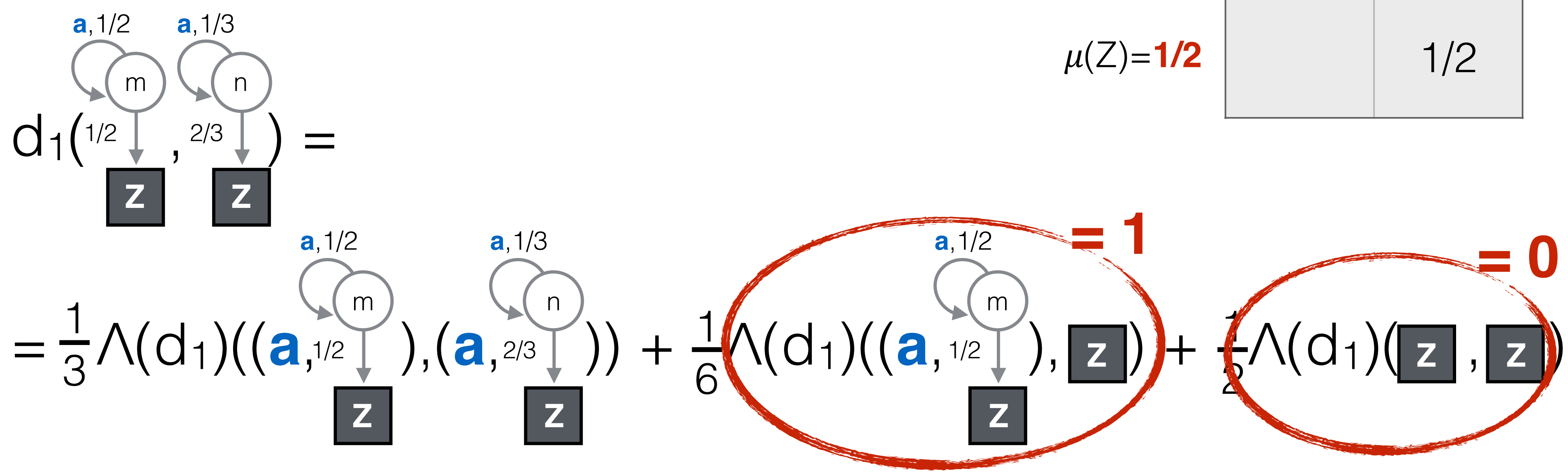
$)) + \frac{1}{6}\Lambda(d_1)((\mathbf{a},$ 

$),$ Z $) + \frac{1}{2}\Lambda(d_1)($ Z $,$ Z $)$

= 1

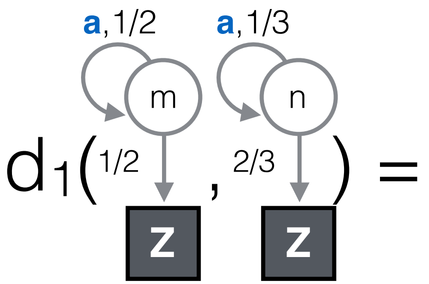
optimal coupling between
transition probabilities
of m and n

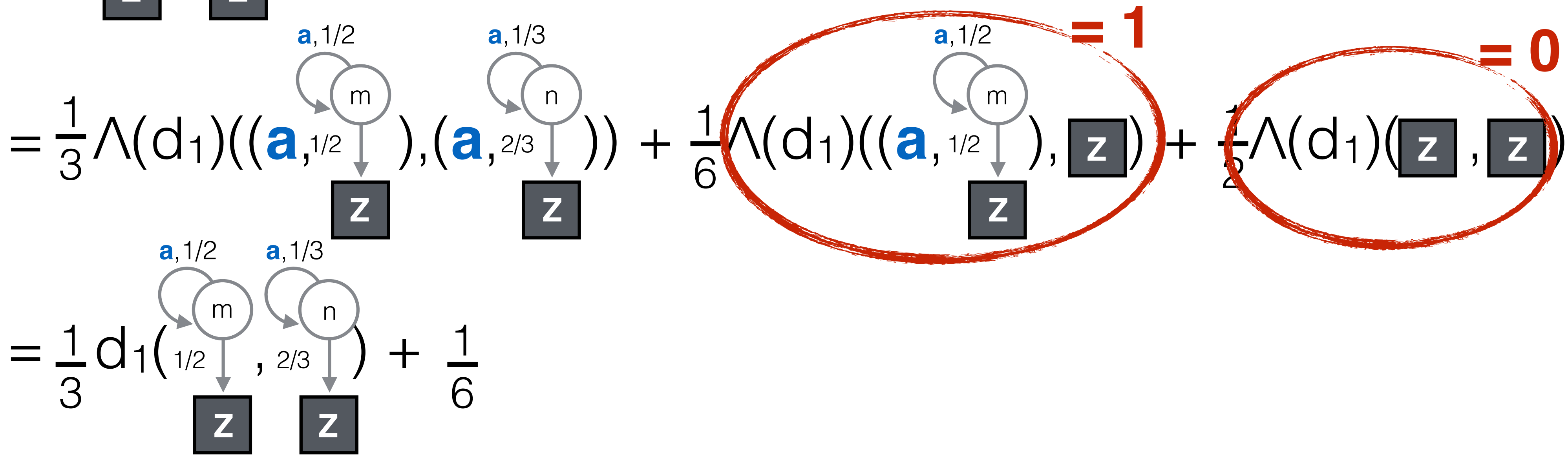
ω^*	$\nu((\mathbf{a},n)) = \mathbf{1/3}$	$\nu(Z) = \mathbf{2/3}$
$\mu((\mathbf{a},m)) = \mathbf{1/2}$	1/3	1/6
$\mu(Z) = \mathbf{1/2}$		1/2



optimal coupling between
transition probabilities
of m and n

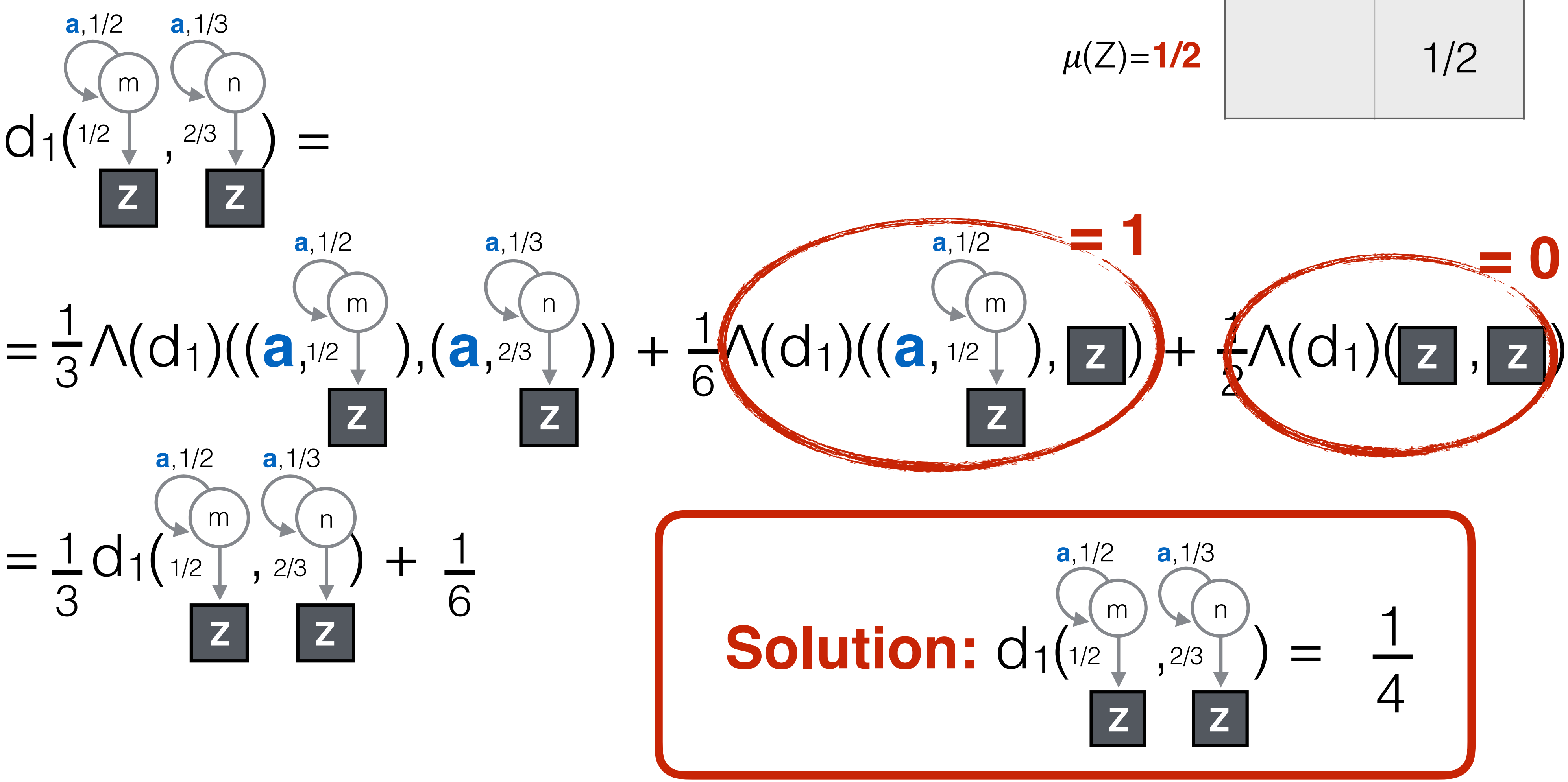
ω^*	$\nu((\mathbf{a},n)) = \mathbf{1/3}$	$\nu(Z) = \mathbf{2/3}$
$\mu((\mathbf{a},m))=\mathbf{1/2}$	1/3	1/6
$\mu(Z)=\mathbf{1/2}$		1/2



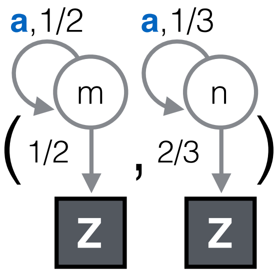


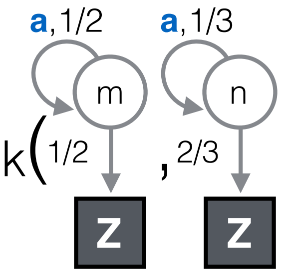
optimal coupling between transition probabilities of m and n

ω^*	$\nu((\mathbf{a},n)) = 1/3$	$\nu(Z) = 2/3$
$\mu((\mathbf{a},m)) = 1/2$	1/3	1/6
$\mu(Z) = 1/2$		1/2

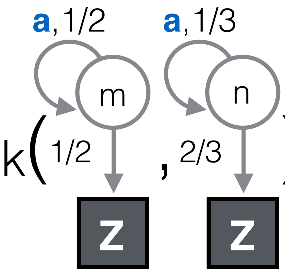


general case (k≥1)

$d_k($ 

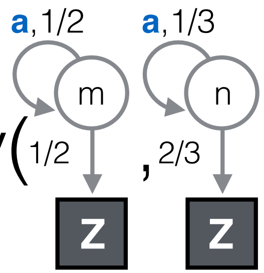
$) = \frac{1}{3^k}\, d_k($ 

$) + \frac{1}{6}$

Solution: $d_k($ 

$) = \frac{3^{k-1}}{2(3^k - 1)}$

by the convergence theorem…

$tv($ 

$) = \lim_{k \to \infty} \frac{3^{k-1}}{2(3^k - 1)} = \mathbf{\frac{1}{6}}$

The Quantitative Equational Theory

rec is problematic…

The quantitative equational framework
of Mardare-Panangaden-Plotkin requires
all operators to be **non-expansive**

(NExp) $\{t_1 =_\varepsilon s_1,\ldots,t_n =_\varepsilon s_n\} \vdash f(t_1,\ldots,t_n) =_\varepsilon f(s_1,\ldots s_n)$ — for $f \in \Sigma$

rec is problematic…

The quantitative equational framework
of Mardare-Panangaden-Plotkin requires
all operators to be **non-expansive**

(NExp) $\{t_1 =_\varepsilon s_1, \ldots, t_n =_\varepsilon s_n\} \vdash f(t_1, \ldots, t_n) =_\varepsilon f(s_1, \ldots s_n)$ — for $f \in \Sigma$

… but the NExp axiom is not sound for recursion

$$\mathcal{A}_{MC} \not\models \big(\{t =_\varepsilon s\} \vdash \text{rec } X.t =_\varepsilon \text{rec } X.s\big)$$

(see paper for the counterexample)

Relaxing non-expansivity

we keep all the axioms of quantitative algebras
but the NExp axiom

(Refl) $\vdash t =_0 t$

(Symm) $\{t =_\varepsilon s\} \vdash s =_\varepsilon t$

(Triang) $\{t =_\varepsilon u, u =_\delta s\} \vdash t =_{\varepsilon+\delta} s$

~~(NExp) $\{t_1 =_\varepsilon s_1, \ldots, t_n =_\varepsilon s_n\} \vdash f(t_1,\ldots,t_n) =_\varepsilon f(s_1,\ldots,s_n)$ — for $f \in \Sigma$~~

(Max) $\{t =_\varepsilon s\} \vdash t =_{\varepsilon+\delta} s$ — for $\delta>0$

(Arch) $\{t =_\delta s \mid \delta>\varepsilon \} \vdash t =_\varepsilon s$

Relaxing non-expansivity

we keep all the axioms of quantitative algebras
but the NExp axiom

(Refl) $\vdash t =_0 t$

(Symm) $\{t =_\varepsilon s\} \vdash s =_\varepsilon t$

(Triang) $\{t =_\varepsilon u, u =_\delta s\} \vdash t =_{\varepsilon+\delta} s$

~~(NExp) $\{t_1 =_\varepsilon s_1, \ldots, t_n =_\varepsilon s_n\} \vdash f(t_1,\ldots,t_n) =_\varepsilon f(s_1,\ldots,s_n)$ — for $f \in \Sigma$~~

(Max) $\{t =_\varepsilon s\} \vdash t =_{\varepsilon+\delta} s$ — for $\delta>0$

(Arch) $\{t =_\delta s \mid \delta>\varepsilon \} \vdash t =_\varepsilon s$

is NOT the original quantitative equational framework!

Relaxing non-expansivity

we keep all the axioms of quantitative algebras
but the NExp axiom

(Refl) $\vdash t =_0 t$

(Symm) $\{t =_\varepsilon s\} \vdash s =_\varepsilon t$

(Triang) $\{t =_\varepsilon u, u =_\delta s\} \vdash t =_{\varepsilon+\delta} s$

~~(NExp) $\{t_1 =_\varepsilon s_1, \ldots, t_n =_\varepsilon s_n\} \vdash f(t_1,\ldots,t_n) =_\varepsilon f(s_1,\ldots,s_n)$ — for $f \in \Sigma$~~

(Max) $\{t =_\varepsilon s\} \vdash t =_{\varepsilon+\delta} s$ — for $\delta>0$

(Arch) $\{t =_\delta s \mid \delta>\varepsilon\} \vdash t =_\varepsilon s$

is NOT the original quantitative equational framework!

the Archimedian axiom will be used to recover completeness

Axiomatization

Interpolative barycentric axioms (Mardare-Panangaden-Plotkin LICS'16)

(B1) $\vdash t +_1 s =_0 t$

(B2) $\vdash t +_e t =_0 t$

(SC) $\vdash t +_e s =_0 s +_{1-e} t$

(SA) $\vdash (t +_e s) +_{e'} u =_0 t +_{ee'} (s +_{\frac{e'-ee'}{1-ee'}} u)$ — for $e,e' \in [0,1)$

(IB) $\{t =_\varepsilon s, t' =_{\varepsilon'} s'\} \vdash t +_e t' =_\delta s +_e s'$ — for $\delta \le e\varepsilon+(1-e)\varepsilon'$

(Top) $\vdash t =_1 s$

Milner's recursion axioms

(Unfold) $\vdash \text{rec } X.t =_0 t[\text{rec } X.t / X]$

(Fix) $\{t = s[t / X]\} \vdash t =_0 \text{rec } X.s$ — for X guarded in t

(Unguard) $\vdash \text{rec } X.(t +_e X) =_0 \text{rec } X.t$

Rabinovich's distributivity axiom

(Dist-pref) $\vdash \mathbf{a}.(t +_e s) = \mathbf{a}.t +_e \mathbf{a}.s$

Weak NExp-axioms

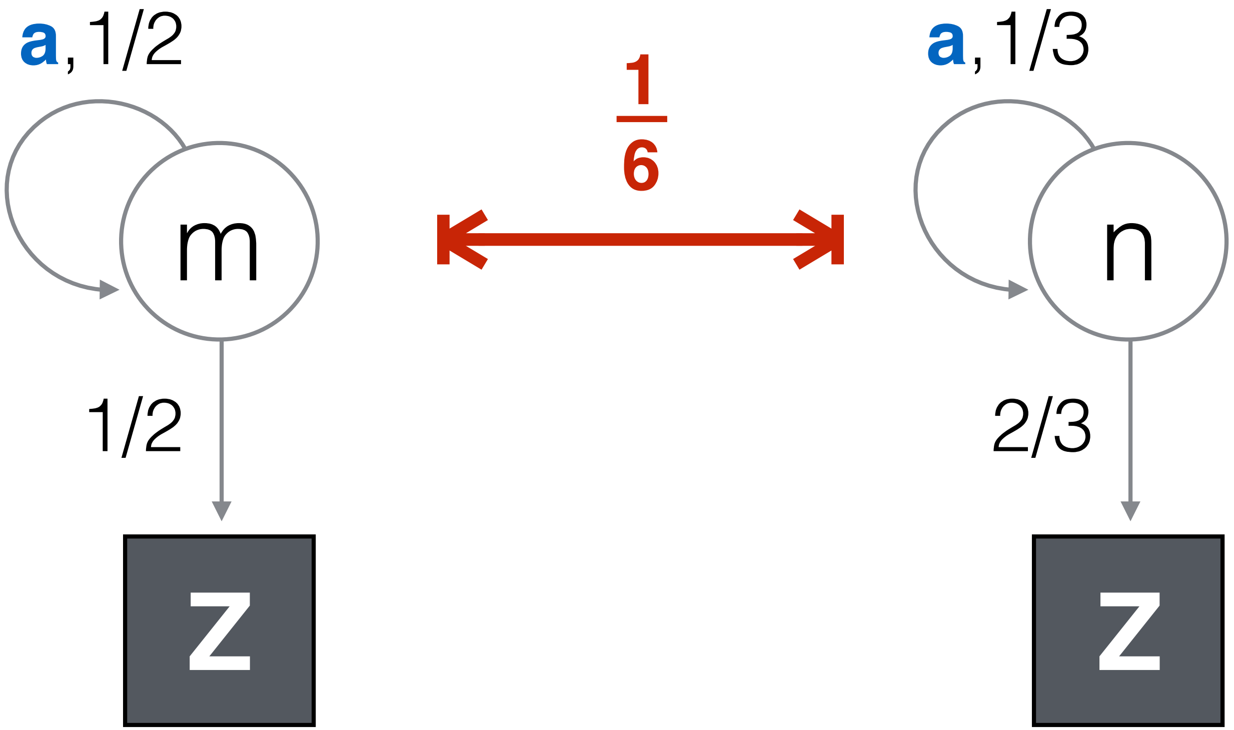
(Pref) $\{t =_\varepsilon s\} \vdash \mathbf{a}.t =_\varepsilon \mathbf{a}.s$

(Cong) $\{t =_0 s\} \vdash \text{rec } X.t =_0 \text{rec } X.s$

proof of completeness

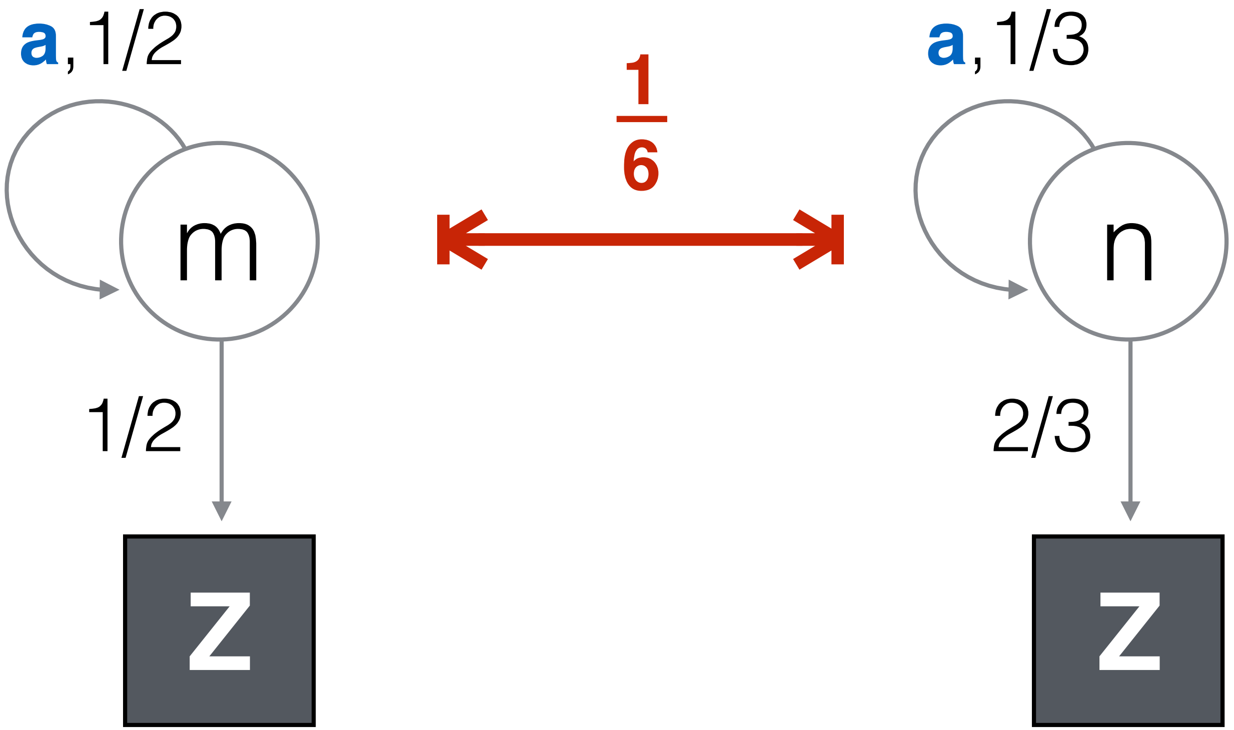
(by example)

Same running example



m = rec X. (**a**.X $+_{1/2}$ Z)   n = rec Y. (**a**.Y $+_{1/3}$ Z)

Same running example



m = rec X. (**a**.X $+_{1/2}$ Z)   n = rec Y. (**a**.Y $+_{1/3}$ Z)

by using (Fix)+(Unfold)…

m = **a**.m $+_{1/2}$ Z   n = **a**.n $+_{1/3}$ Z

Kantorovich via IB

(IB) $\{t =_{\varepsilon} s,\ t' =_{\varepsilon'} s'\} \vdash t +_e t' =_{\delta} s +_e s'$

— for $\delta \leq e\varepsilon + (1-e)\varepsilon'$

Kantorovich via IB

(IB) $\{t =_{\varepsilon} s,\ t' =_{\varepsilon'} s'\} \vdash t +_e t' =_{\delta} s +_e s'$

— for $\delta \leq e\varepsilon + (1-e)\varepsilon'$

$$\mathbf{a}.m +_{1/2} Z =_0 (\mathbf{a}.m +_{1/3} \mathbf{a}.m) +_{1/2} Z \quad \text{(B2)}$$

$$=_0 \mathbf{a}.m +_{\mathbf{1/6}} (\mathbf{a}.m +_{2/5} Z) \quad \text{(SA)}$$

Kantorovich via IB

(IB) $\{t =_\varepsilon s,\ t' =_{\varepsilon'} s'\} \vdash t +_e t' =_\delta s +_e s'$
— for $\delta \le e\varepsilon+(1-e)\varepsilon'$

$$\mathbf{a}.m +_{1/2} Z =_0 (\mathbf{a}.m +_{1/3} \mathbf{a}.m) +_{1/2} Z \quad \text{(B2)}$$
$$=_0 \mathbf{a}.m +_{\mathbf{1/6}} (\mathbf{a}.m +_{2/5} Z) \quad \text{(SA)}$$

$$\mathbf{a}.n +_{1/3} Z =_0 Z +_{2/3} \mathbf{a}.n \quad \text{(SC)}$$
$$=_0 (Z +_{1/4} Z) +_{2/3} \mathbf{a}.n \quad \text{(B2)}$$
$$=_0 Z +_{\mathbf{1/6}} (\mathbf{a}.n +_{2/5} Z) \quad \text{(SA)+(SC)}$$

Kantorovich via IB

(IB) $\{t =_\varepsilon s,\ t' =_{\varepsilon'} s'\} \vdash t +_e t' =_\delta s +_e s'$
— for $\delta \leq e\varepsilon+(1\text{-}e)\varepsilon'$

$\mathbf{a}.m +_{1/2} Z =_0 (\mathbf{a}.m +_{1/3} \mathbf{a}.m) +_{1/2} Z$ (B2)

$=_0 \mathbf{a}.m +_{\mathbf{1/6}} (\mathbf{a}.m +_{2/5} Z)$ (SA)

ω^*	($\mathbf{a}$,Y)	Z
($\mathbf{a}$,X)	1/3	**1/6**
Z		1/2

$\mathbf{a}.n +_{1/3} Z =_0 Z +_{2/3} \mathbf{a}.n$ (SC)

$=_0 (Z +_{1/4} Z) +_{2/3} \mathbf{a}.n$ (B2)

$=_0 Z +_{\mathbf{1/6}} (\mathbf{a}.n +_{2/5} Z)$ (SA)+(SC)

Kantorovich via IB

(IB) $\{t =_{\varepsilon} s,\ t' =_{\varepsilon'} s'\} \vdash t +_{e} t' =_{\delta} s +_{e} s'$

— for $\delta \leq e\varepsilon + (1-e)\varepsilon'$

$\mathbf{a}.m +_{1/2} Z =_{0} (\mathbf{a}.m +_{1/3} \mathbf{a}.m) +_{1/2} Z$ (B2)

$=_{0} \mathbf{a}.m +_{\mathbf{1/6}} (\mathbf{a}.m +_{2/5} Z)$ (SA)

ω*	(**a**,Y)	Z
(**a**,X)	1/3	**1/6**
Z		1/2

$\mathbf{a}.n +_{1/3} Z =_{0} Z +_{2/3} \mathbf{a}.n$ (SC)

$=_{0} (Z +_{1/4} Z) +_{2/3} \mathbf{a}.n$ (B2)

$=_{0} Z +_{\mathbf{1/6}} (\mathbf{a}.n +_{2/5} Z)$ (SA)+(SC)

Kantorovich via IB

(IB) $\{t =_{\varepsilon} s,\ t' =_{\varepsilon'} s'\} \vdash t +_{e} t' =_{\delta} s +_{e} s'$

— for $\delta \leq e\varepsilon + (1-e)\varepsilon'$

$\mathbf{a}.m +_{1/2} Z =_0 (\mathbf{a}.m +_{1/3} \mathbf{a}.m) +_{1/2} Z$ (B2)

$=_0 \mathbf{a}.m +_{1/6} (\mathbf{a}.m +_{2/5} Z)$ (SA)

ω^*	($\mathbf{a}$,Y)	Z
($\mathbf{a}$,X)	1/3	**1/6**
Z		1/2

$\mathbf{a}.n +_{1/3} Z =_0 Z +_{2/3} \mathbf{a}.n$ (SC)

$=_0 (Z +_{1/4} Z) +_{2/3} \mathbf{a}.n$ (B2)

$=_0 Z +_{1/6} (\mathbf{a}.n +_{2/5} Z)$ (SA)+(SC)

from what we have seen before and
(Pref)+(Top)+(IB) we obtain:

$$\{m =_{\varepsilon} n\} \vdash m =_{1/3\varepsilon+1/6} n$$

from what we have seen before and
(Pref)+(Top)+(IB) we obtain:

$$\{m =_{\varepsilon} n\} \vdash m =_{1/3\varepsilon+1/6} n$$

$$\textbf{(Top)} \vdash m =_{1} n$$

from what we have seen before and
(Pref)+(Top)+(IB) we obtain:

$$\{m =_{\varepsilon} n\} \vdash m =_{1/3\varepsilon+1/6} n$$

$$\textbf{(Top)} \vdash m =_1 n$$

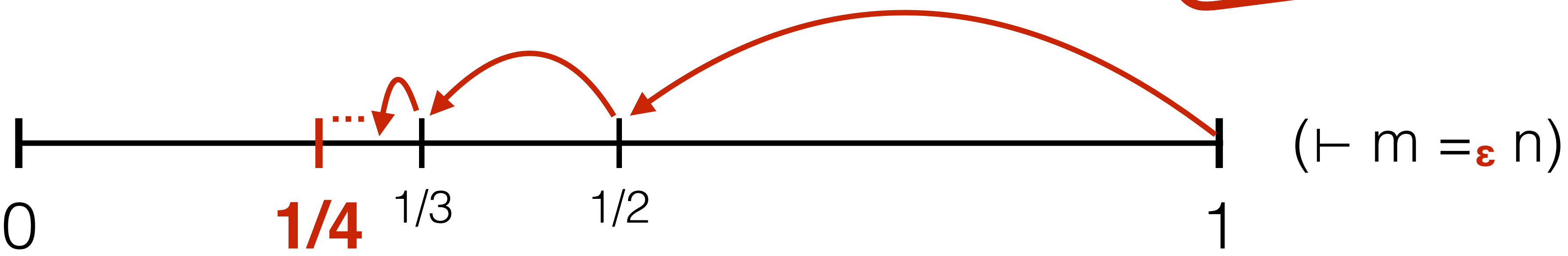
greatest fixed point operator

from what we have seen before and
(Pref)+(Top)+(IB) we obtain:

$$\{m =_{\varepsilon} n\} \vdash m =_{1/3\varepsilon+1/6} n$$

$$\textbf{(Top)} \vdash m =_1 n$$

greatest fixed point operator



(Max) $\{t =_{\varepsilon} s\} \vdash t =_{\varepsilon+\delta} s$ — for $\delta>0$

(Arch) $\{t =_{\delta} s \mid \delta>\varepsilon\} \vdash t =_{\varepsilon} s$

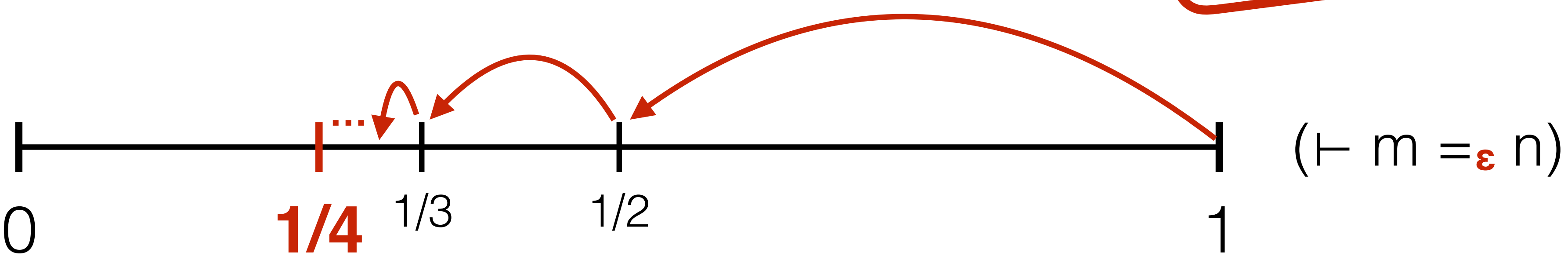
$\Rightarrow$ $\vdash m =_{1/4} n$

from what we have seen before and
(Pref)+(Top)+(IB) we obtain:

$$\{m =_{\varepsilon} n\} \vdash m =_{1/3\varepsilon+1/6} n$$

$$\textbf{(Top)} \vdash m =_1 n$$

greatest fixed point operator



(Max) $\{t =_{\varepsilon} s\} \vdash t =_{\varepsilon+\delta} s$ — for $\delta>0$

(Arch) $\{t =_{\delta} s \mid \delta>\varepsilon\} \vdash t =_{\varepsilon} s$

$\Rightarrow$ $\vdash m =_{1/4} n$

$d_1(m,n)$

What about generic k≥1?

$$m =_0 \mathbf{a}.m +_{1/2} Z \xrightarrow{\text{(Pref)}} \mathbf{a}.m =_0 \mathbf{a}.(\mathbf{a}.m +_{1/2} Z)$$

$$=_0 \mathbf{a}.\mathbf{a}.m +_{1/2} \mathbf{a}.Z \quad \text{(Dist-pref)}$$

$$m =_0 (\mathbf{a}.\mathbf{a}.m +_{1/2} \mathbf{a}.Z) +_{1/2} Z$$

What about generic k≥1?

$$m =_0 \mathbf{a}.m +_{1/2} Z \quad \xrightarrow{\text{(Pref)}} \quad \mathbf{a}.m =_0 \mathbf{a}.(\mathbf{a}.m +_{1/2} Z)$$

(k=1)

$$=_0 \mathbf{a}.\mathbf{a}.m +_{1/2} \mathbf{a}.Z \quad \text{(Dist-pref)}$$

+1

$$m =_0 (\mathbf{a}.\mathbf{a}.m +_{1/2} \mathbf{a}.Z) +_{1/2} Z$$

(k=2)

What about generic k≥1?

(Pref)

$m =_0 \mathbf{a}.m +_{1/2} Z$ (k=1) → $\mathbf{a}.m =_0 \mathbf{a}.(\mathbf{a}.m +_{1/2} Z)$

$=_0 \mathbf{a}.\mathbf{a}.m +_{1/2} \mathbf{a}.Z$ (Dist-pref)

+1 → $m =_0 (\mathbf{a}.\mathbf{a}.m +_{1/2} \mathbf{a}.Z) +_{1/2} Z$ (k=2)

+1 → $m =_0 ((\mathbf{a}.\mathbf{a}.\mathbf{a}.m +_{1/2} \mathbf{a}.\mathbf{a}.Z) +_{1/2} \mathbf{a}.Z) +_{1/2} Z$ (k=3)

+1 → …

What about generic k≥1?

(Pref)

$m =_0 \mathbf{a}.m +_{1/2} Z$ $\longrightarrow$ $\mathbf{a}.m =_0 \mathbf{a}.(\mathbf{a}.m +_{1/2} Z)$

(k=1)

$=_0 \mathbf{a}.\mathbf{a}.m +_{1/2} \mathbf{a}.Z$ (Dist-pref)

+1

$m =_0 (\mathbf{a}.\mathbf{a}.m +_{1/2} \mathbf{a}.Z) +_{1/2} Z$

(k=2)

+1

$m =_0 ((\mathbf{a}.\mathbf{a}.\mathbf{a}.m +_{1/2} \mathbf{a}.\mathbf{a}.Z) +_{1/2} \mathbf{a}.Z) +_{1/2} Z$

(k=3)

+1 …

1. for each k≥1 we proceed as before to compute d_k
2. by (Arch)+(Max) we converge to **tv**

A quantitative Kleene's theorem

finite MCs

$(\mathrm{MC}/{\approx}, \mathrm{tv})$

$(\mathrm{Exp}/{=}, d_{\vdash})$

A quantitative Kleene's theorem

finite MCs

$(MC/\approx, tv)$

$(Exp/=, d_{\vdash})$

$d_{\vdash}([t],[s]) = \inf\{ \varepsilon \mid \vdash t =_{\varepsilon} s \}$

A quantitative Kleene's theorem

finite MCs

$$(MC/\approx,\ tv)$$

$$\cong$$ isometric isomorphism

$$(Exp/\equiv,\ d_{\vdash})$$

$$d_{\vdash}([t],[s]) = \inf\{\ \varepsilon \mid\ \vdash t =_{\varepsilon} s\ \}$$

Conclusions

Conclusions

- Sound&Complete Axiomatization
- Quantitative Kleene's Theorem

Conclusions

- Sound&Complete Axiomatization
- Quantitative Kleene's Theorem

future work…

Conclusions

- Sound&Complete Axiomatization
- Quantitative Kleene's Theorem

future work…

- Different models? (e.g., non-determinism)

Conclusions

- Sound&Complete Axiomatization
- Quantitative Kleene's Theorem

future work…

- Different models? (e.g., non-determinism)
- Coalgebraic generalization?
  (like Silva-Bonchi-Bonsangue-Rutten'11)

Conclusions

- Sound&Complete Axiomatization
- Quantitative Kleene's Theorem

future work…

- Different models? (e.g., non-determinism)
- Coalgebraic generalization?
  (like Silva-Bonchi-Bonsangue-Rutten'11)
- Beyond non-expansive operators

Thank you
for your attention