Converging from Branching to Linear Metrics on MCs

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ICTAC 2015

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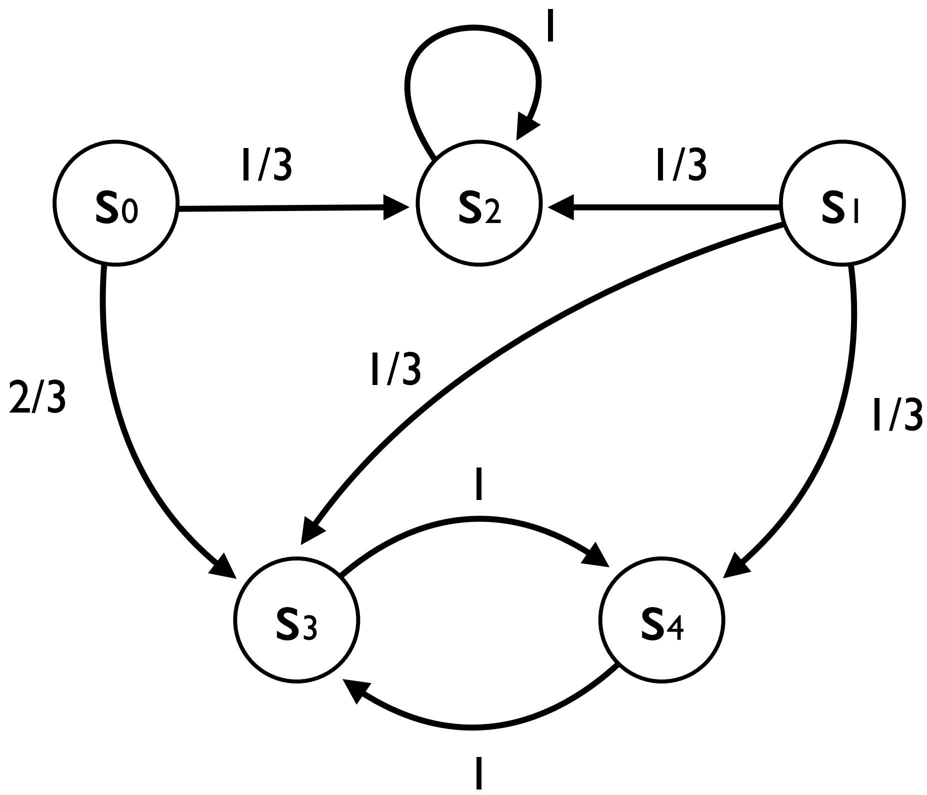
- in particular: ***Linear-time Properties***
    - observables are execution runs (no internal access!)
    - **Why?** --systems biology, machine learning, artificial intelligence, security, etc.

Markov Chains



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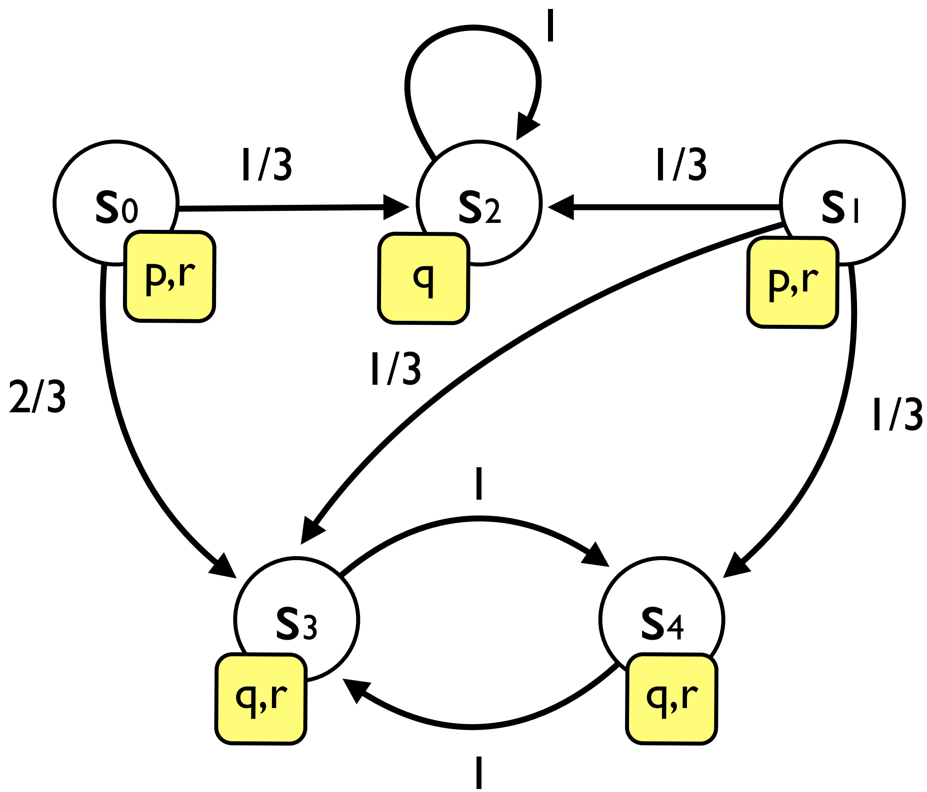

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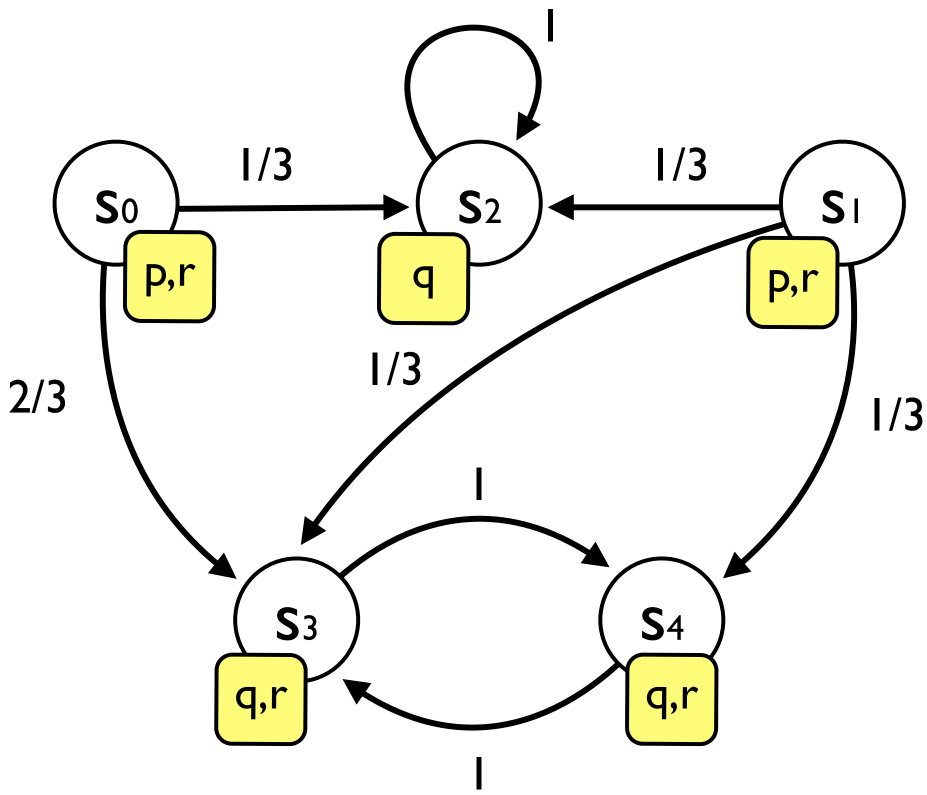

Given an initial state, MCs can be interpreted as “machines” that emit infinite traces of states with a certain probability

Measurable Events

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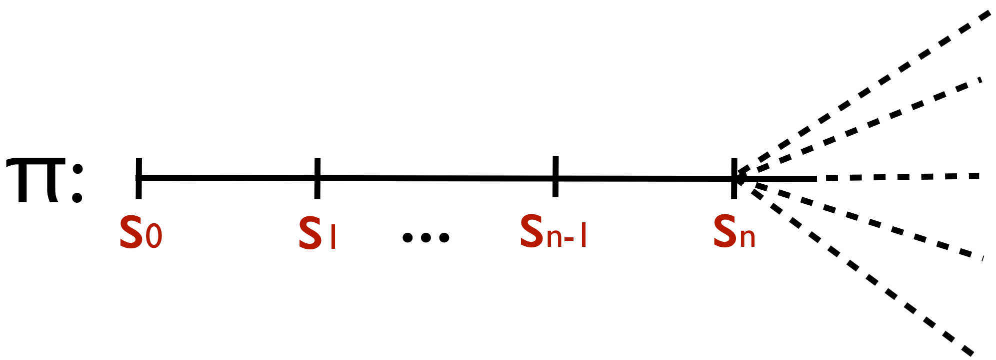
Cylinder set (or cone)
(with prefix $s_0 \ldots s_n \in S_0 \ldots S_n$)



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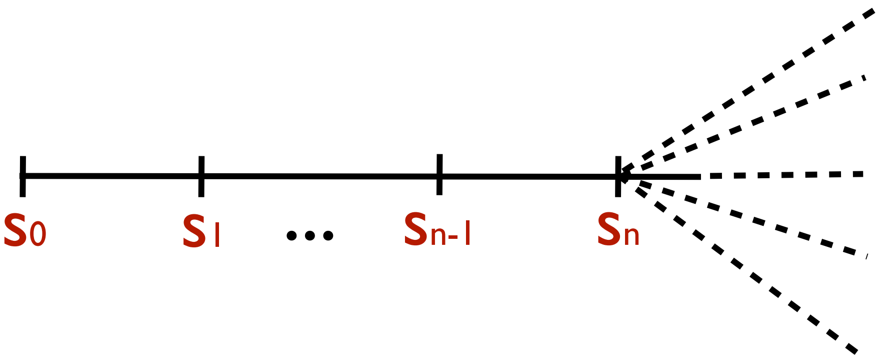

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$P(s)(\mathfrak{C}(S_0 \ldots S_n)) =$ "probability that, *starting from s*, the MC emits a path with prefix in $S_0 \ldots S_n$"

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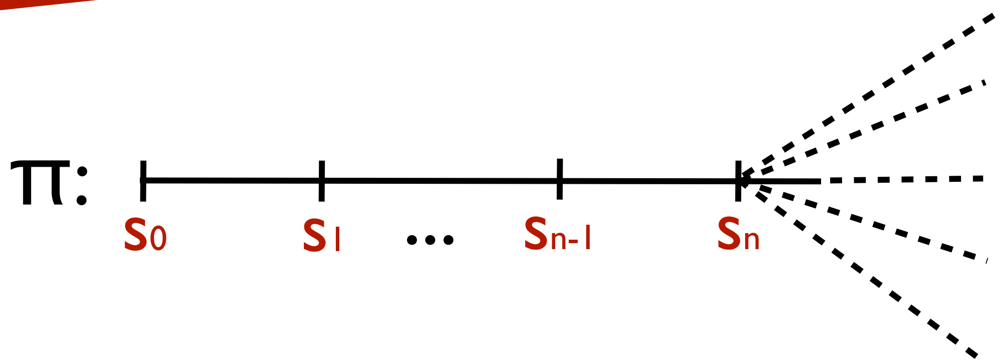

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(Pnueli)

Linear Temporal Logic

Atomic prop. — Next — Until

$$\varphi \coloneqq p \mid \bot \mid \varphi \rightarrow \varphi \mid X\varphi \mid \varphi U \varphi$$

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$$P(s)([\varphi]) = ?$$

What is the probabability that the MC with initial state s satisfies the formula φ?

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- Model Checking does not scale to large systems (even after model reduction, symbolic tecn., partial-order reduction, etc.)
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- **Proposed solution:**
  Behavioral metrics to quatify the error

A distance for approx. Model Checking

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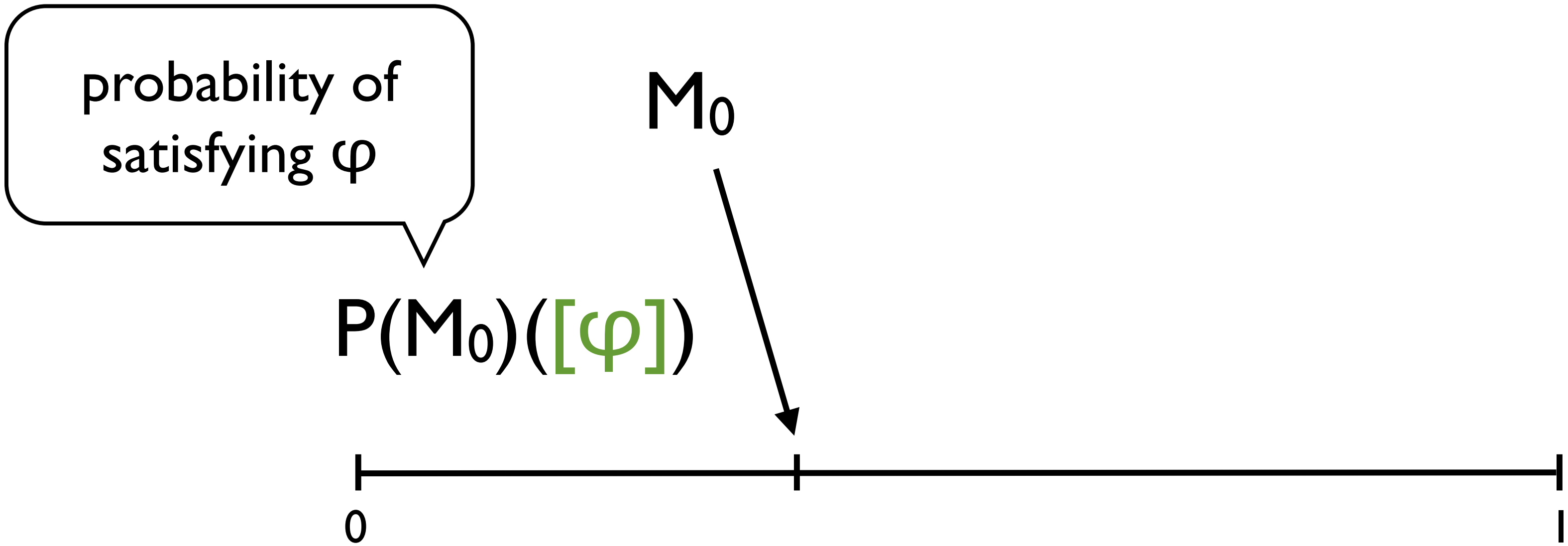

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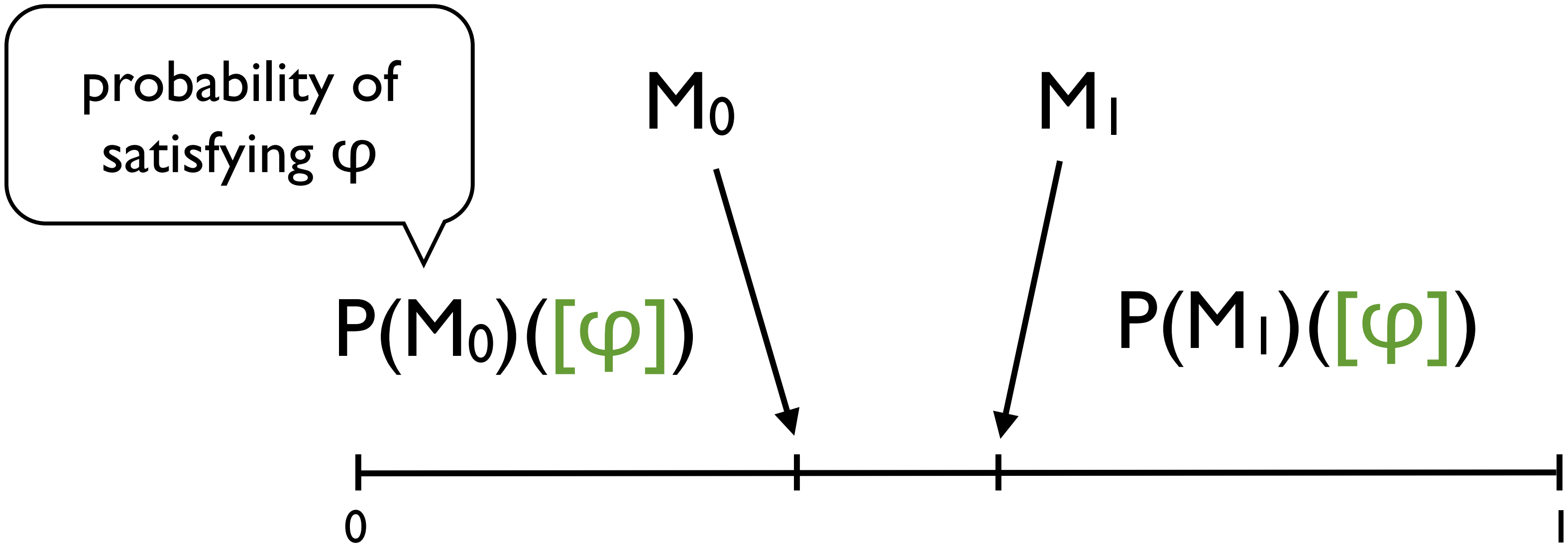

$$|P(M_0)([\varphi]) - P(M_1)([\varphi])|$$

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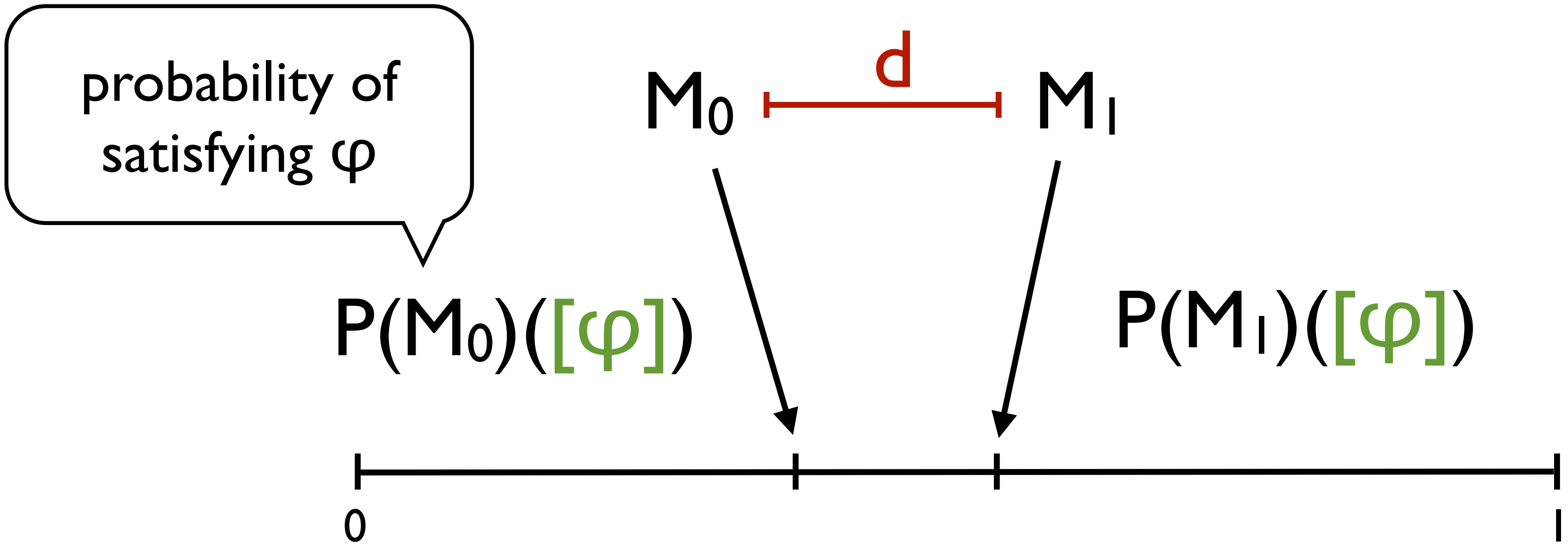

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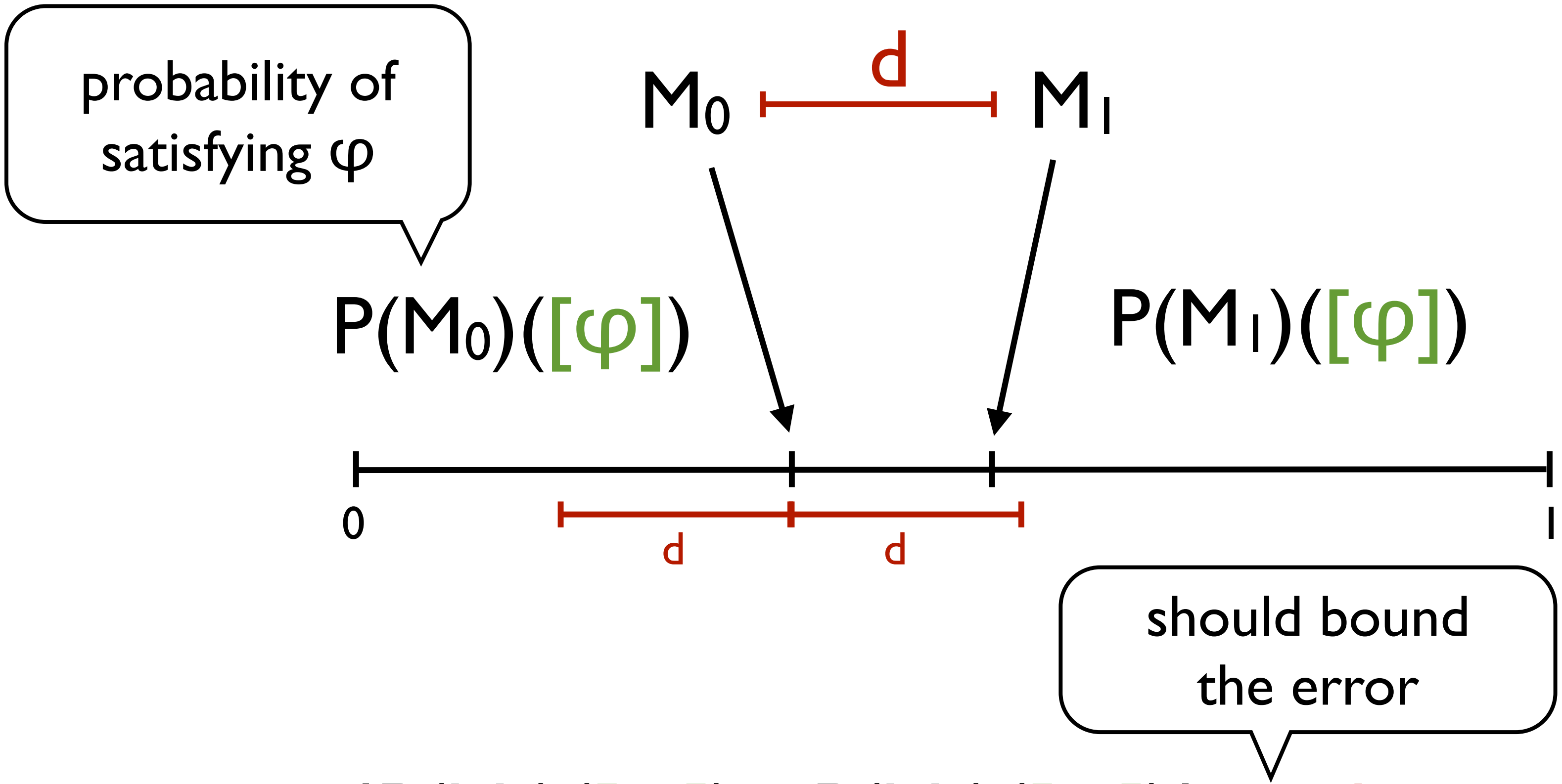

$$|P(M_0)([\varphi]) - P(M_1)([\varphi])| \leq d$$

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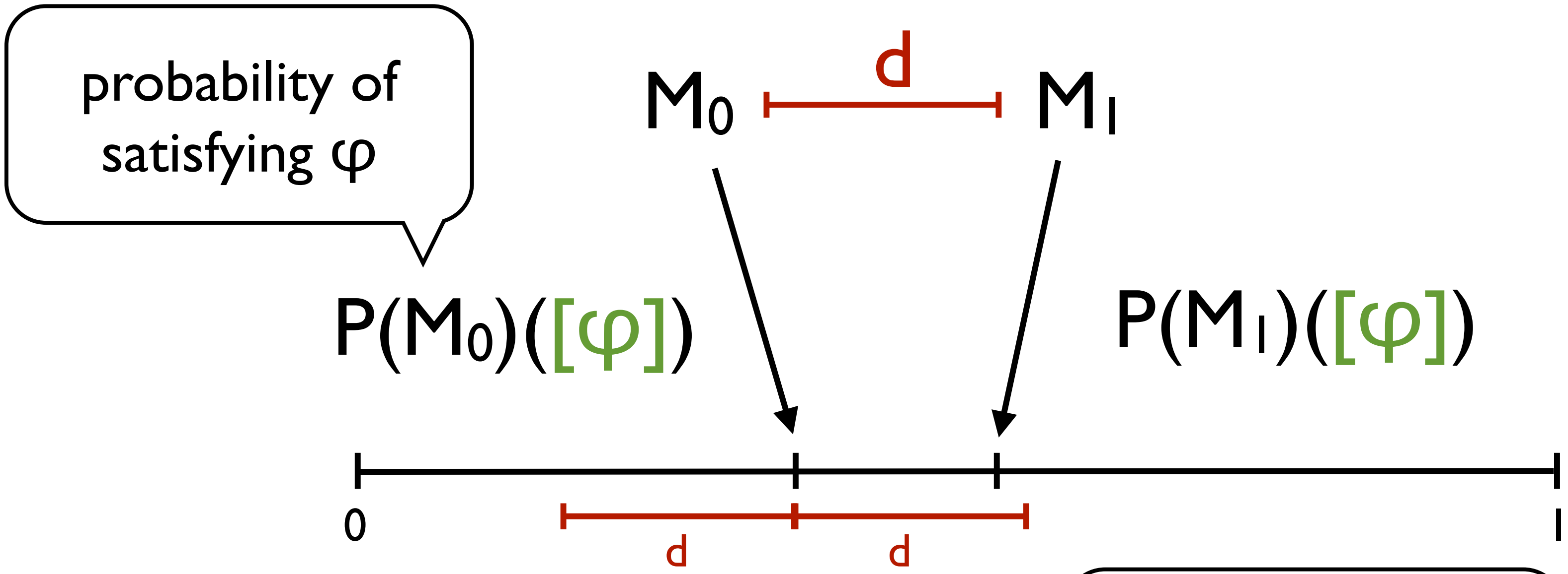

should bound the error

$$|P(M_0)([\varphi]) - P(M_1)([\varphi])| \leq d$$

for all formulas!

Two logical distances

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the LTL distance

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LTL without next operator

Two logical distances

Three natural questions

Q1: Can we compute the two metrics?

Q2: Can we compute them exactly?
If not, can we approximate them
to any arbitrary precision?

Q3: What about complexity?

Characterizations

Trace distance

$$T(s,t) = \sup_{E \in \sigma(\mathcal{T})} |P(s)(E) - P(t)(E)|$$

Stutter-trace distance

$$ST(s,t) = \sup_{E \in \sigma(\mathcal{ST})} |P(s)(E) - P(t)(E)|$$

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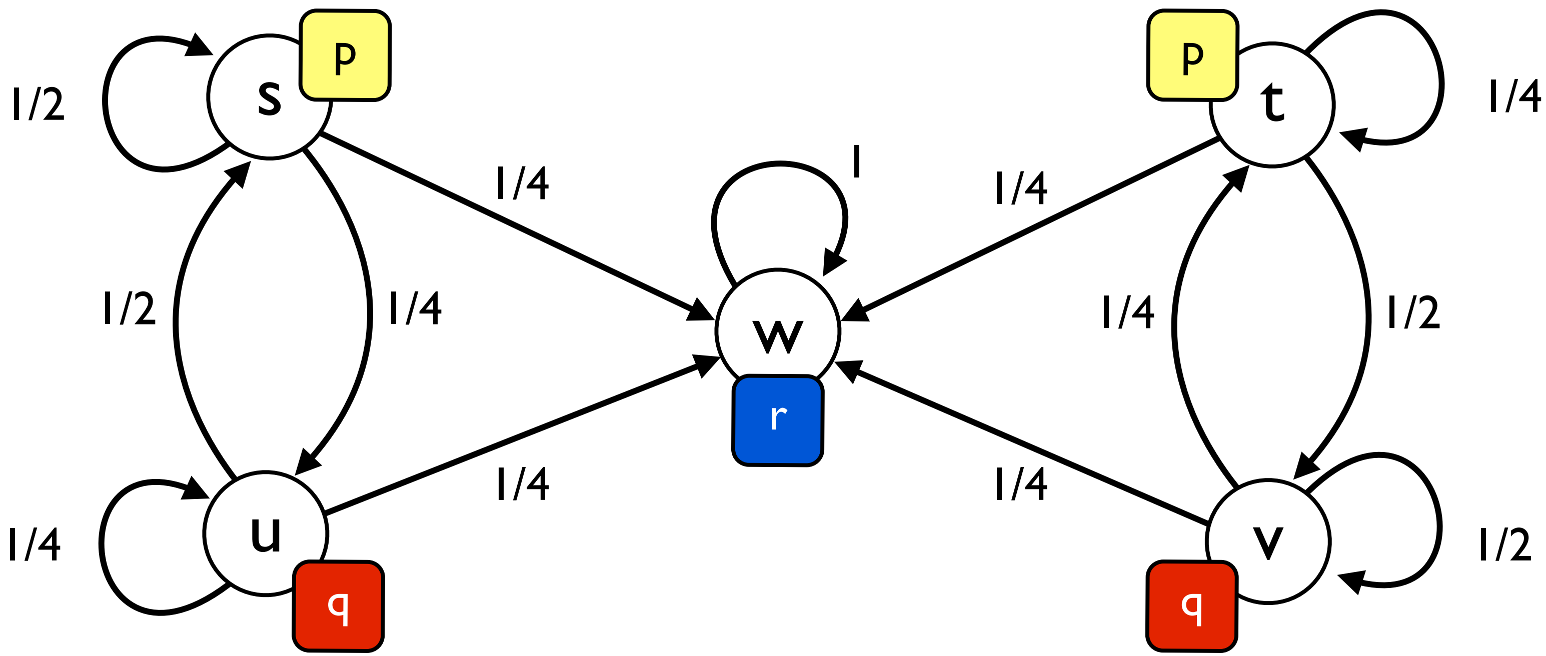
Characterization Theorem

$LTL(s,t) = T(s,t)$ and $LTL^{-x}(s,t) = ST(s,t)$

A tiny yet tricky example

(from Chen-Kiefer LICS'14)



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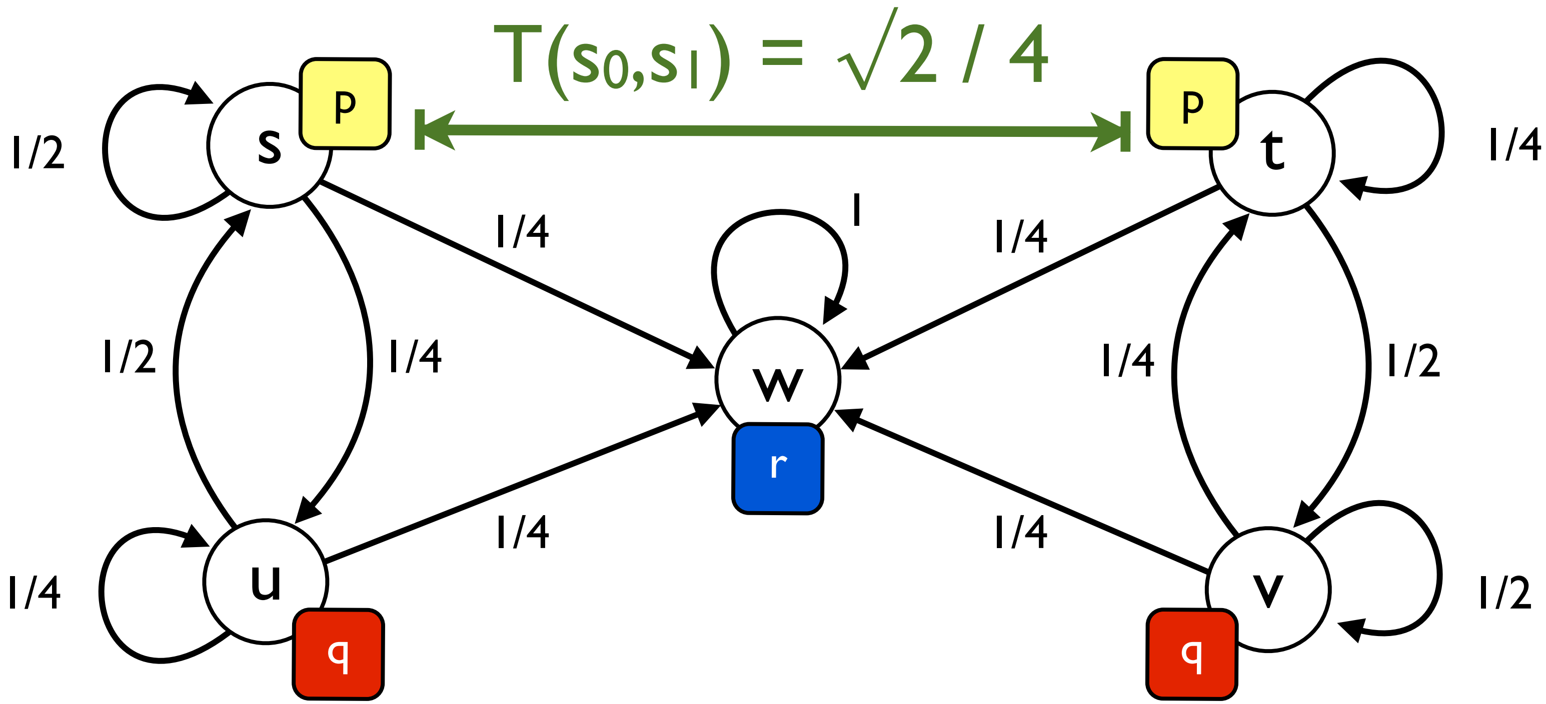

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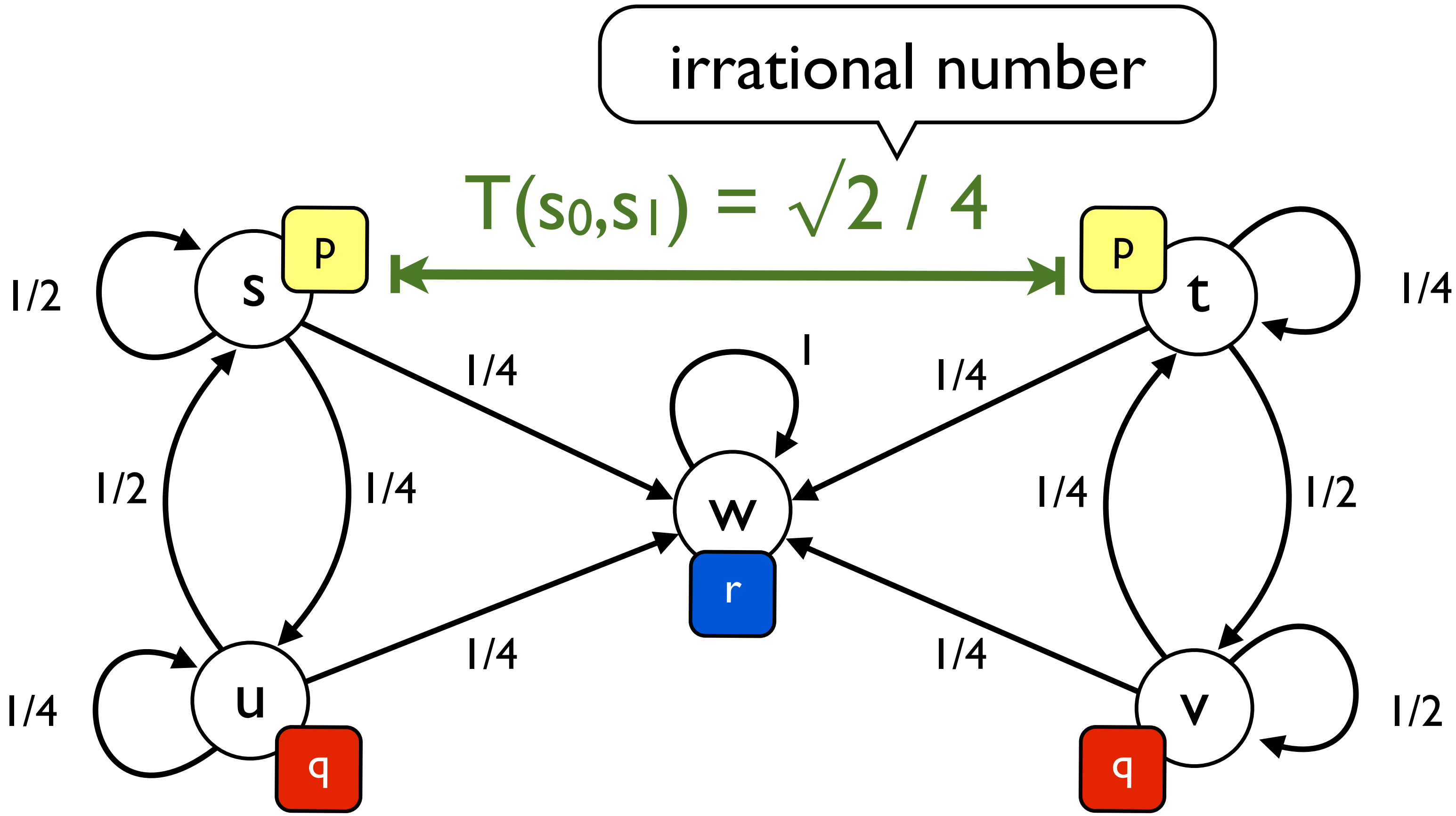

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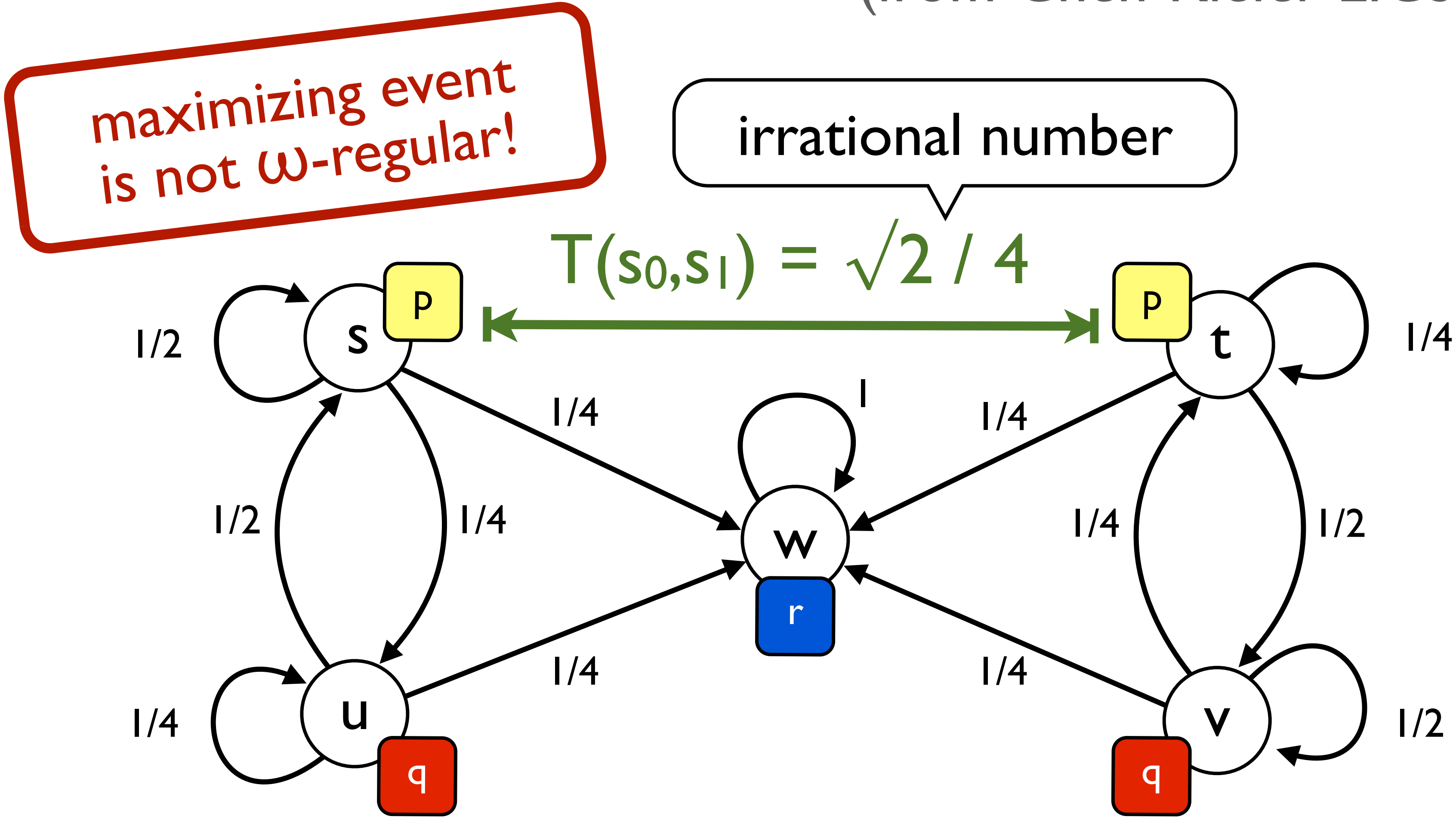

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Q: Can we approximate the logical/trace distances up to any arbitrary precision?

Approximation Algorithm

(in the slides only for the Trace Distance)

generalizes / improves
Chen-Kiefer LICS'14

Approximation Algorithm

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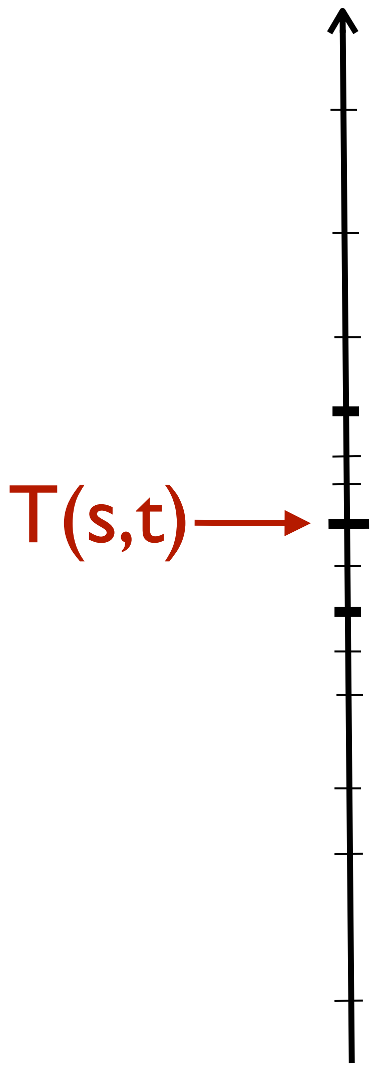
Approximation Schema
(general idea)



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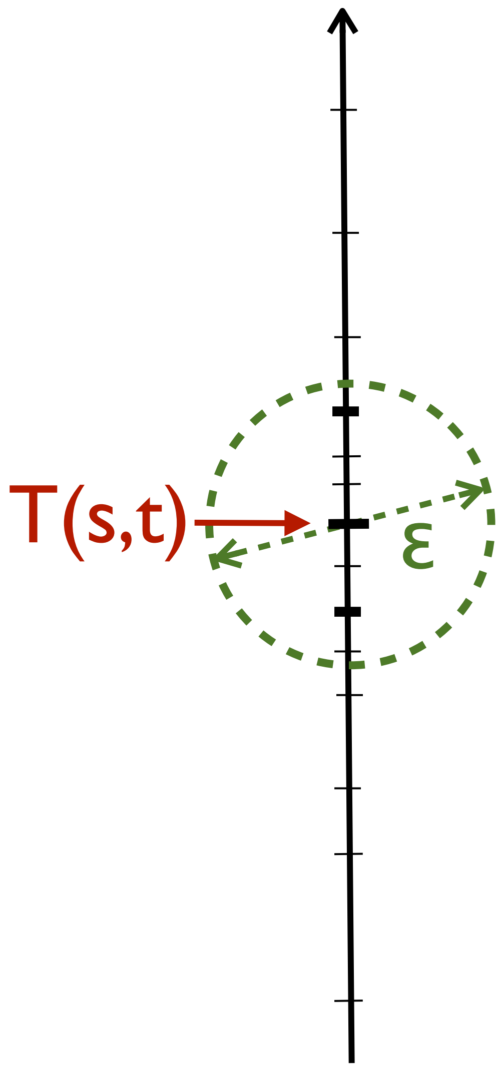

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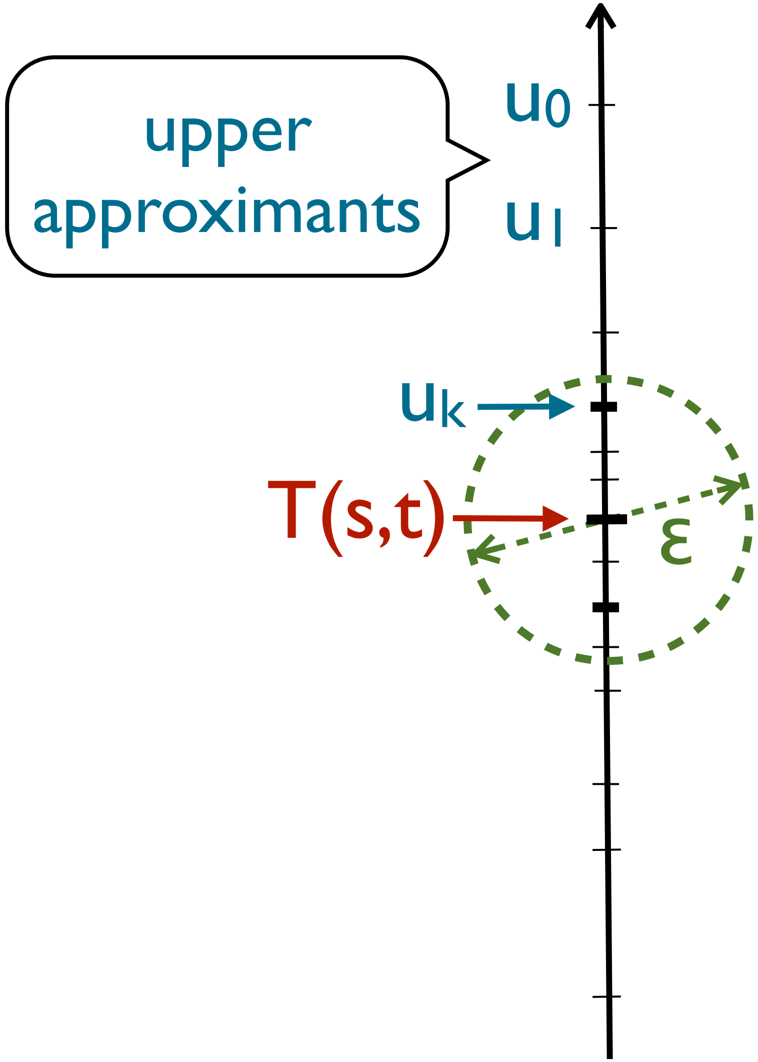

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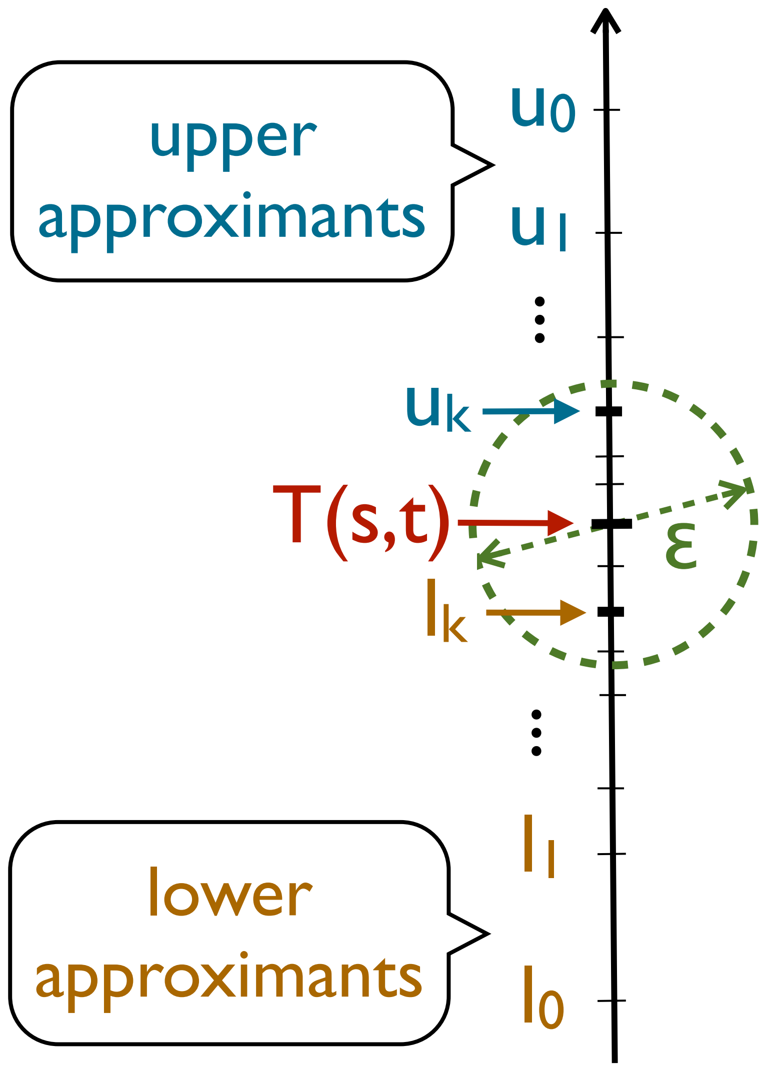

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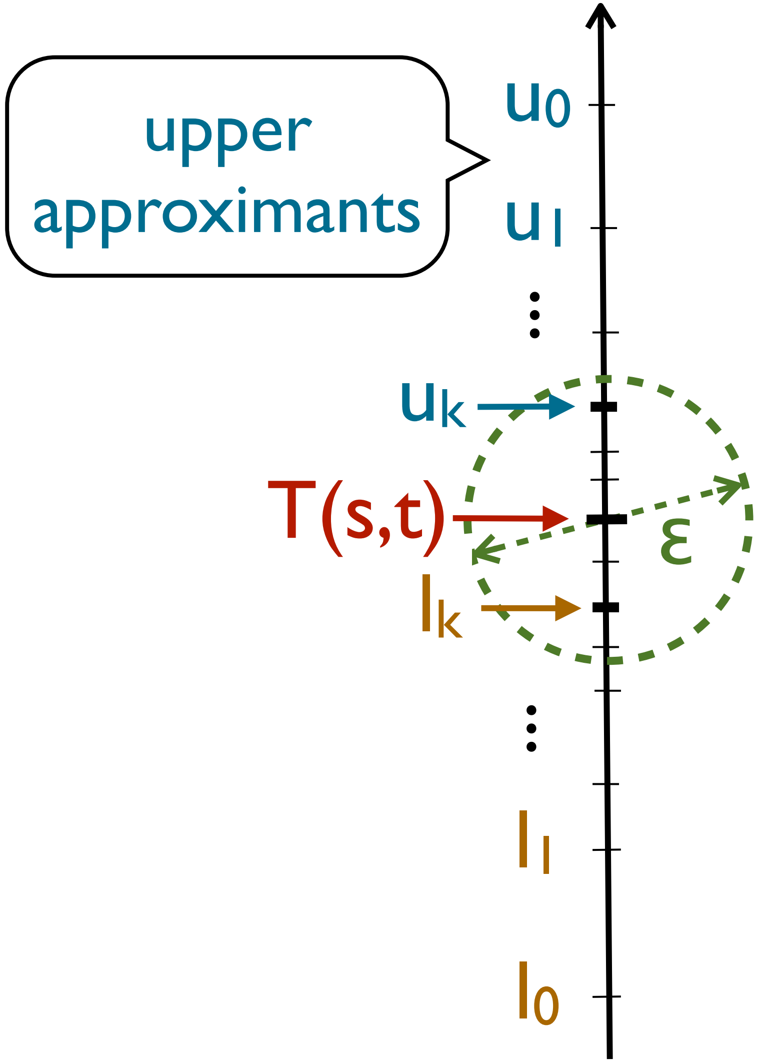

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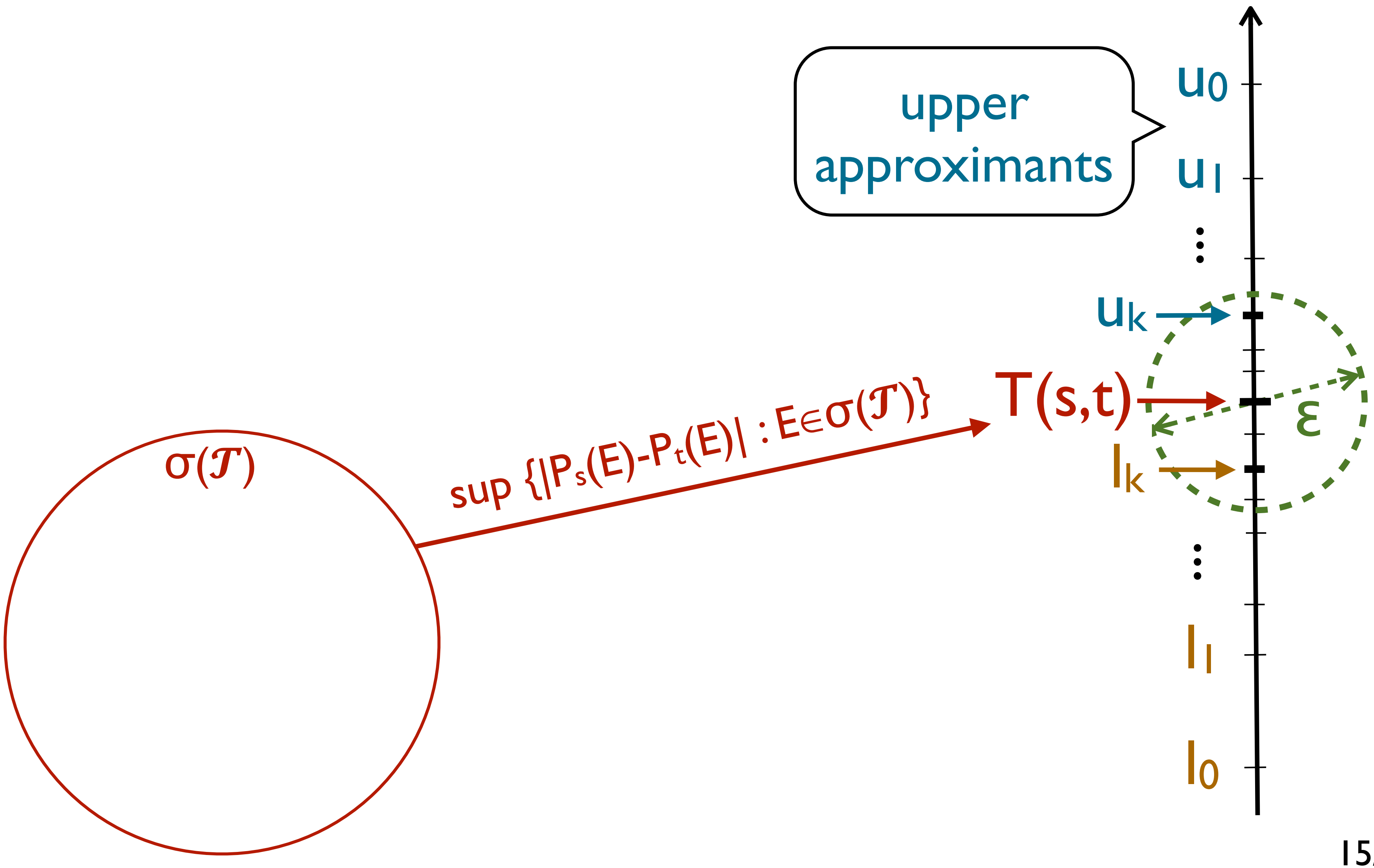

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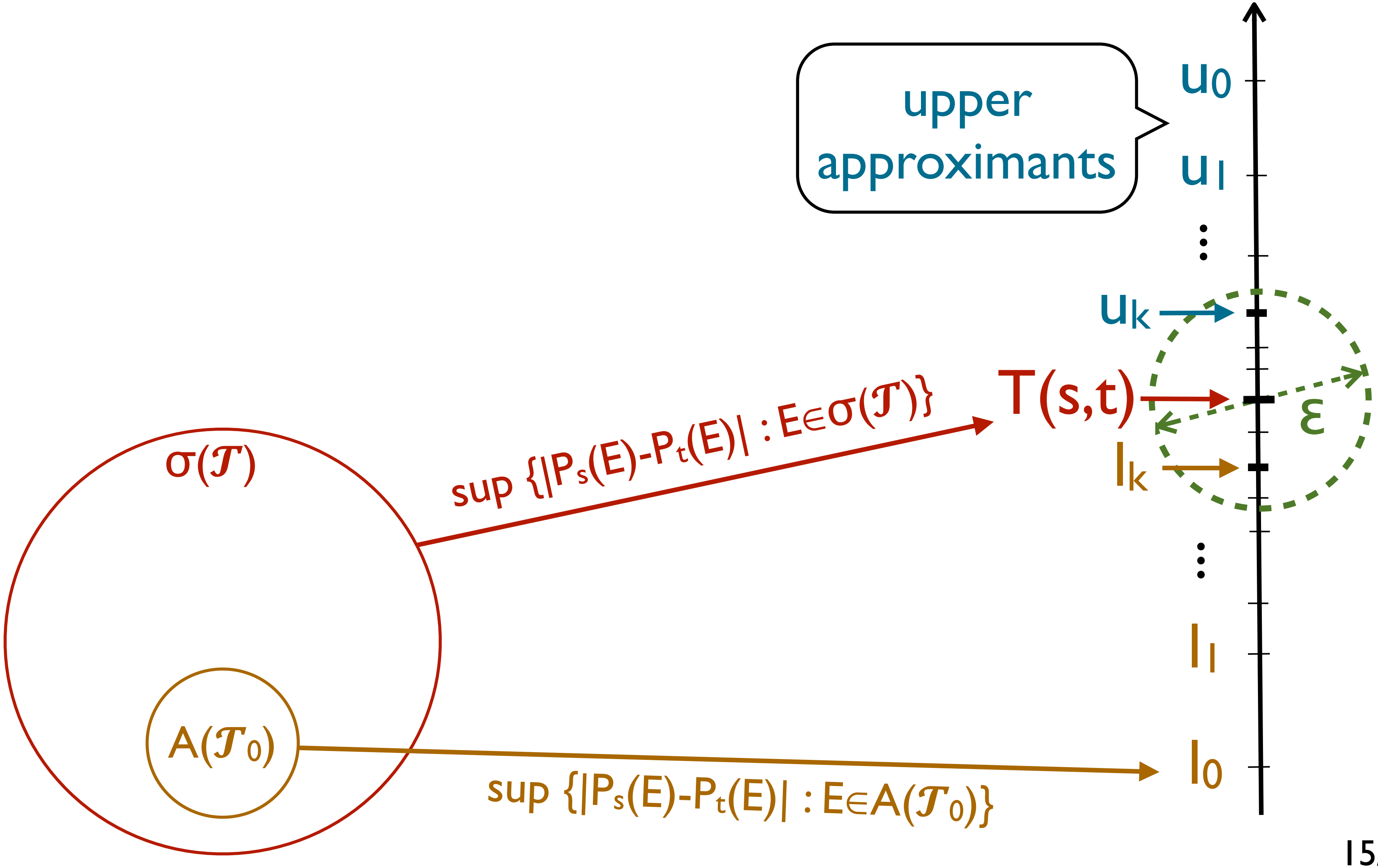

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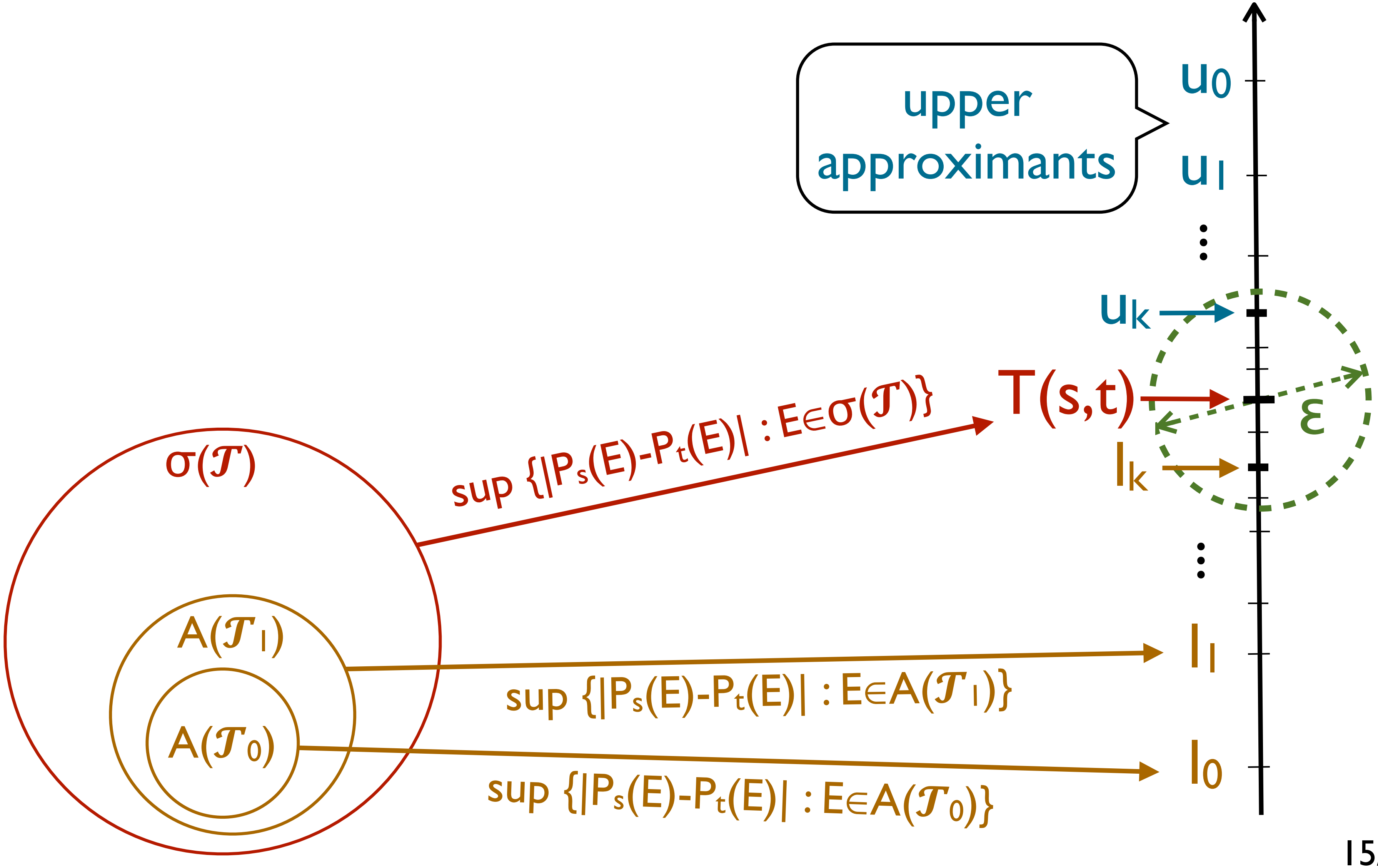

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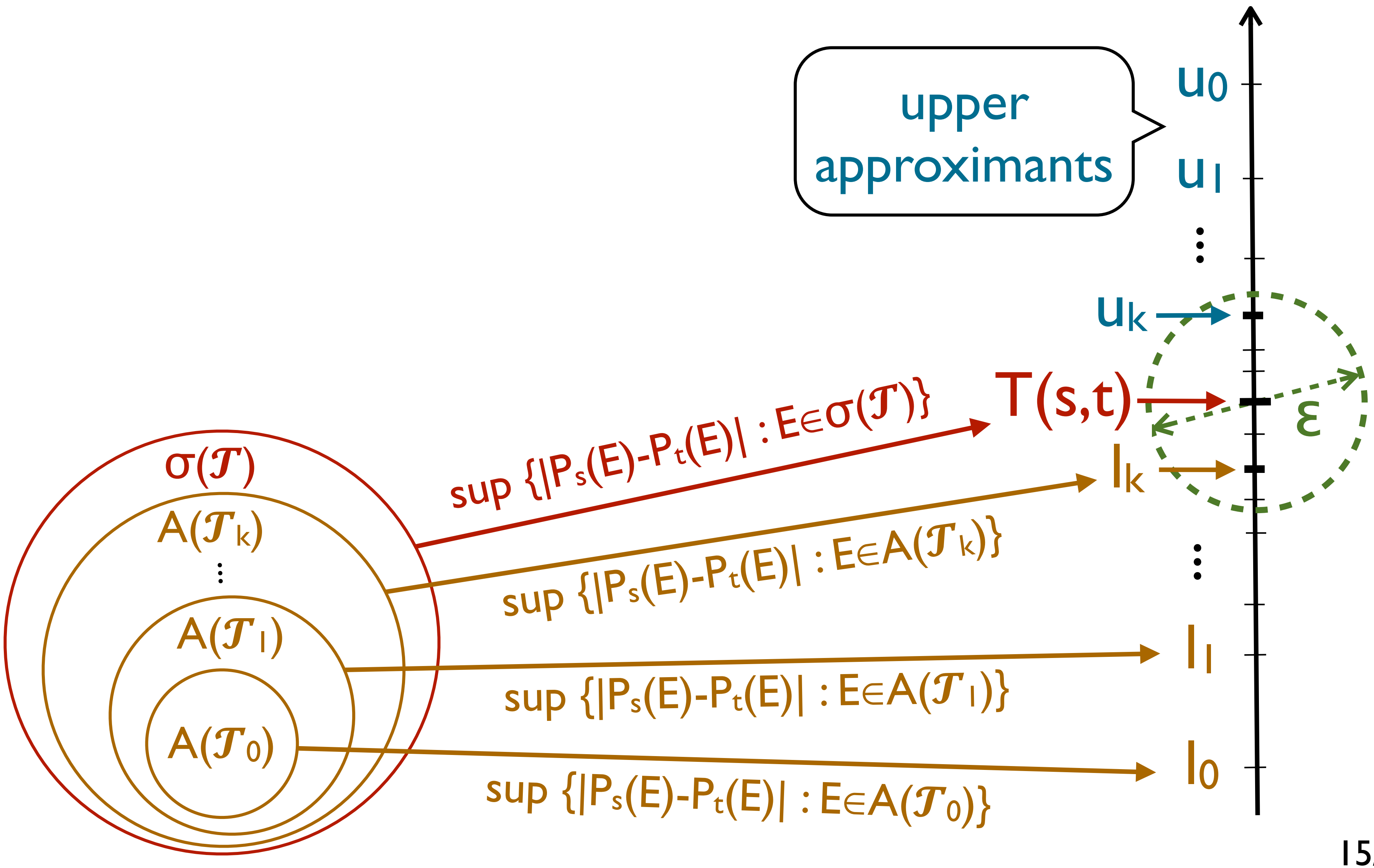

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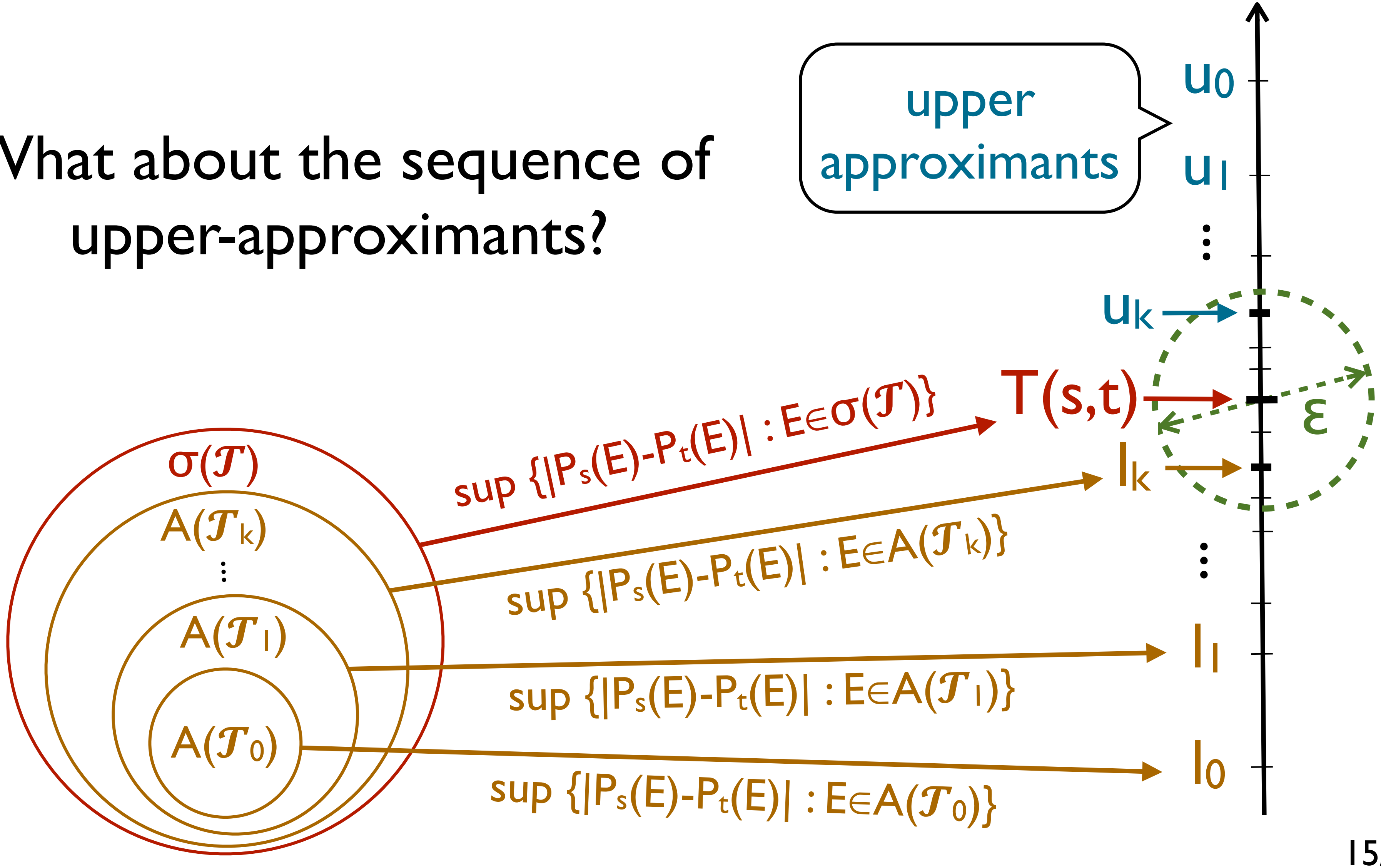
Approximation Schema

What about the sequence of upper-approximants?



Coupling Characterization

(as total variation distance)

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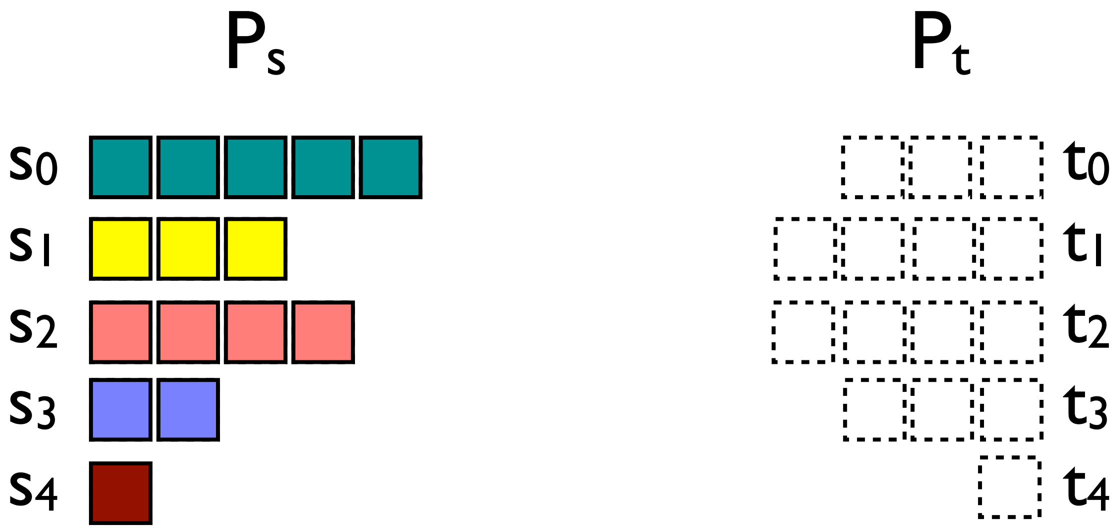

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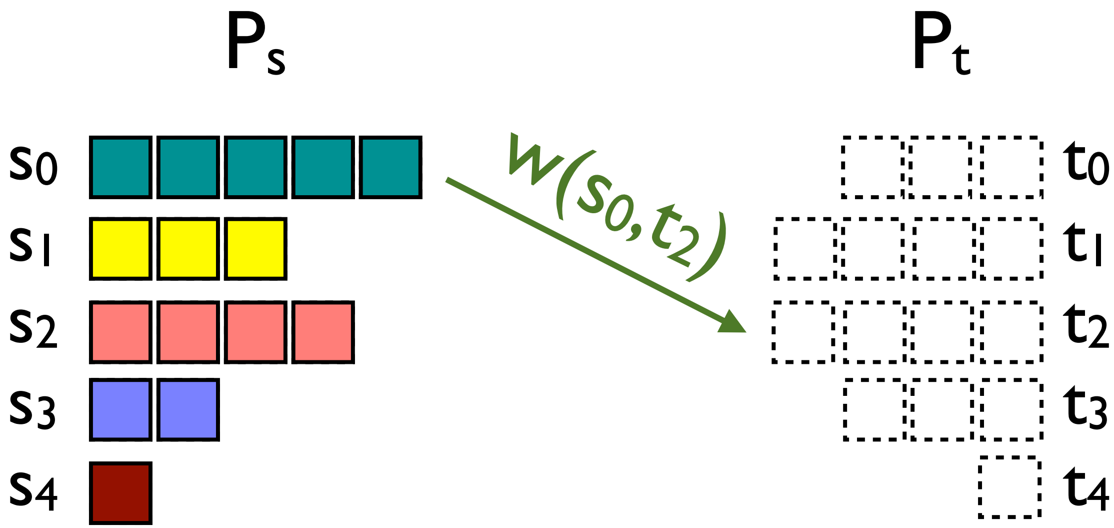

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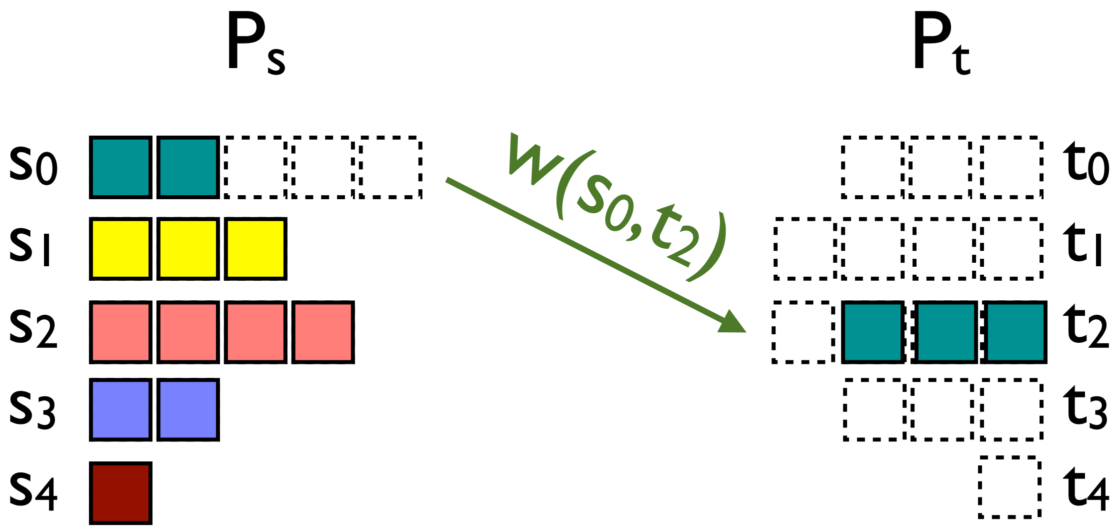

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trace inequivalence

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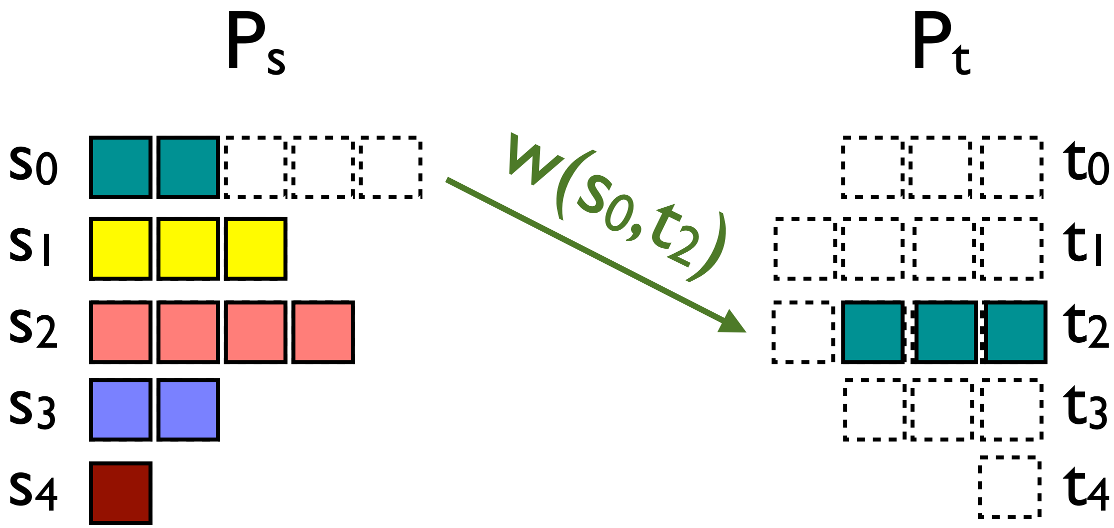

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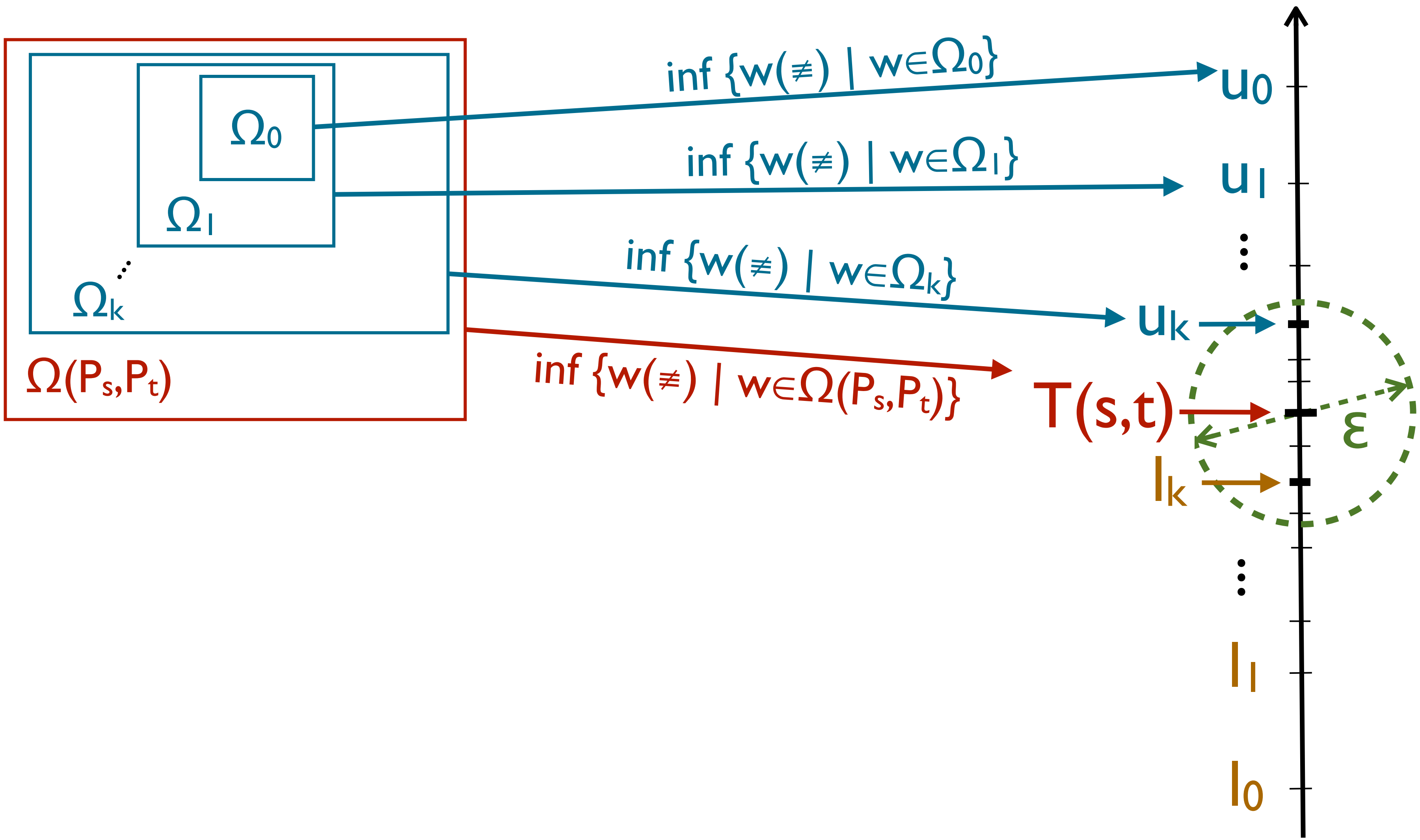

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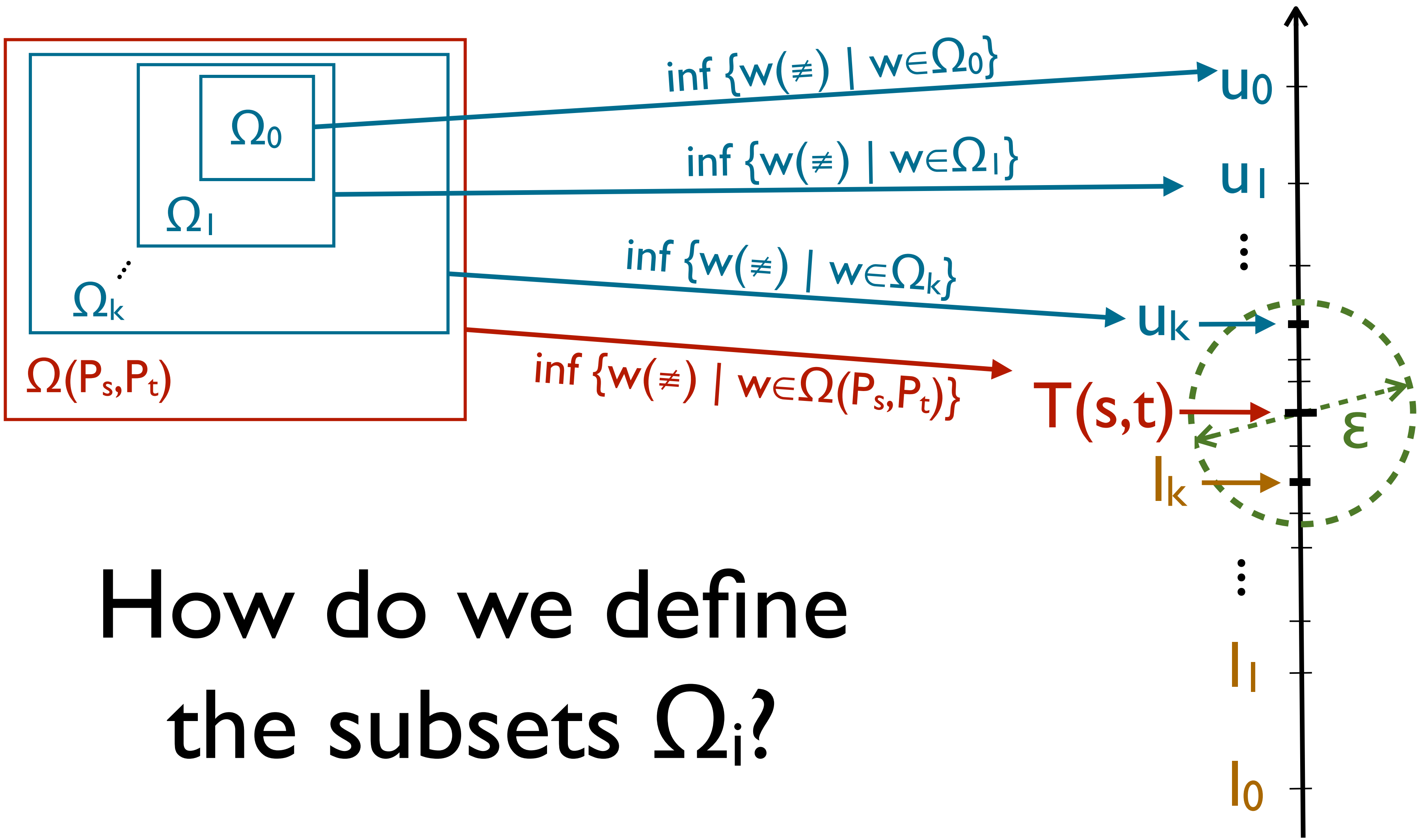

How do we define the subsets Ω_i?

Coupling Structure

Coupling Structure of rank k

$$\mathcal{C}: S \times S \rightarrow \Delta(S^k \times S^k)$$

such that $\mathcal{C}(s,t) \in \Omega(P(s)^k, P(t)^k)$

Stochastic process generating pairs of paths divided in multisteps of length k

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Lemma

(i) $\Omega_k \subseteq \Omega(P(s),P(t))$, (ii) $\Omega_k \subseteq \Omega_{hk}$ (for all $k,h>0$)

(iii) $\bigcup_k \Omega_k$ is dense in $\Omega(P(s),P(t))$

Computing the Approximants

(*) MC with rational transition probabilities

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proved via alternative characterizations

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proved via alternative characterizations

the threshold problem for T(s,t) is still NP-hard!

(*) MC with rational transition probabilities

Upper approx. are Branching Metrics!

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its kernel is Larsen-Skou probabilistic bisimilarity!

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Kantorovich lifting + ≢-selector

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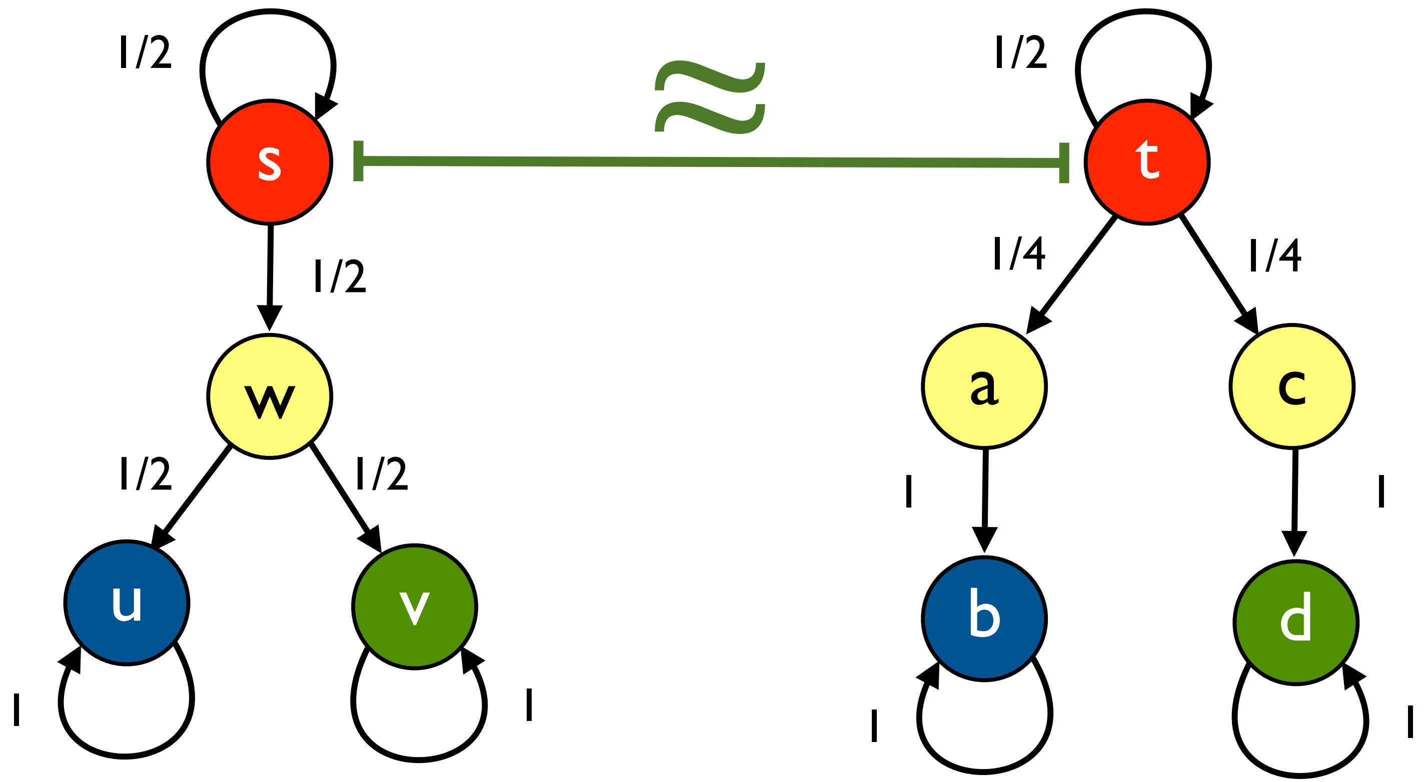
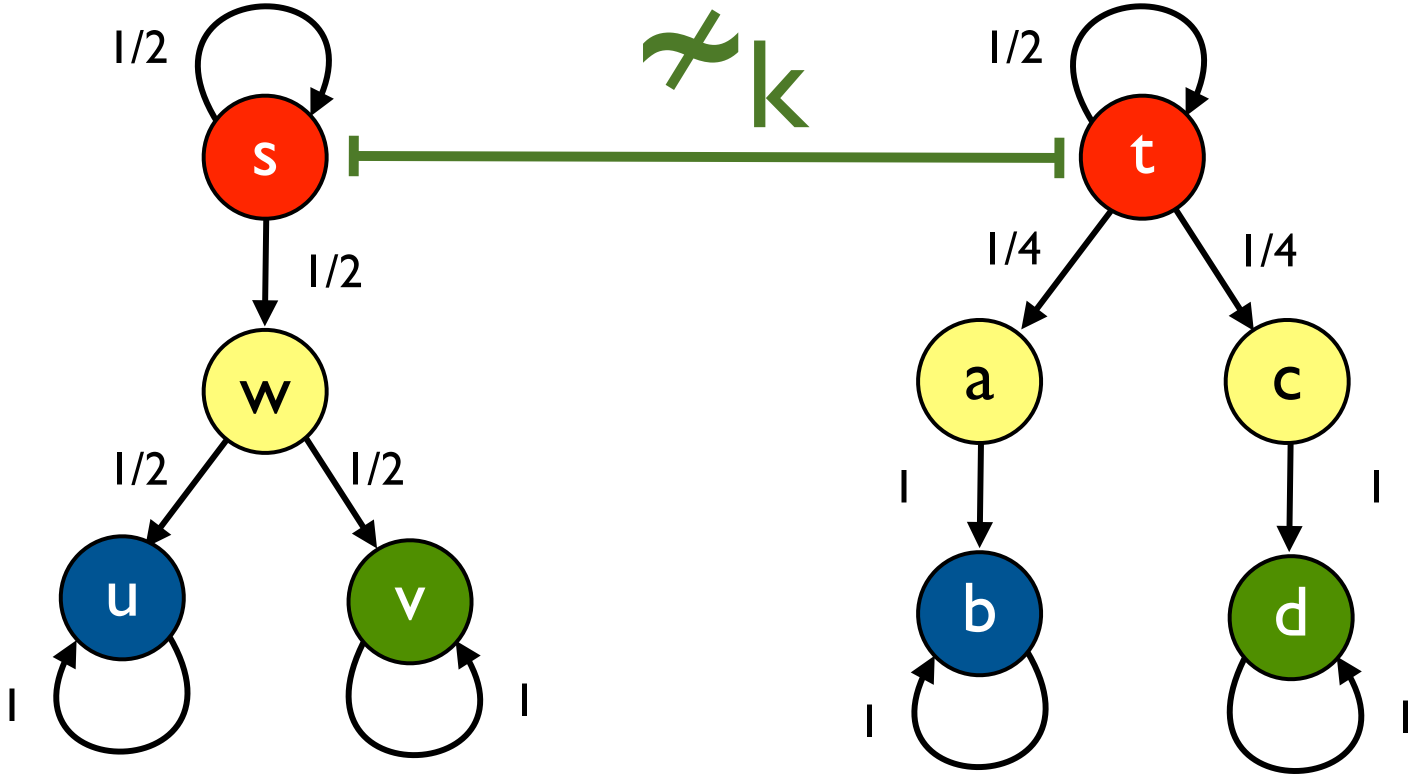
Exact semantics do NOT converge

	metric-based	equiv.-based
(monotone)	$u_k \geq u_{hk}$	$\sim_k \subseteq \sim_{hk}$
(bound)	$u_k \geq T$	$\sim_k \subseteq \approx$
(convergence)	$\inf_k u_k = T$	$\bigcup_k \sim_k \neq \approx$

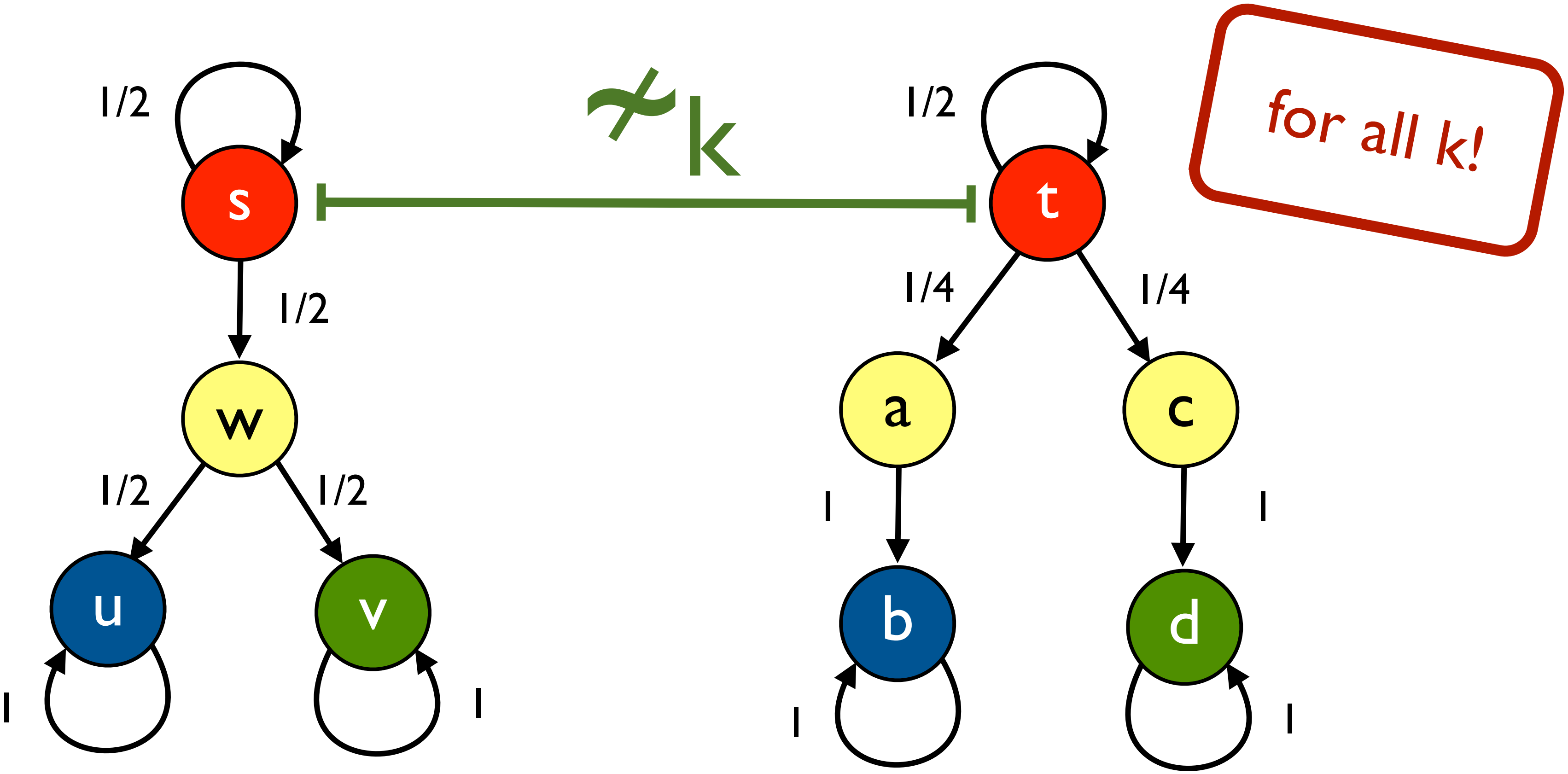
The Counterexample



The Counterexample



The Counterexample



Concluding Remarks

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- Metrics for Model Checking
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Future Work

- Better algorithms? (on-the-fly techniques)
- different kind of models (non-determinism?)
- explore topological properties

Thank you
for the attention

Appendix

The theorem behind...

For $\mu, \nu : \Sigma \to \mathbb{R}_+$ finite measures on (X, Σ)
and $F \subseteq \Sigma$ field such that $\sigma(F) = \Sigma$

Representation Theorem

$$\|\mu - \nu\| = \sup_{E \in F} |\mu(E) - \nu(E)|$$

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For $\mu, \nu : \Sigma \to \mathbb{R}_+$ finite measures on (X,Σ)
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Representation Theorem

$$\|\mu - \nu\| = \sup_{E \in F} |\mu(E) - \nu(E)|$$

F is much simpler than Σ, nevertheless
it suffices to attain the supremum!