On the Total Variation Distance of SMCs

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FoSSaCS '15

Outline

- Motivations
- Semi-Markov Chains (SMCs)
- Trace Distance vs Model Checking of SMCs
- Approximation Algorithm for Trace Distance
- Concluding Remarks

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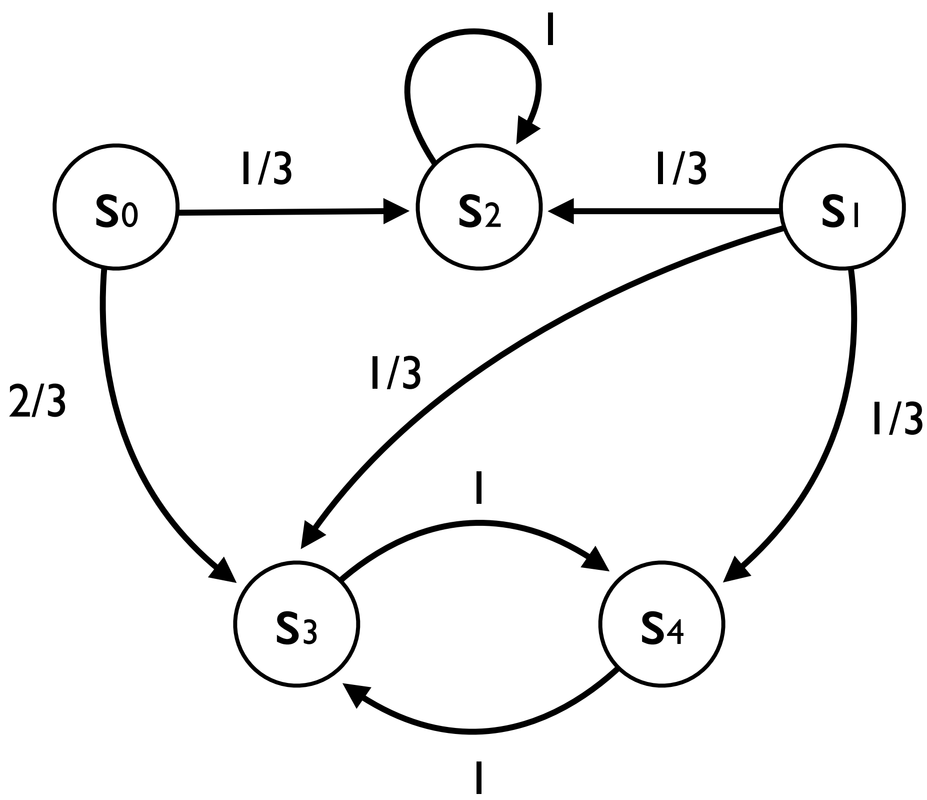
  tests over execution runs (no internal access!)
  - Example: systems biology, machine learning, artificial intelligence, security, ect.

semi-Markov Chains



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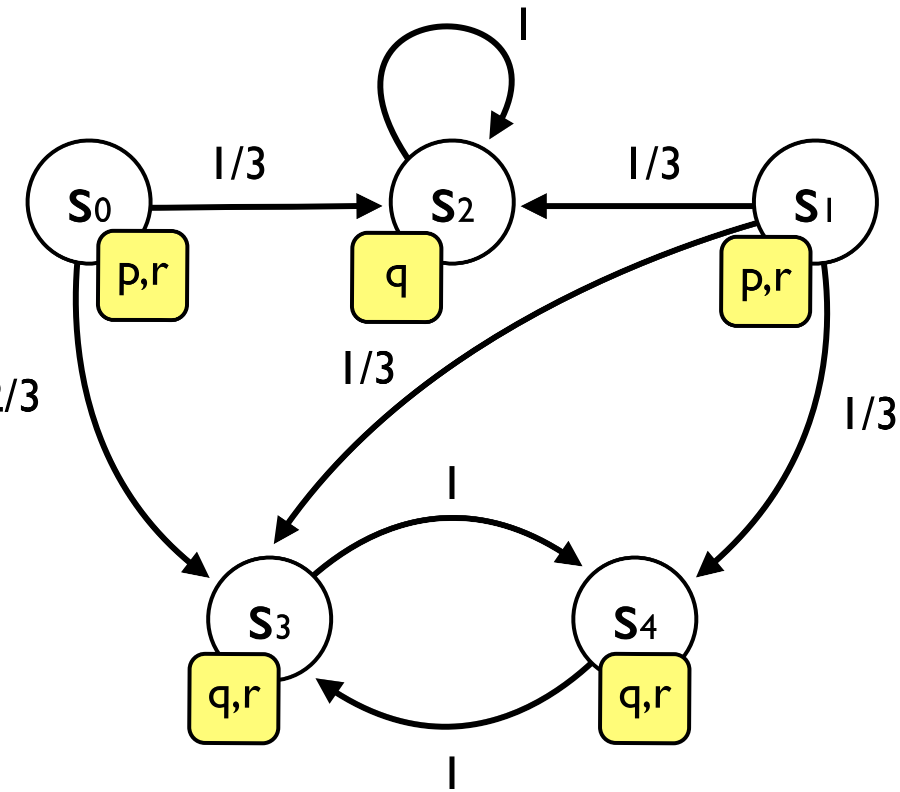

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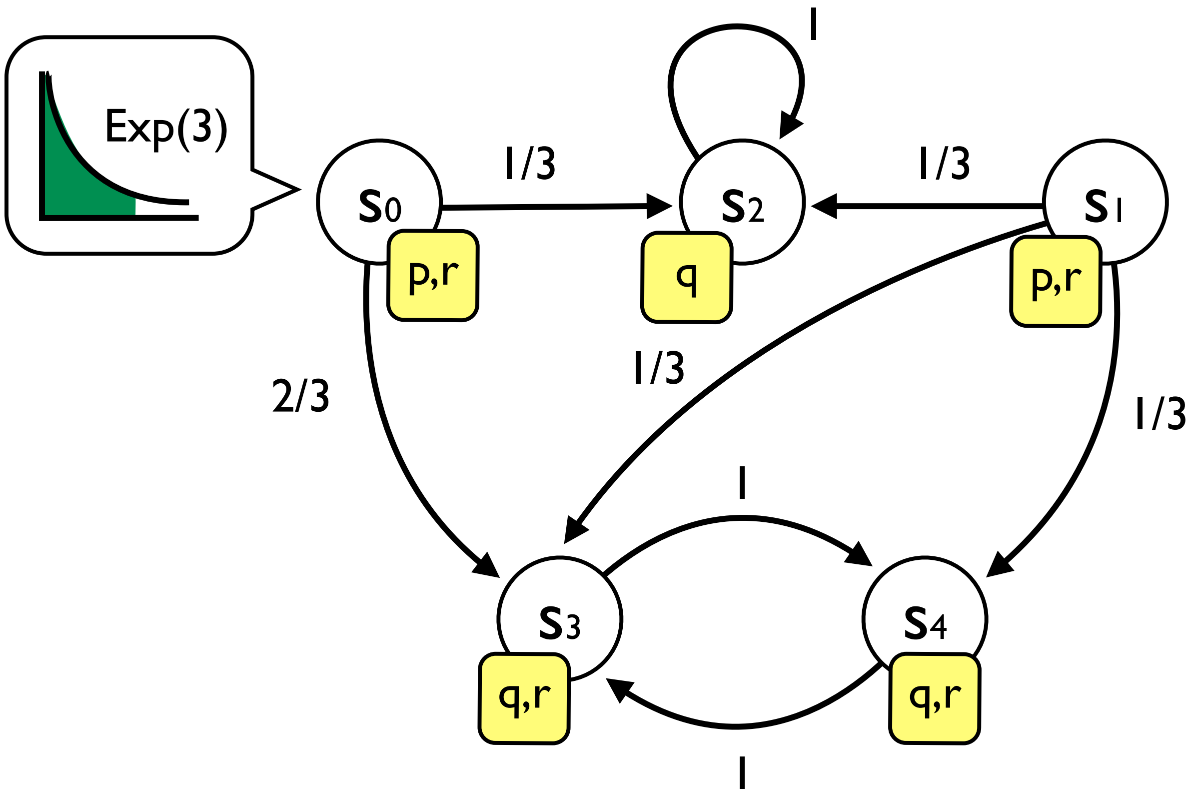

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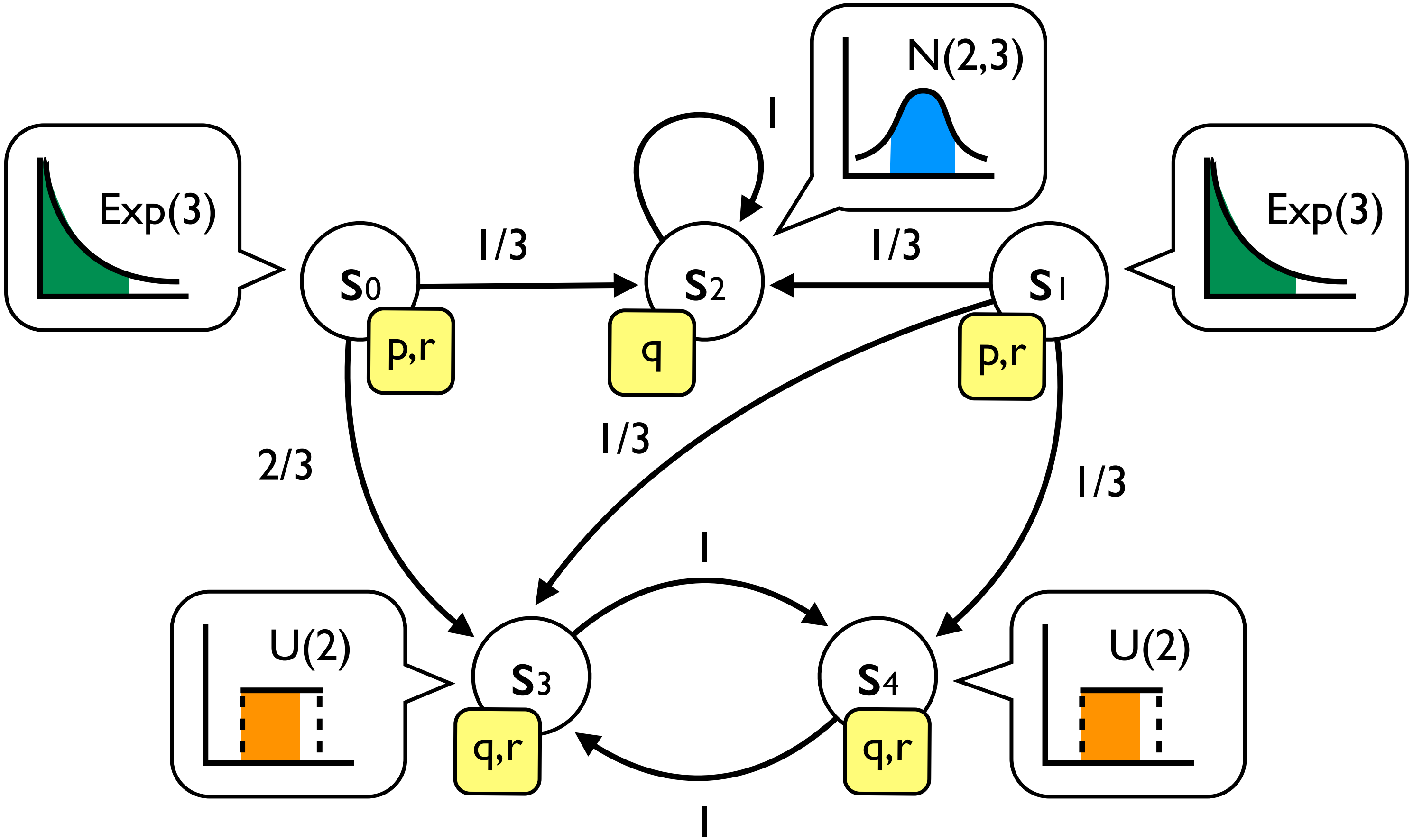

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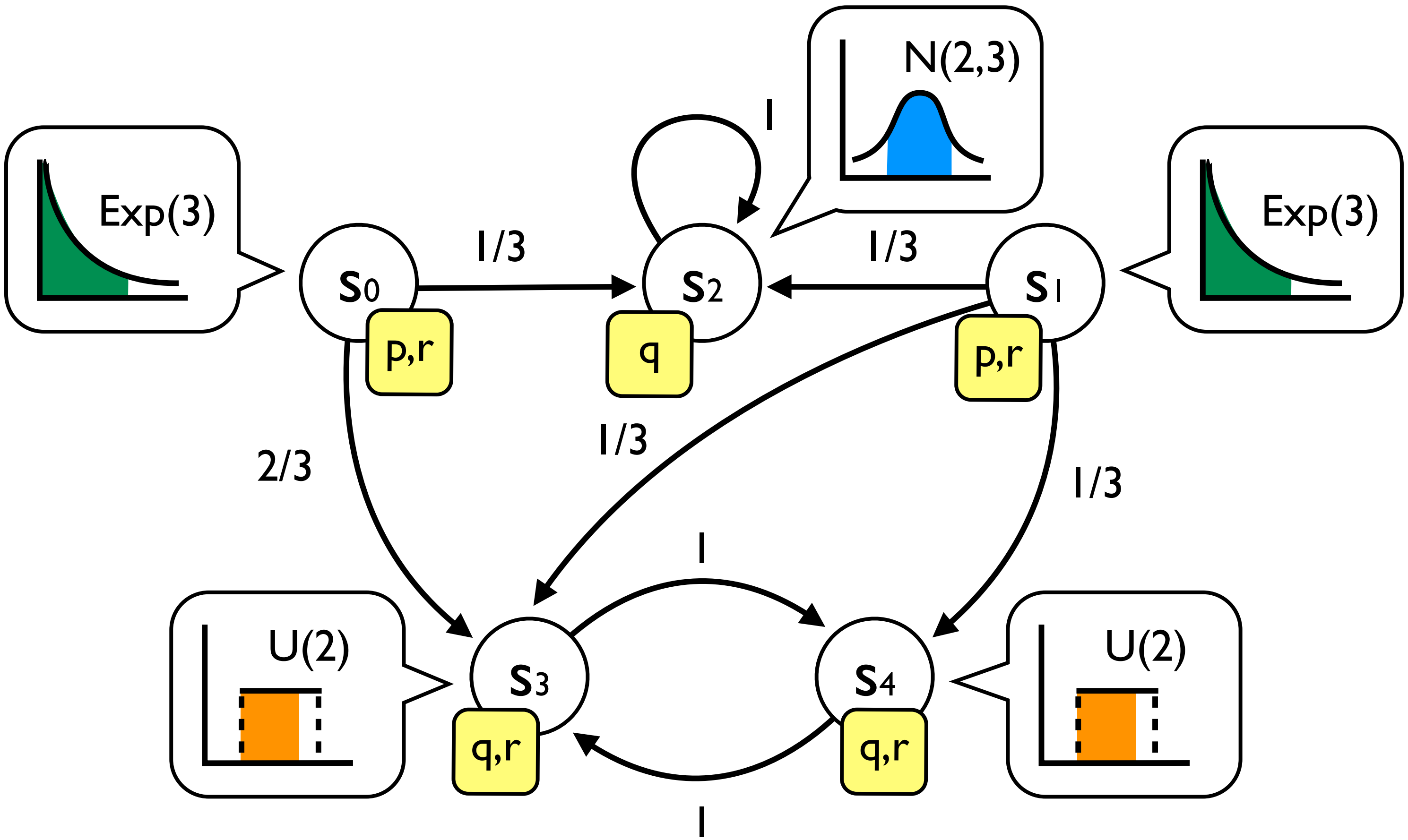
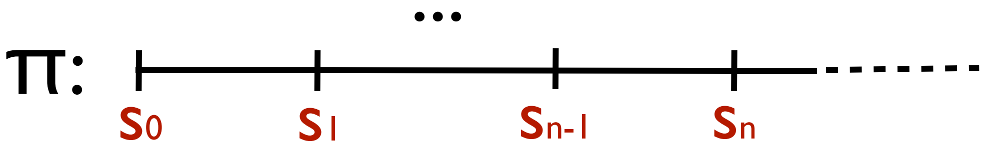


Given an initial state, SMCs can be interpreted as “machines” that emit timed traces of states with a certain probability

Events: Timed paths



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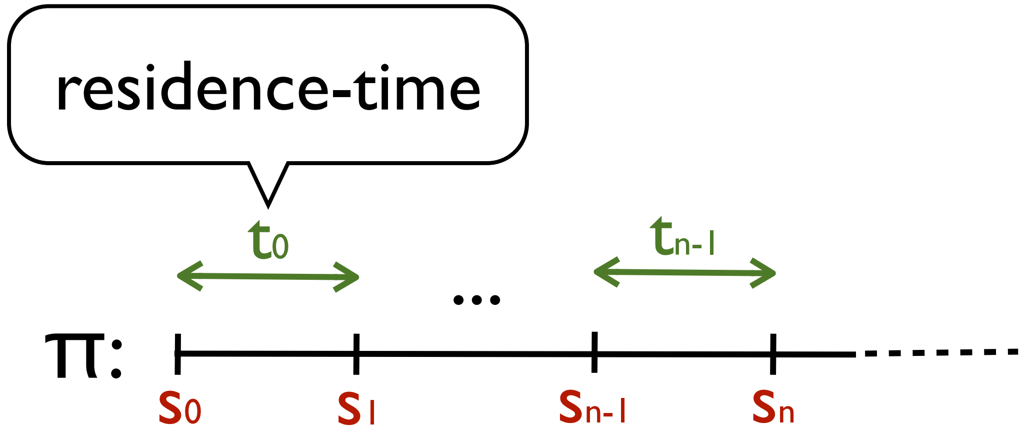

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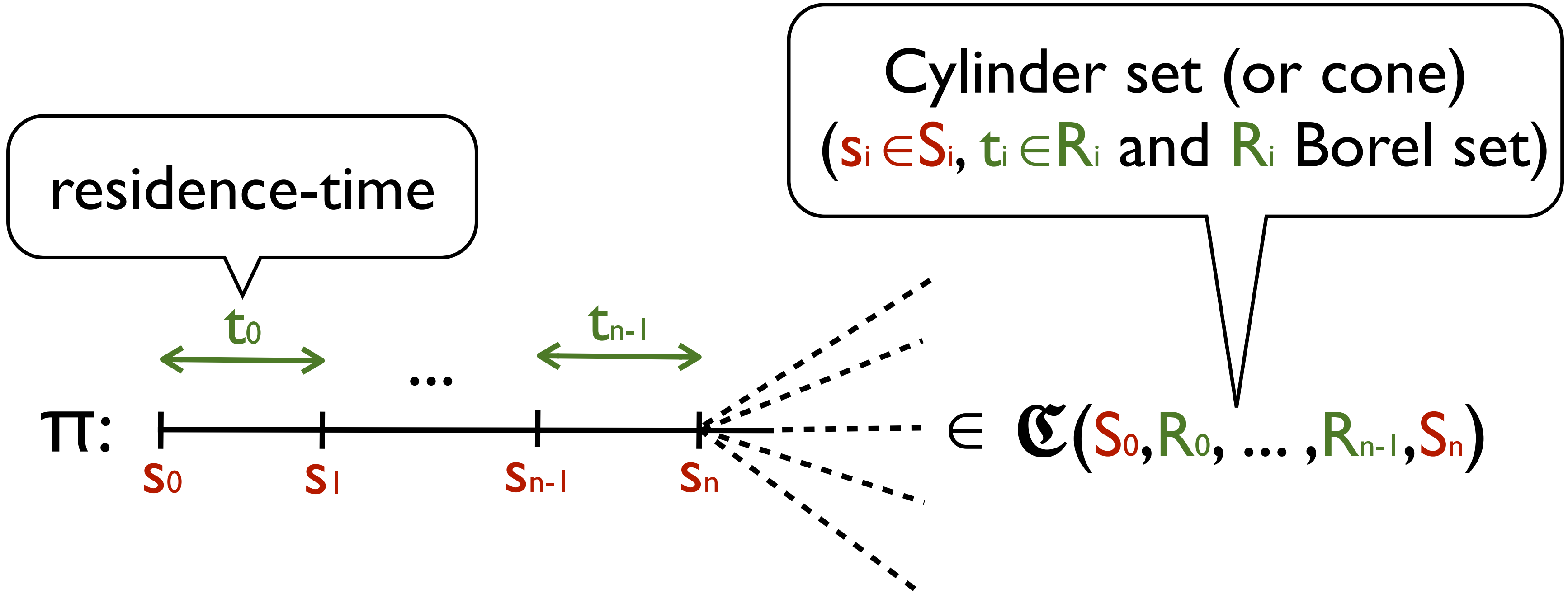

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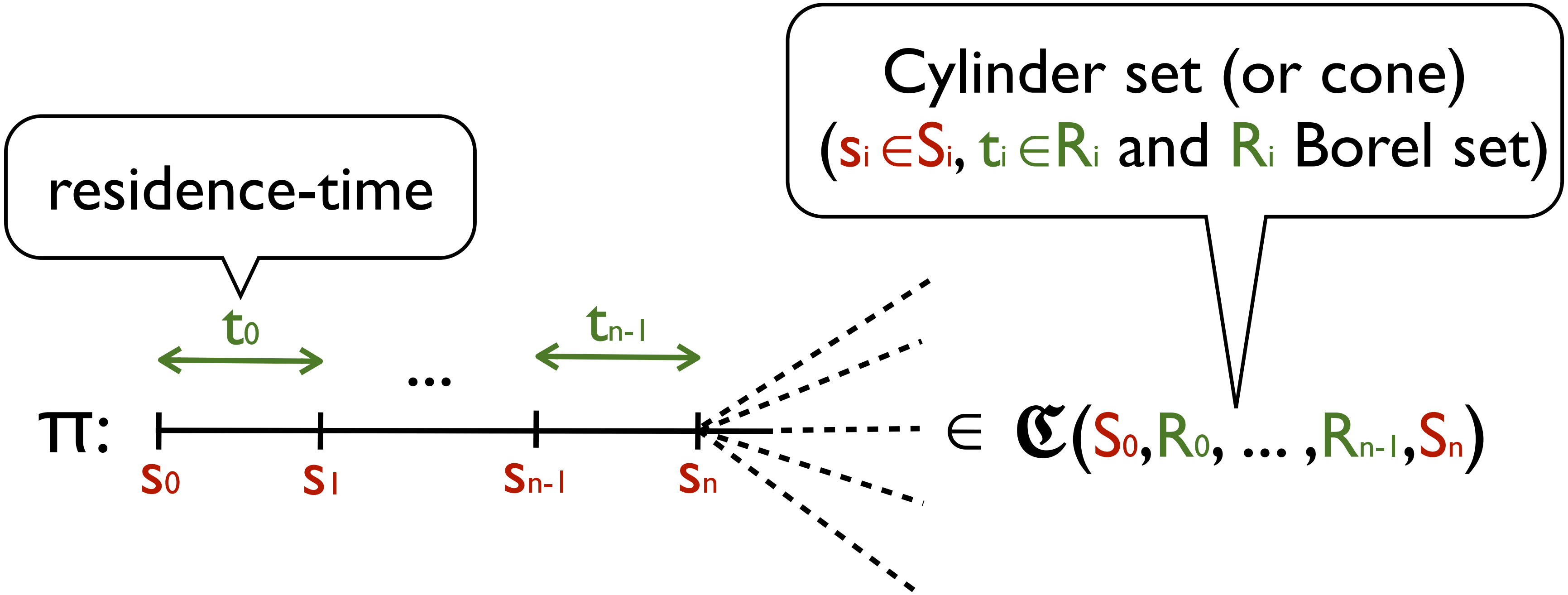
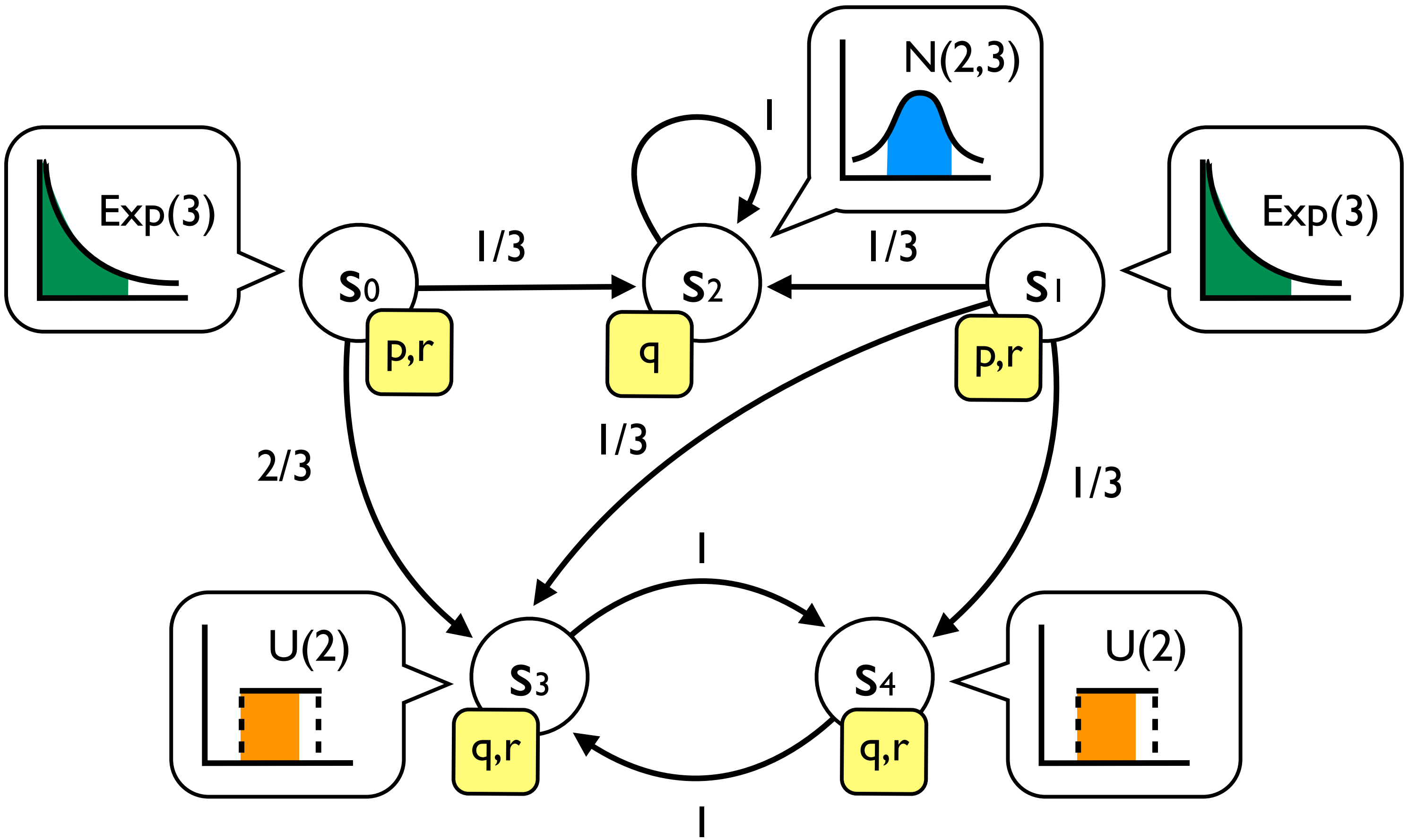


$P[s](\mathfrak{C}(S_0, R_0, \ldots, R_{n-1}, S_n)) =$ "probability that, *starting from s*, the SMC emits a timed path with prefix in $S_0 \times R_0 \times \ldots \times R_{n-1} \times S_n$"

Prob. Trace Equivalence



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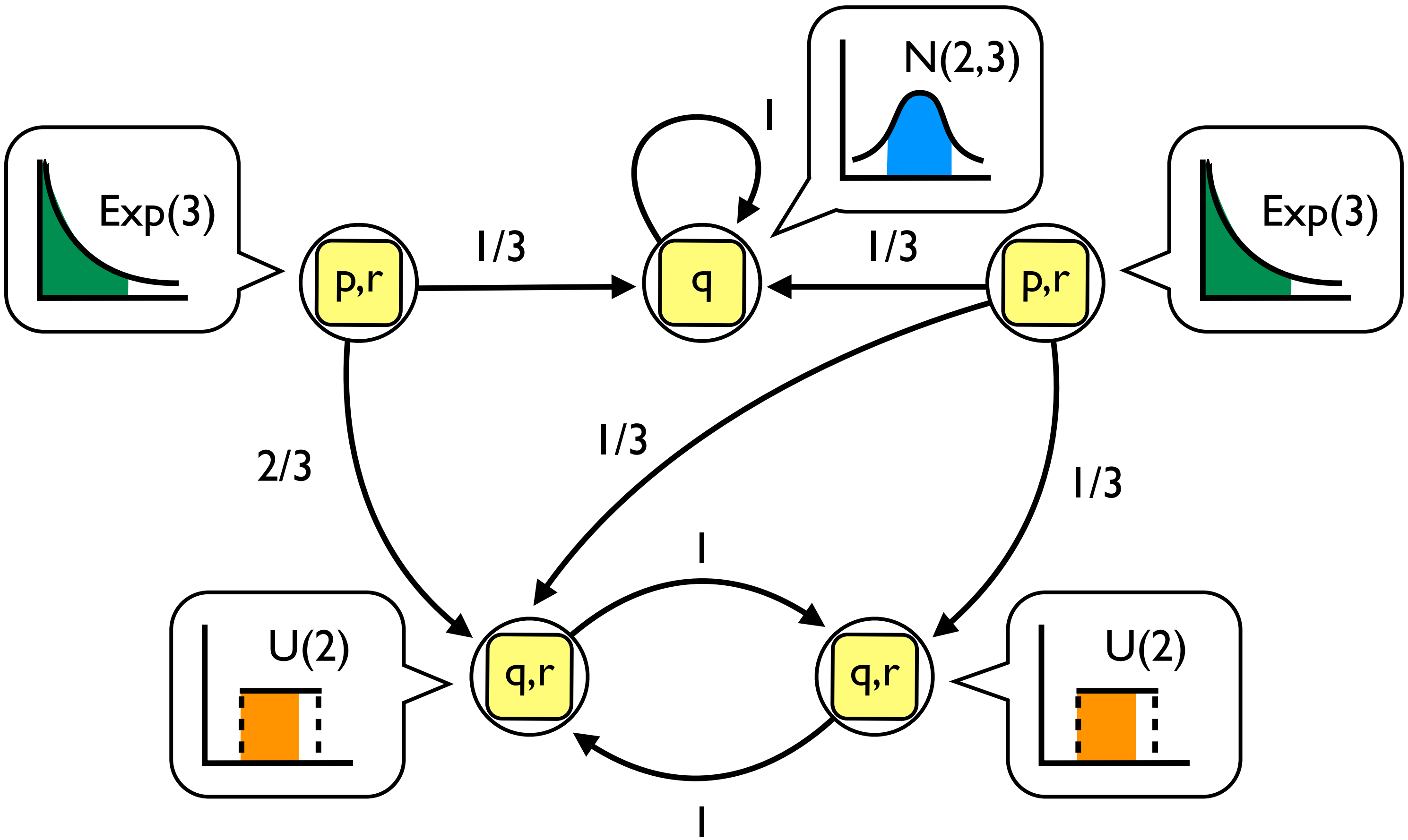

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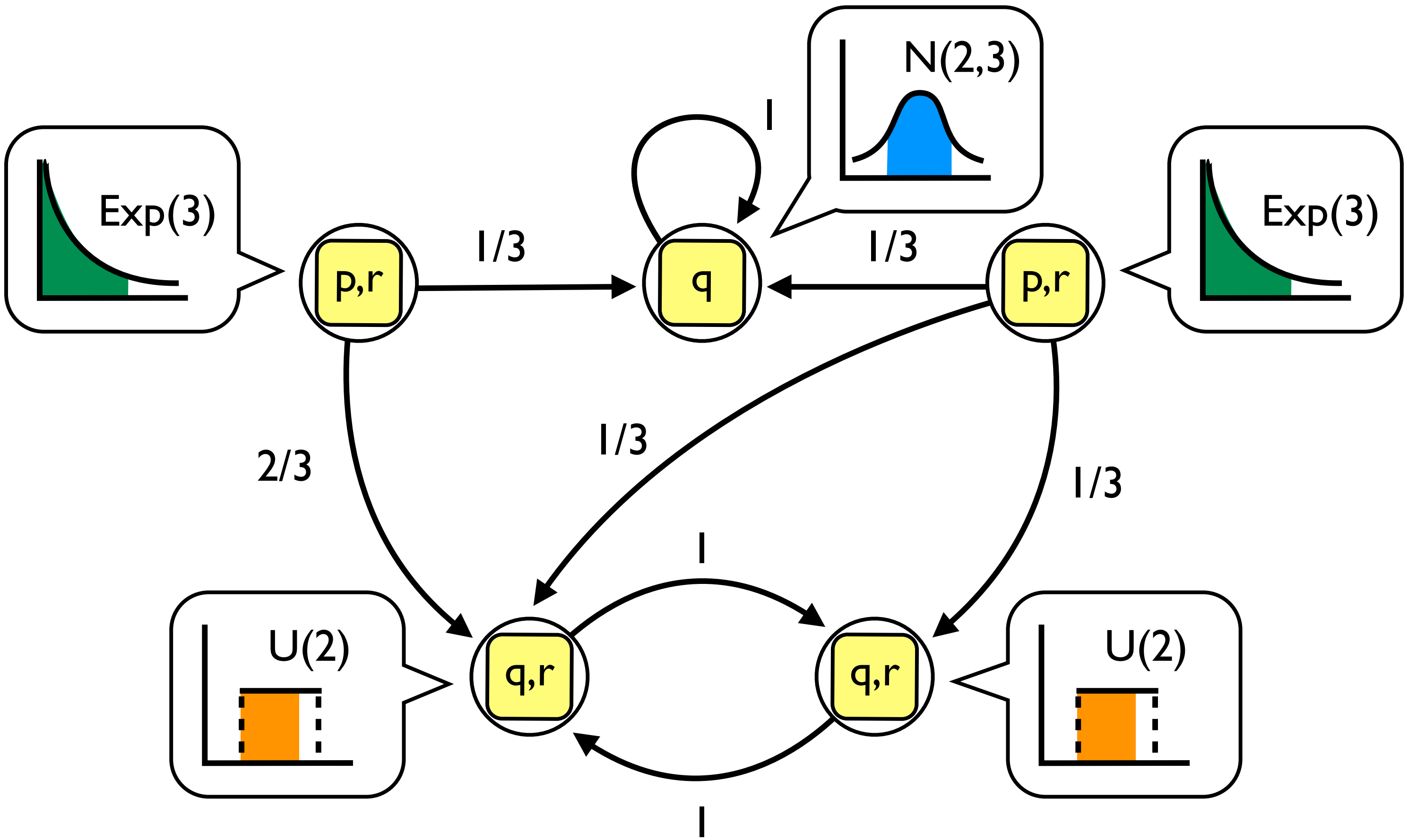

$$P[s_0](\mathfrak{C}(L_0, R_0, \ldots, R_{n-1}, L_n)) = P[s_1](\mathfrak{C}(L_0, R_0, \ldots, R_{n-1}, L_n))$$

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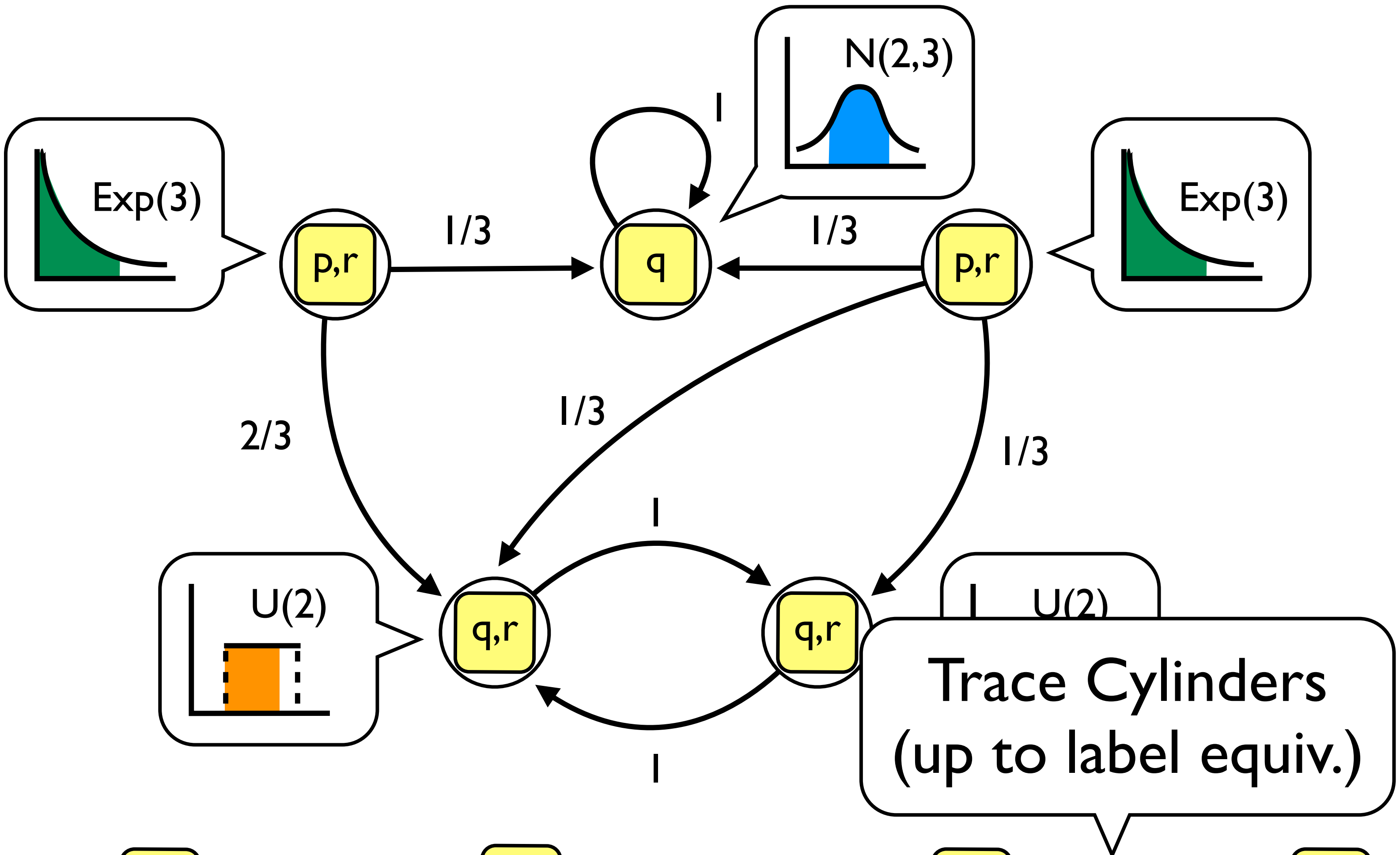

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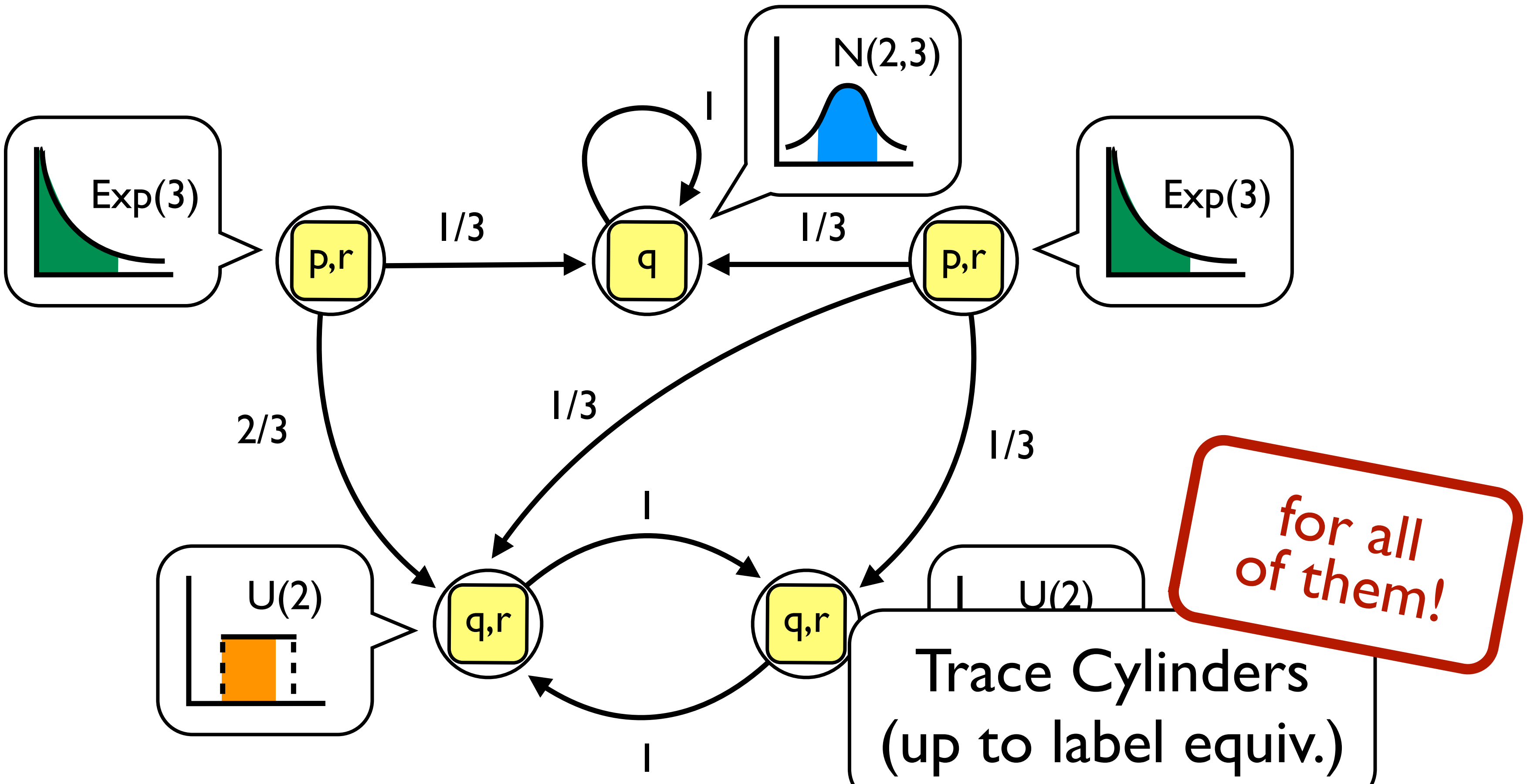

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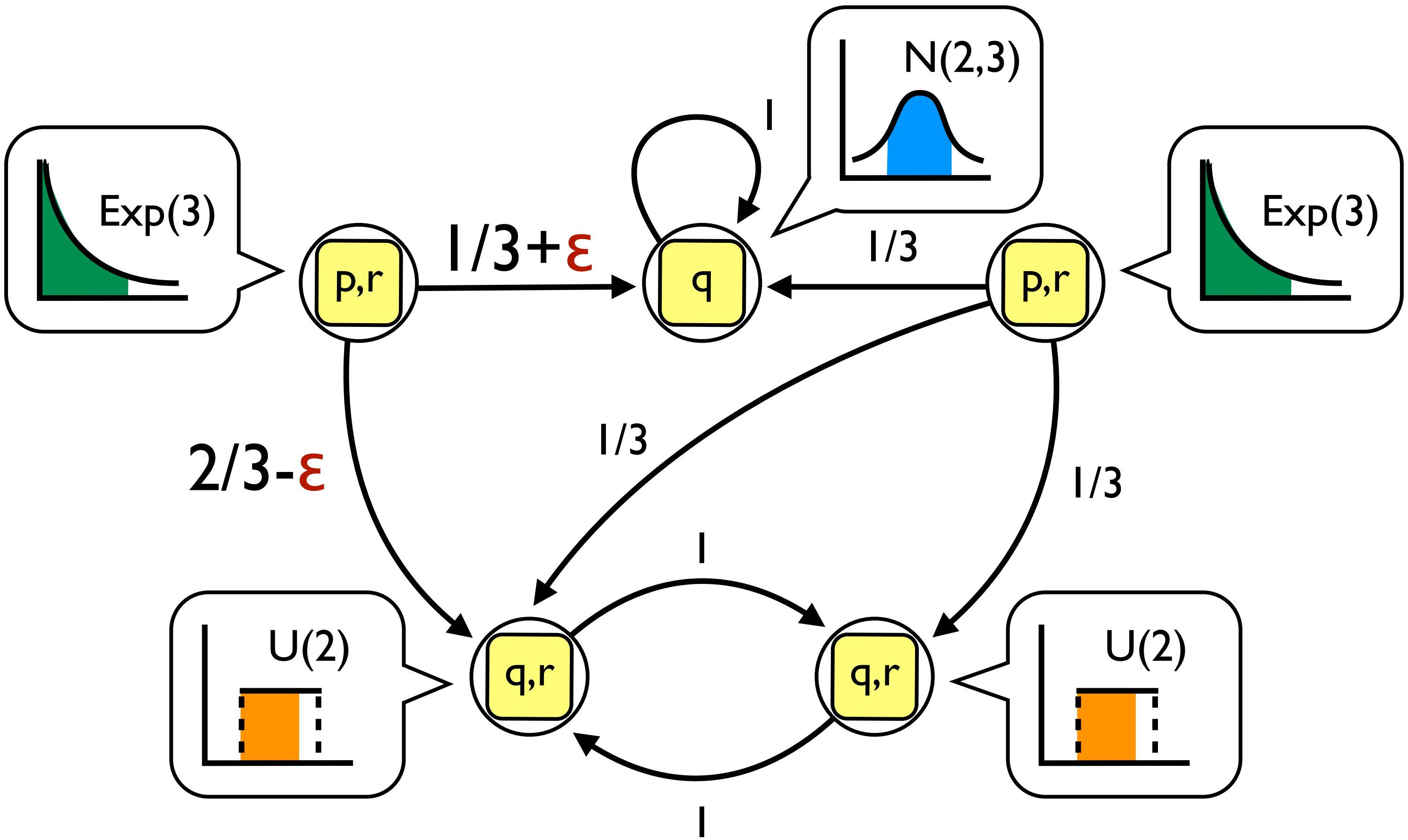

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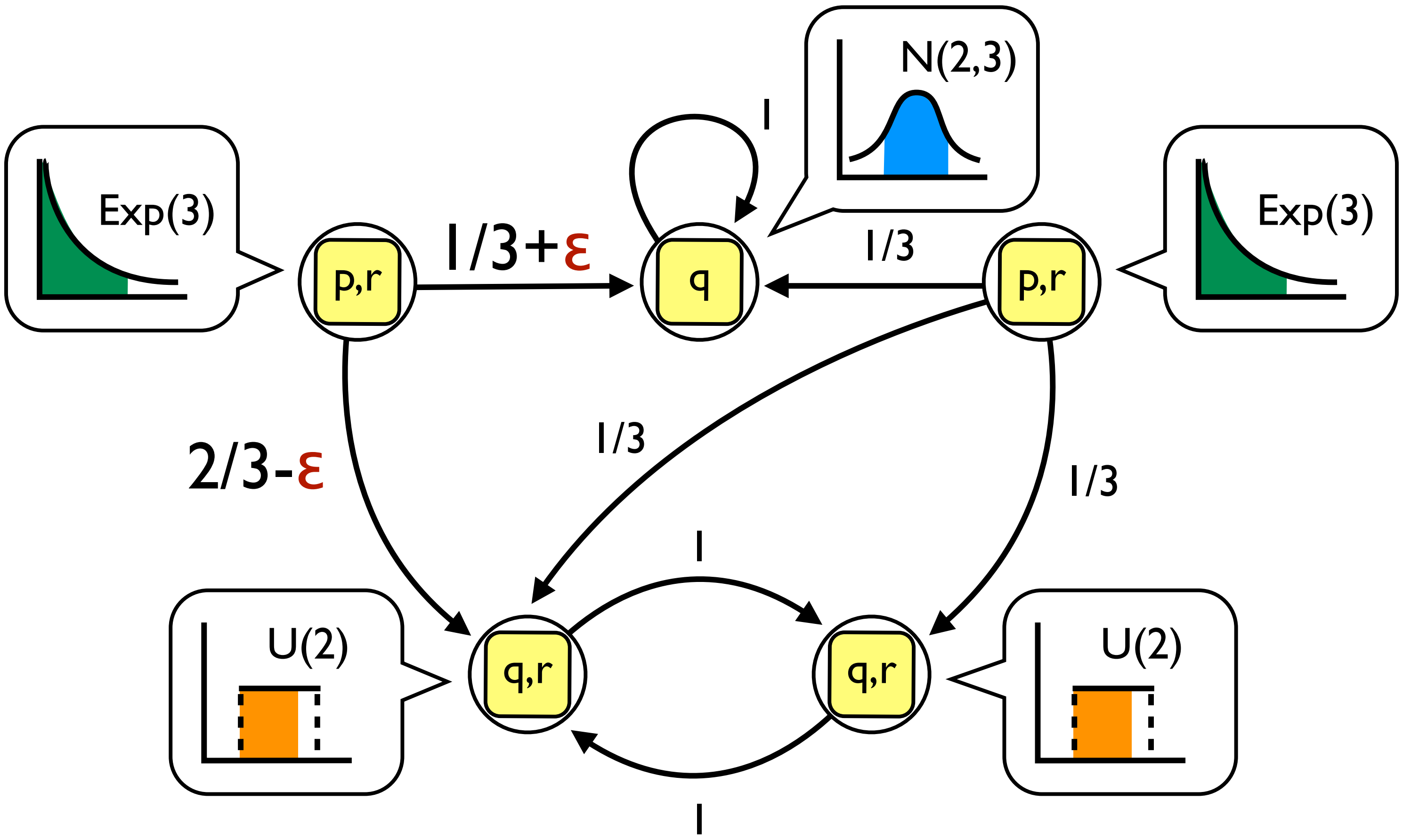
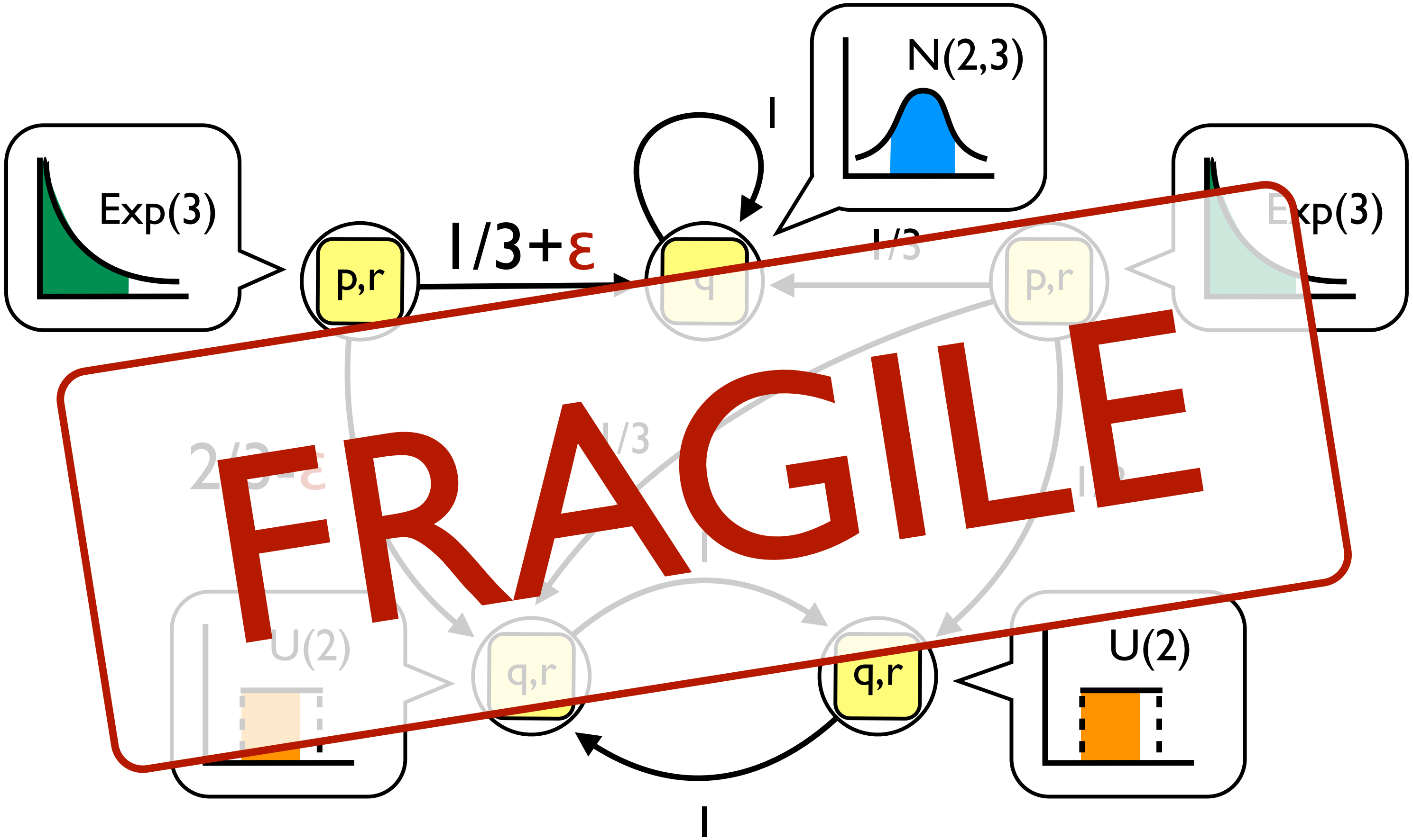


P[s0](ℭ(p,r, ℝ, q)) = 1/3+ε ≠ 1/3 = P[s1] (ℭ(p,r, ℝ, q))

Prob. Trace Equivalence



$$P[s_0](\mathfrak{C}(\boxed{p,r},\mathbb{R},\boxed{q})) = 1/3+\varepsilon \neq 1/3 = P[s_1](\mathfrak{C}(\boxed{p,r},\mathbb{R},\boxed{q}))$$

Trace Pseudometric

$$d(s,s') = \sup_{E \in \sigma(\mathcal{T})} |P[s](E) - P[s'](E)|$$

σ-algebra generated by Trace Cylinders

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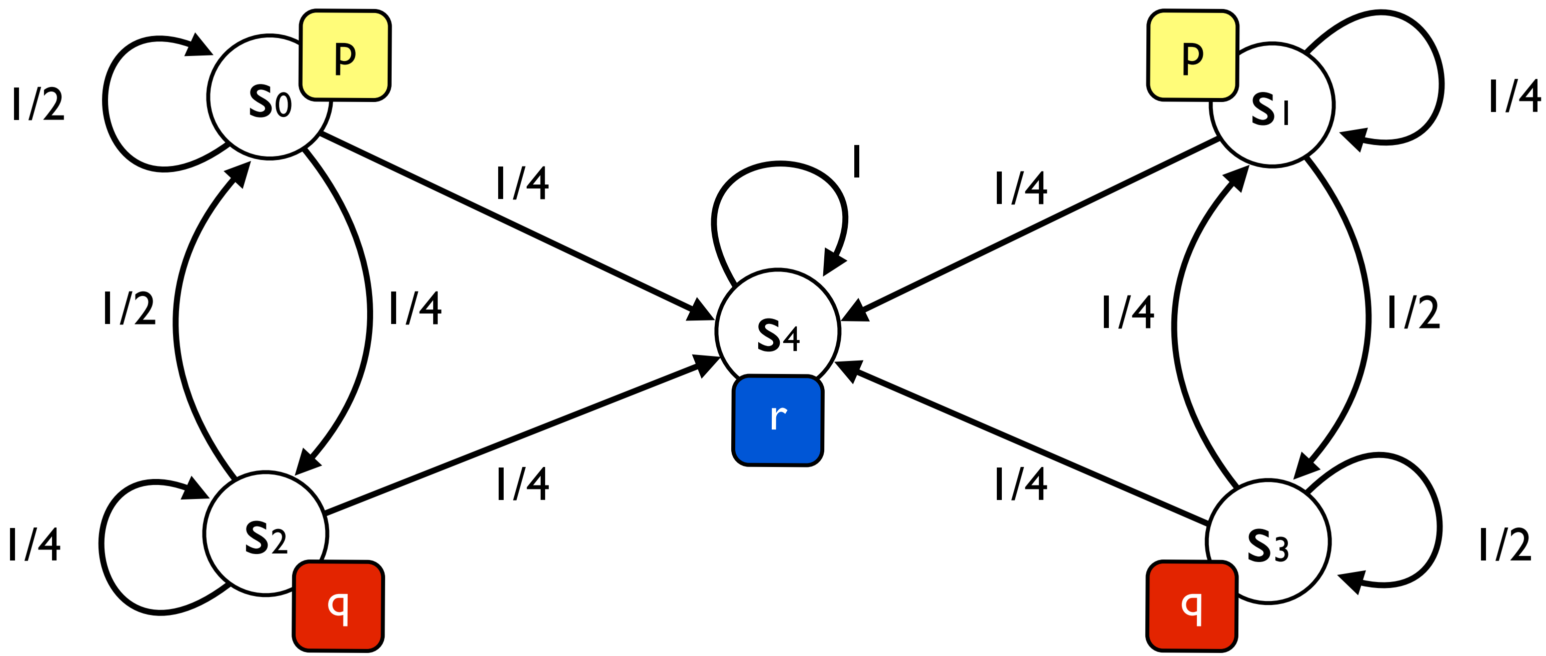
σ-algebra generated by Trace Cylinders

It's a Behavioral Distance!

$$d(s,s') = 0 \quad \text{iff} \quad s \approx_T s'$$

A tiny yet tricky example

(from Chen-Kiefer LICS'14)



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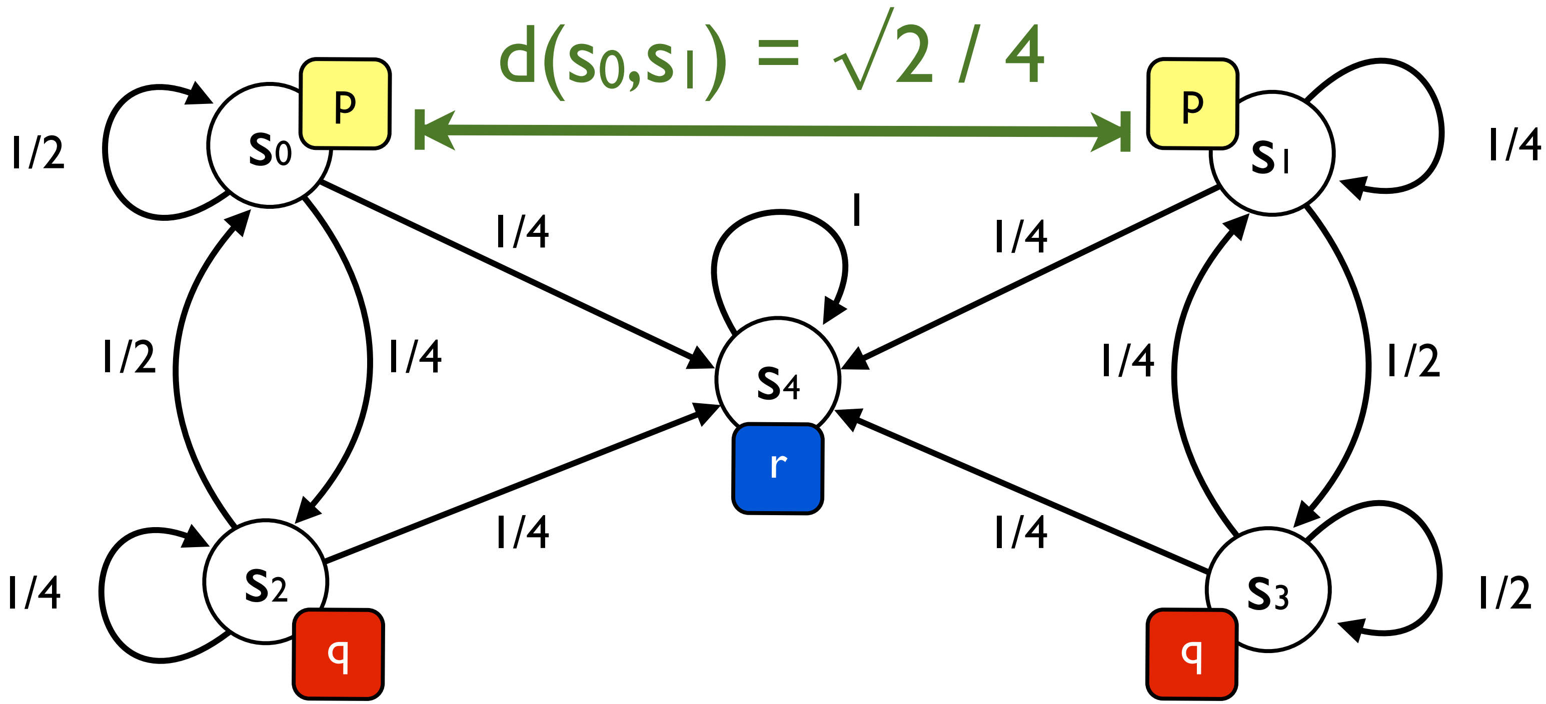

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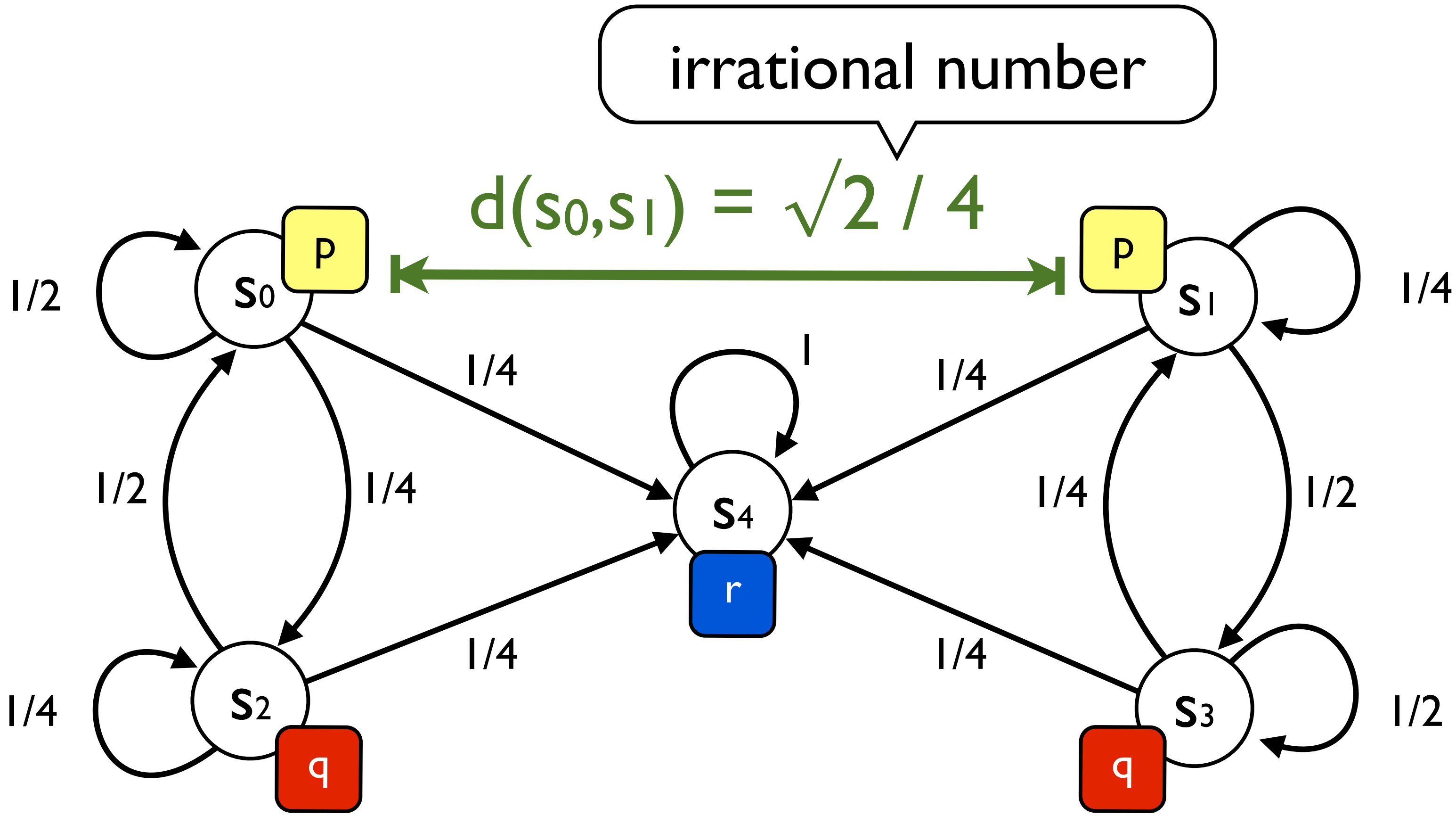

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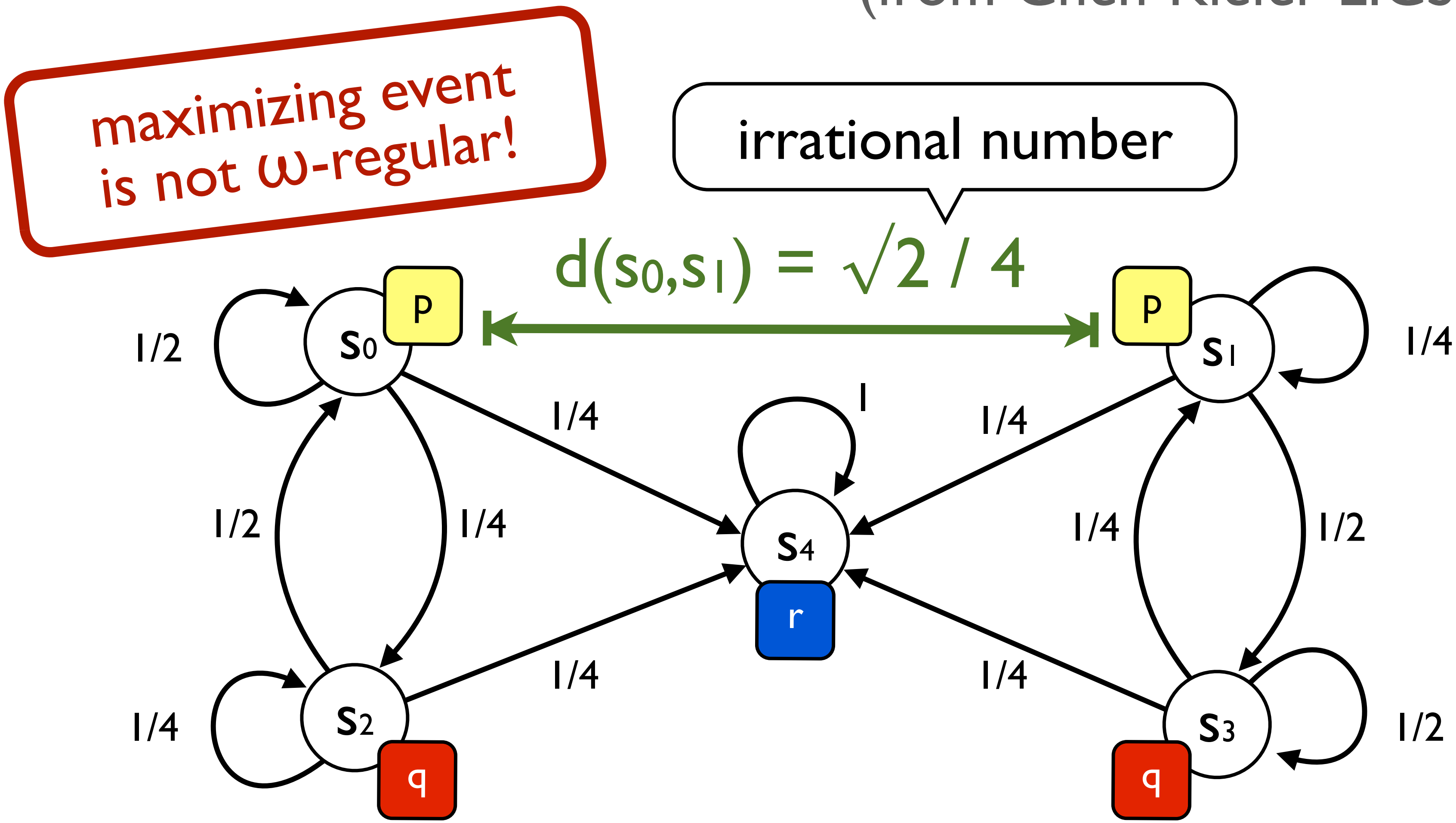

It's a Total Variation!

(a.k.a. supremum norm)

Given $\mu, \nu : \Sigma \to \mathbb{R}_+$ measures on (X, Σ)

Total Variation Distance

$$||\mu - \nu|| = \sup_{E \in \Sigma} |\mu(E) - \nu(E)|$$

The largest possible difference that μ and ν assign to the same event

Distance = Approx. Error

Distance $\stackrel{?}{=}$ Approx. Error

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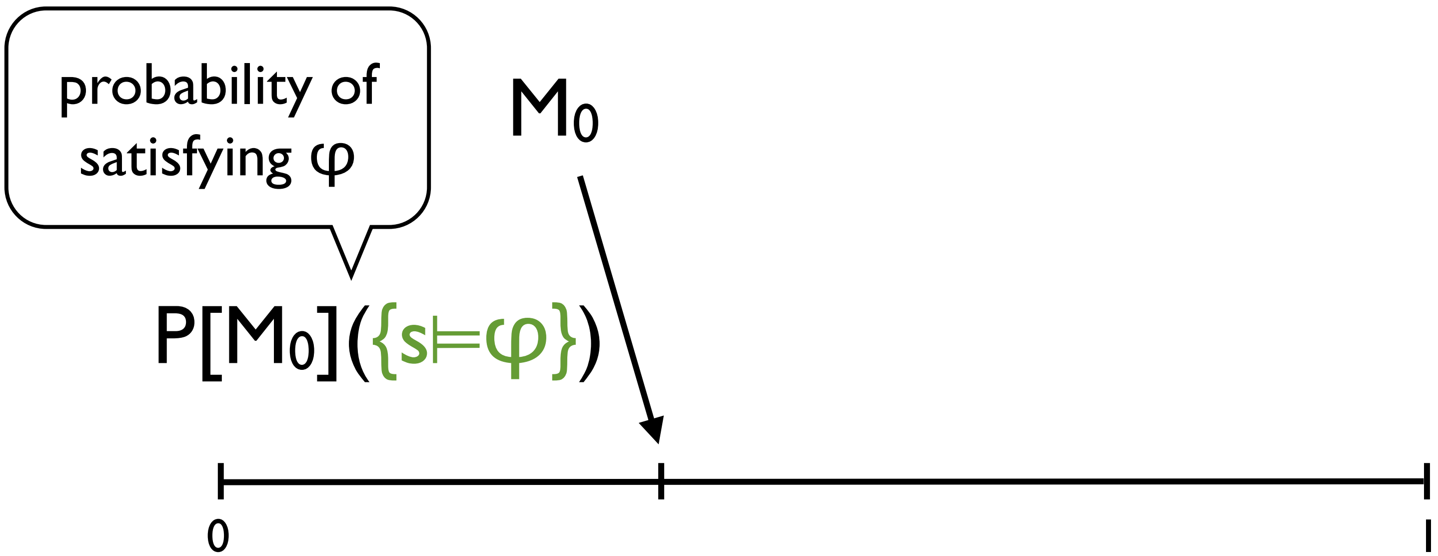
Application: Probabilistic Model Checking



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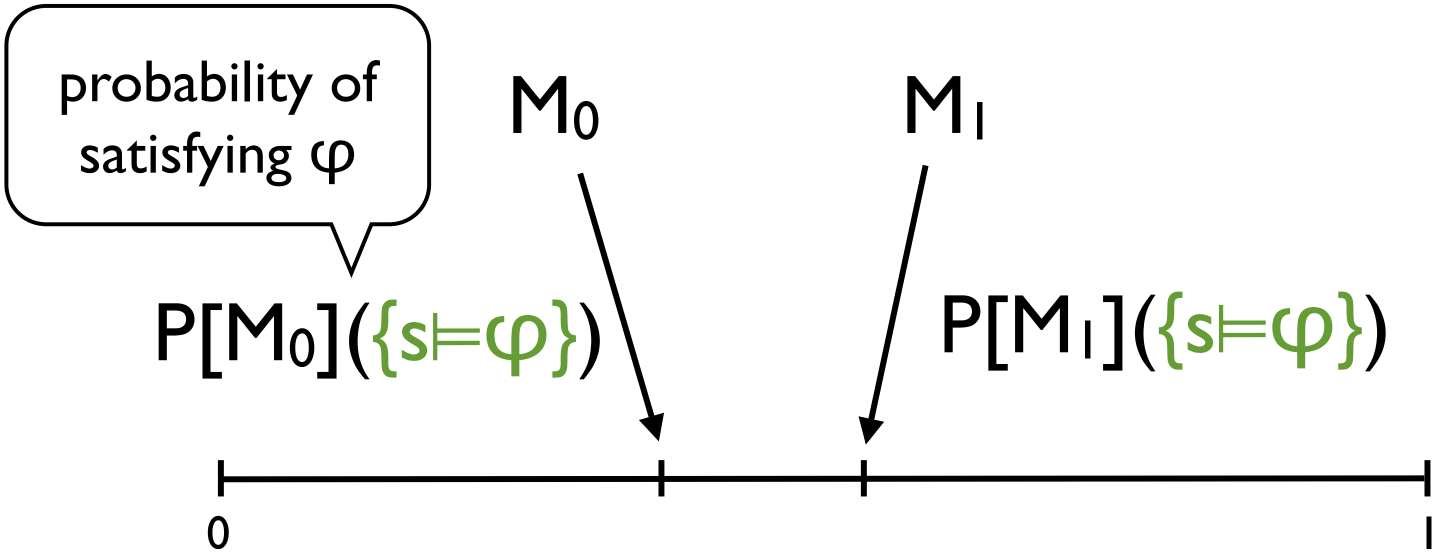

$$|P[M_0](\{s \models \varphi\}) - P[M_1](\{s \models \varphi\})|$$

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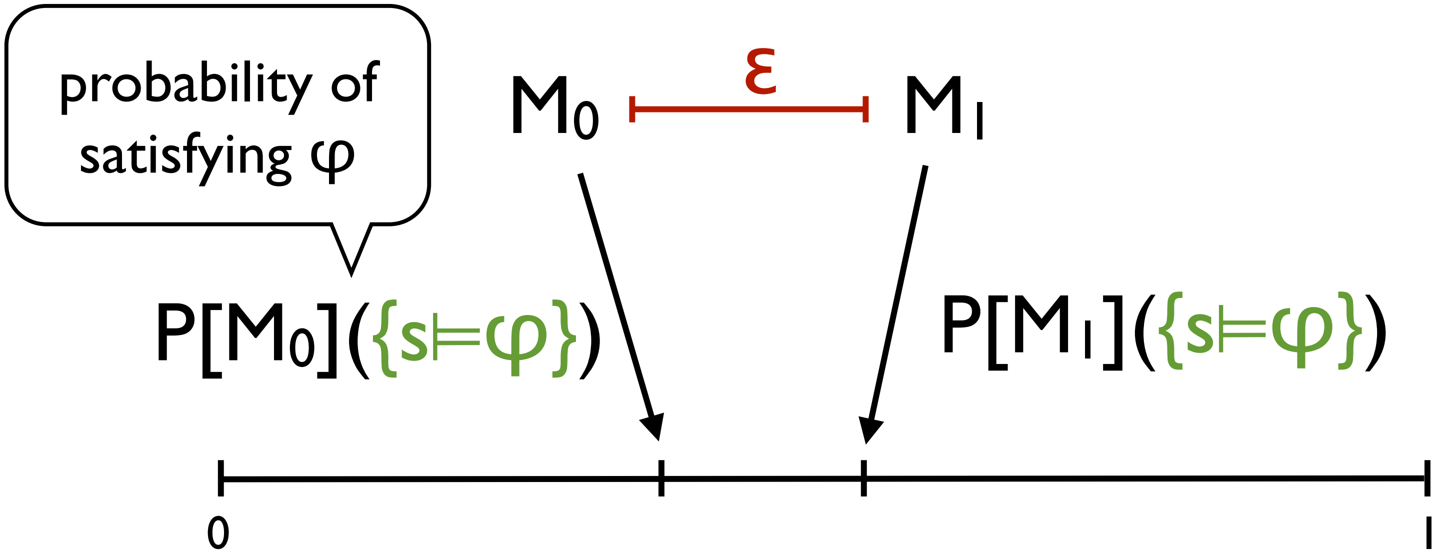

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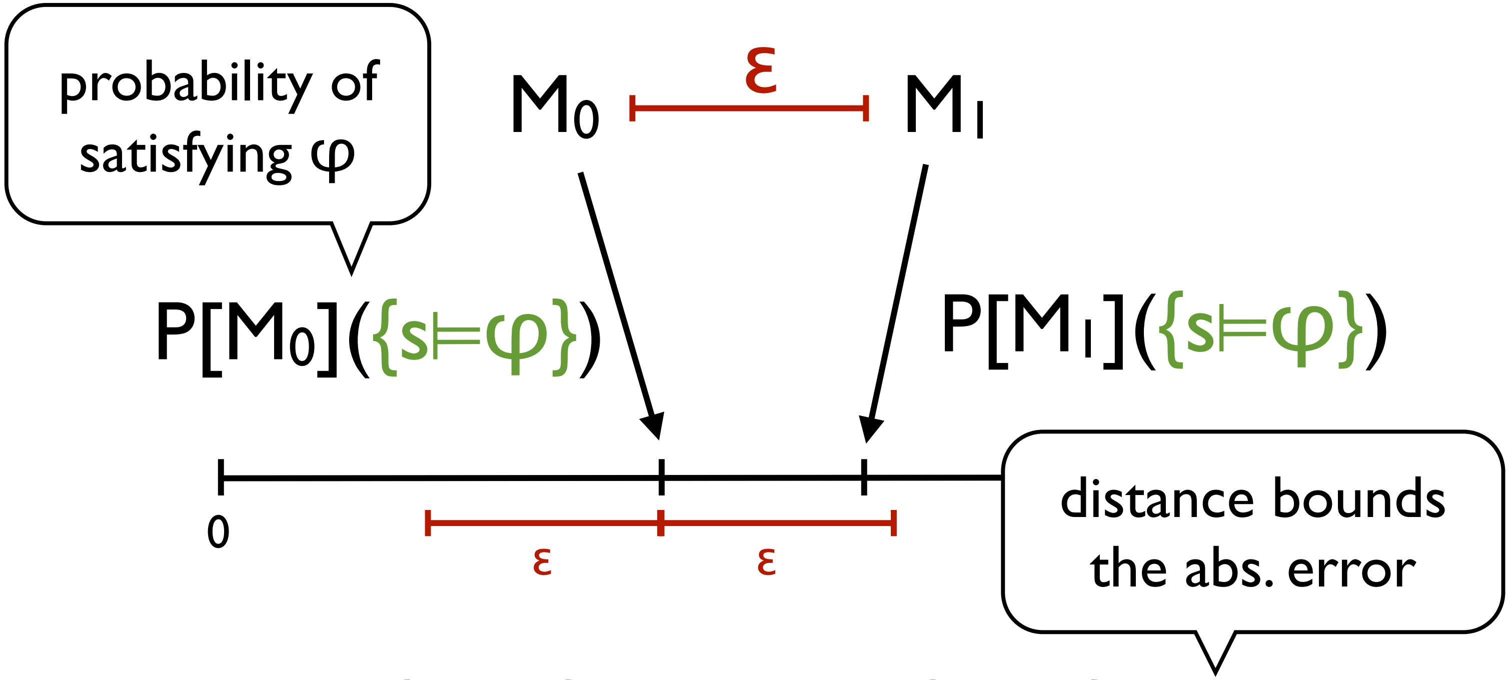

$$|P[M_0](\{s \models \varphi\}) - P[M_1](\{s \models \varphi\})| \leq \varepsilon$$

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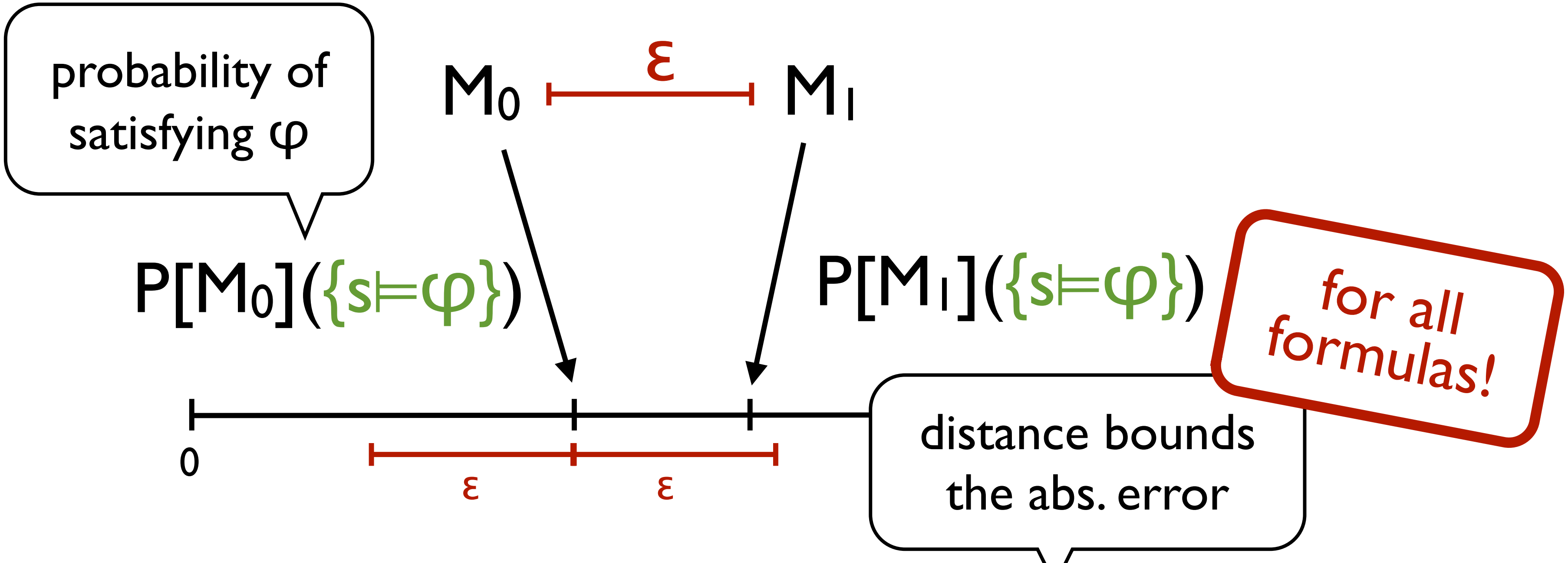

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Trace Distance
vs.
Model Checking

(i.e., does it provide a good approximation error?)

Probabilistic Model Checking

i.e., measuring the likelihood that
a property is satisfied by the probabilistic model

SMC $\models$ Linear Real-time Spec.

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... or languages
recognized
by Timed Automata

(Alur-Henzinger)

Metric Temporal Logic

Next

Until

$$\varphi := p \mid \bot \mid \varphi \to \varphi \mid X^{I}\varphi \mid \varphi U^{I} \varphi$$

(*) $I \subseteq \mathbb{R}$ closed interval with *rational* endpoints

(Alur-Henzinger)

Metric Temporal Logic

Next   Until

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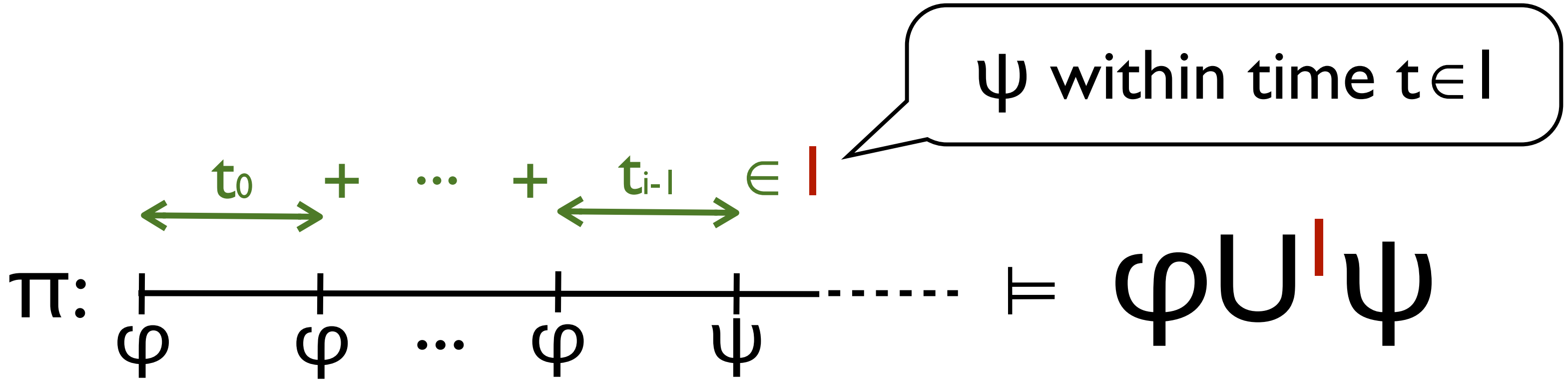

MTL distance

(max error w.r.t. MTL properties)

set of timed paths
that satisfy φ

$$MTL(s,s') = \sup_{\varphi \in MTL} |P[s](\{\pi \models \varphi\}) - P[s'](\{\pi \models \varphi\})|$$

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Relation with Trace Distance

$$MTL(s,s') \leq d(s,s') = \sup_{E \in \sigma(\mathcal{T})} |P[s](E) - P[s'](E)|$$

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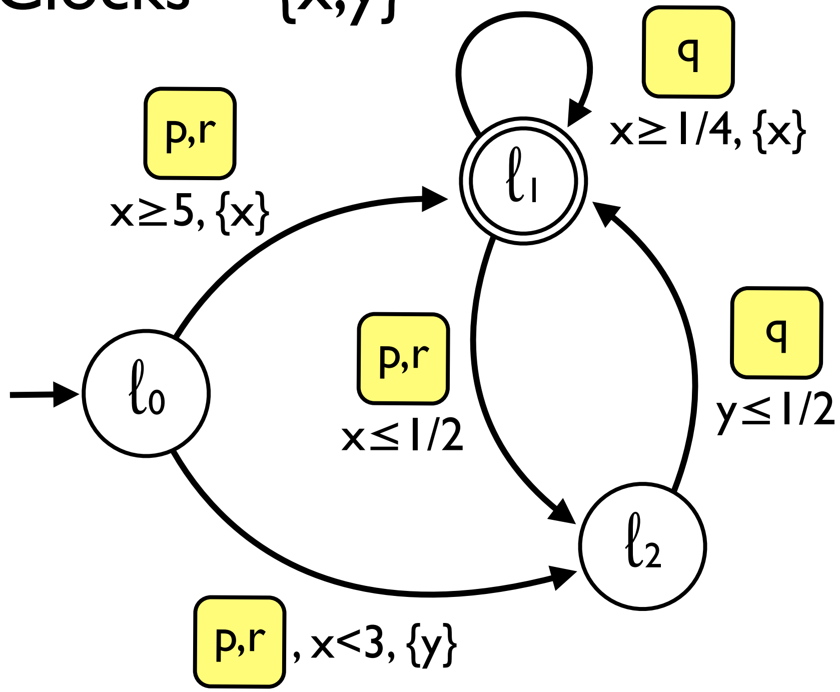
(Alur-Dill)

(Muller) Timed Automata

without invariants

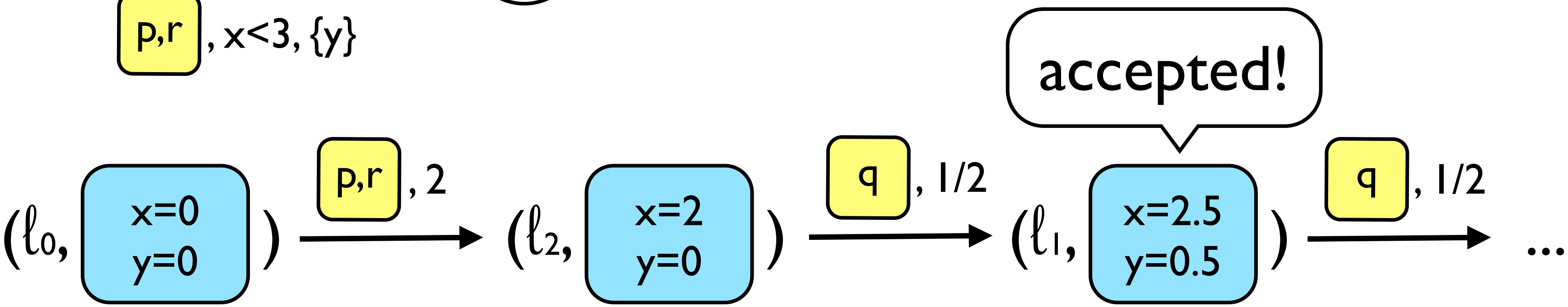
Clocks = {x,y}



Clock Guards

$$g := x \bowtie q \mid g \wedge g$$

for $\bowtie \in \{<,\leq,>,\geq\}$, $q \in \mathbb{Q}$



TA distance

(max error w.r.t. timed regular properties)

$$TA(s,s') = \sup_{\mathcal{A} \in TA} |P[s](\{\pi \in L(\mathcal{A})\}) - P[s'](\{\pi \in L(\mathcal{A})\})|$$

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set of timed paths accepted by $\mathcal{A}$

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The theorem behind...

For $\mu,\nu : \Sigma \to \mathbb{R}_+$ finite measures on (X,Σ)
and $F \subseteq \Sigma$ field such that $\sigma(F)=\Sigma$

Representation Theorem

$$\|\mu - \nu\| = \sup_{E \in F} |\mu(E) - \nu(E)|$$

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Representation Theorem

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F is much simpler than Σ, nevertheless
it suffices to attain the supremum!

A Series of Characterizations

$$MTL(s,s') = MTL^{\neg U}(s,s')$$

$$TA(s,s') = DTA(s,s') = 1\text{-}DTA(s,s') = 1\text{-}RDTA(s,s')$$

A Series of Characterizations

max error w.r.t. $\varphi \in$MTL
without Until

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Deterministic TAs

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Deterministic TAs

$$\mathrm{TA}(s,s') = \mathrm{DTA}(s,s') = \text{1-DTA}(s,s') = \text{1-RDTA}(s,s')$$

max error w.r.t.
single-clock DTAs

A Series of Characterizations

max error w.r.t. φ∈MTL without Until

$$MTL(s,s') = MTL^{\neg U}(s,s')$$

max error w.r.t. Deterministic TAs

max error w.r.t. Resetting 1-DTAs

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Approximation Algorithm for the Trace Distance

generalizes
Chan-Kiefer LICS'14
with timed-event

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Approximation Algorithm for the Trace Distance

NP-hardness [Lyngsø-Pedersen JCSS'02]

Approximating the trace distance up to any ε>0 whose size is polynomial in the size of the Interval MC is NP-hard.

easy to adapt to MCs...

generalizes
Chan-Kiefer LICS'14
with timed-event

Decidability
still an open
problem!

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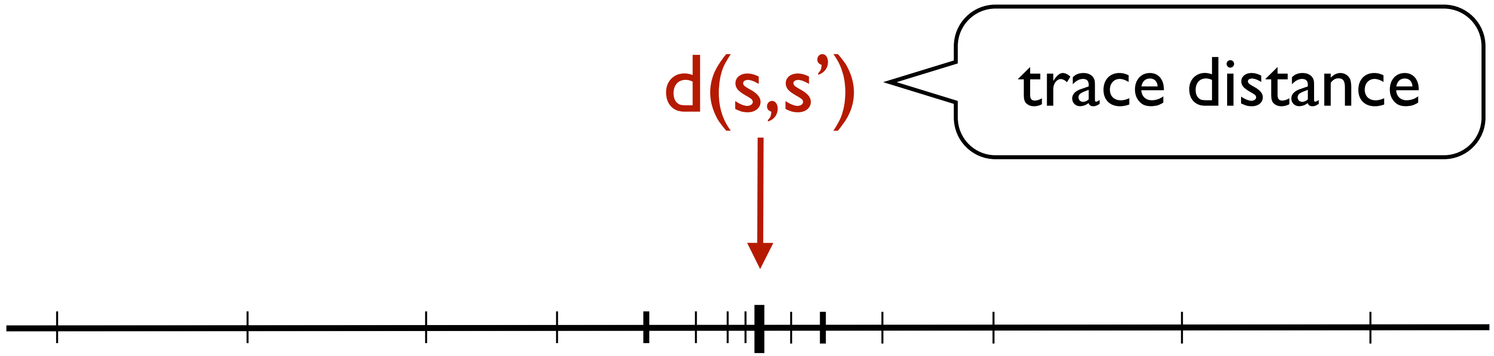
Approximation Algorithm



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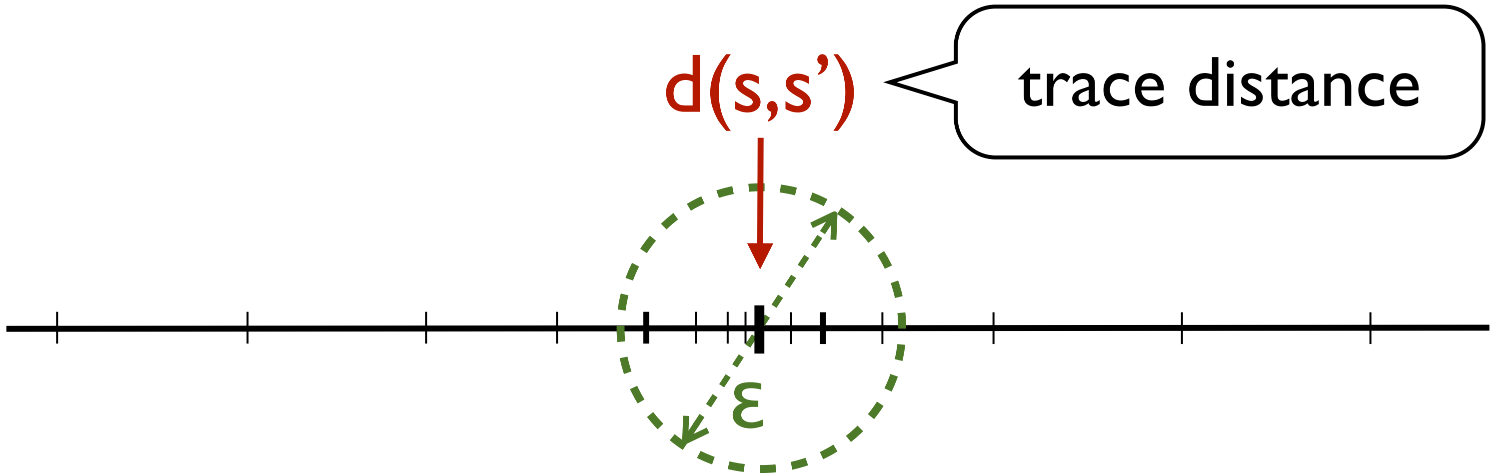

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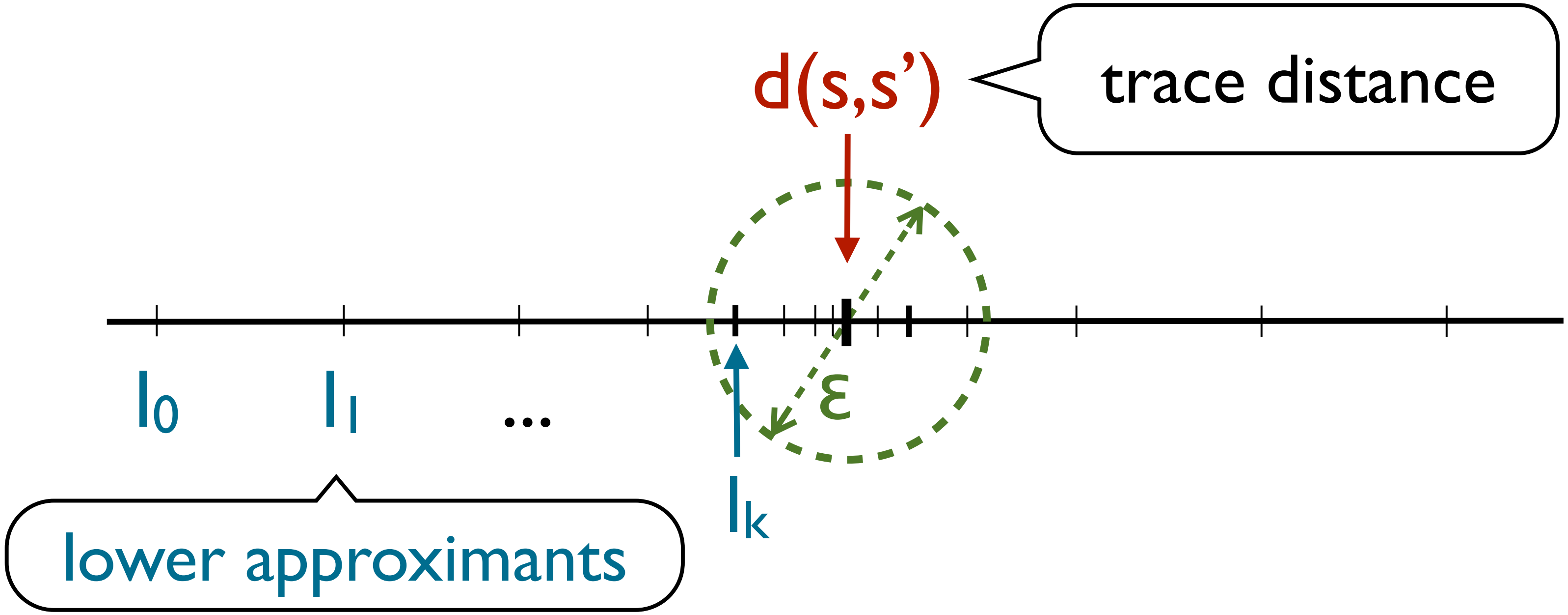

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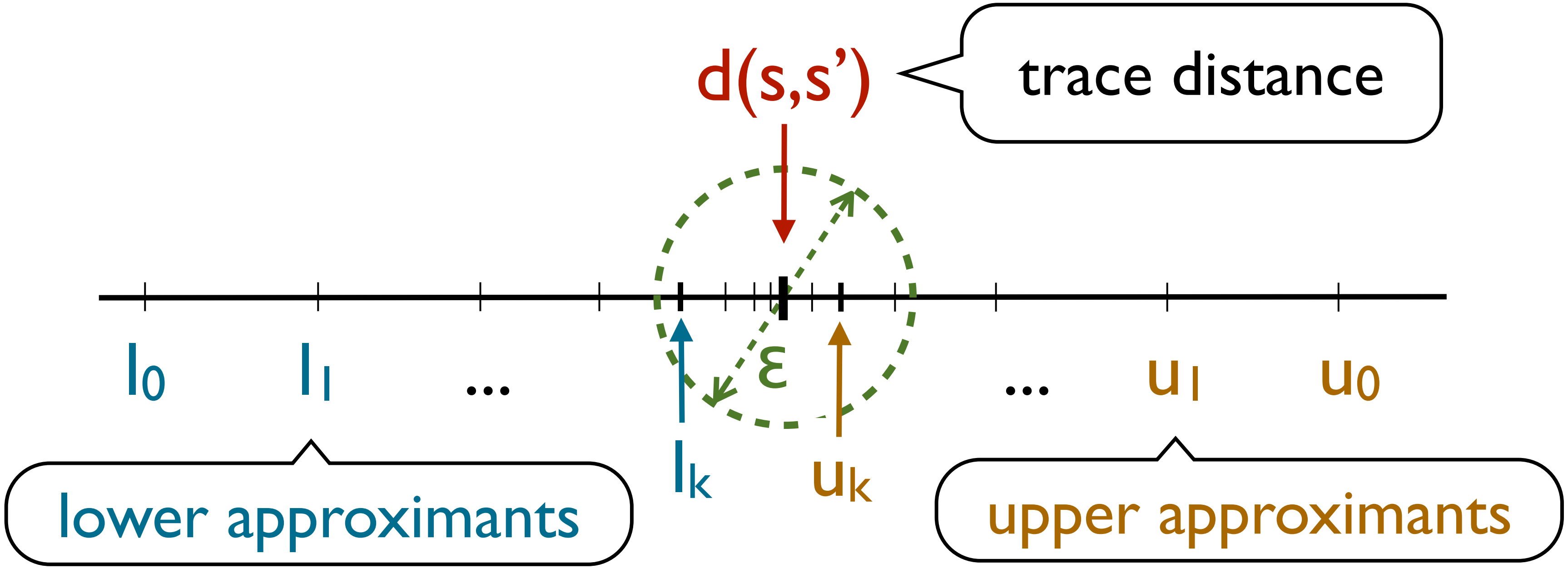

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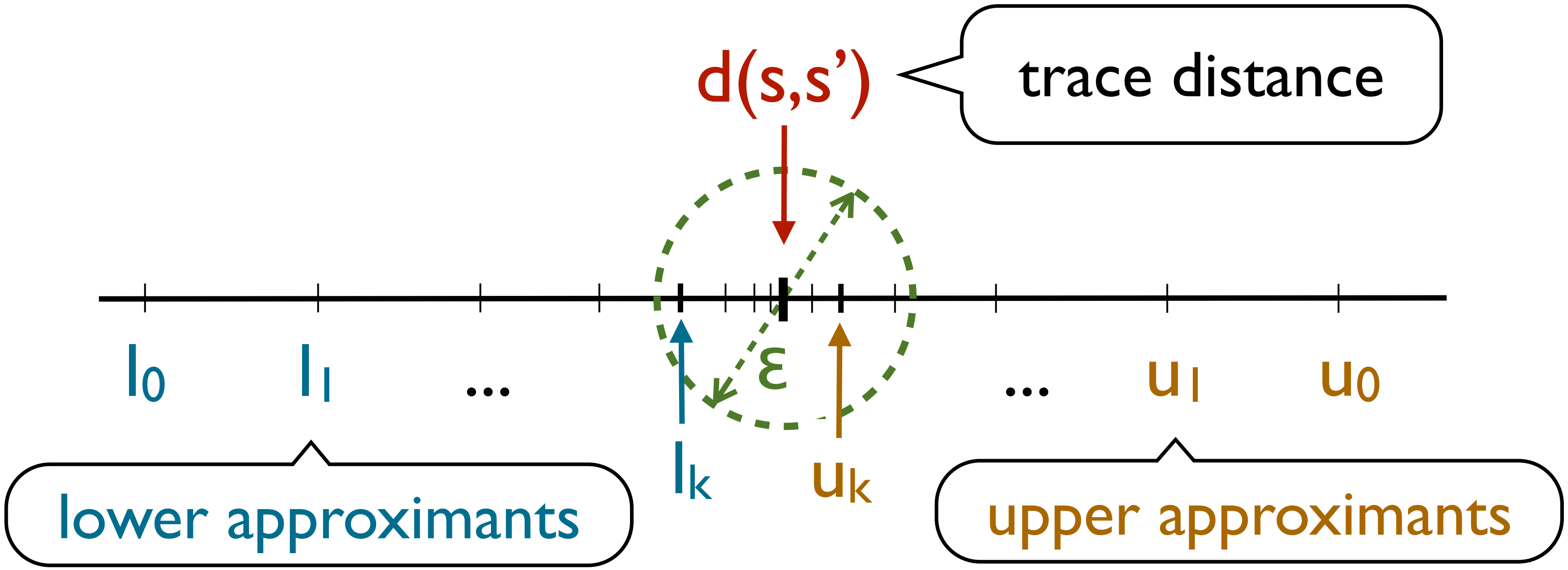
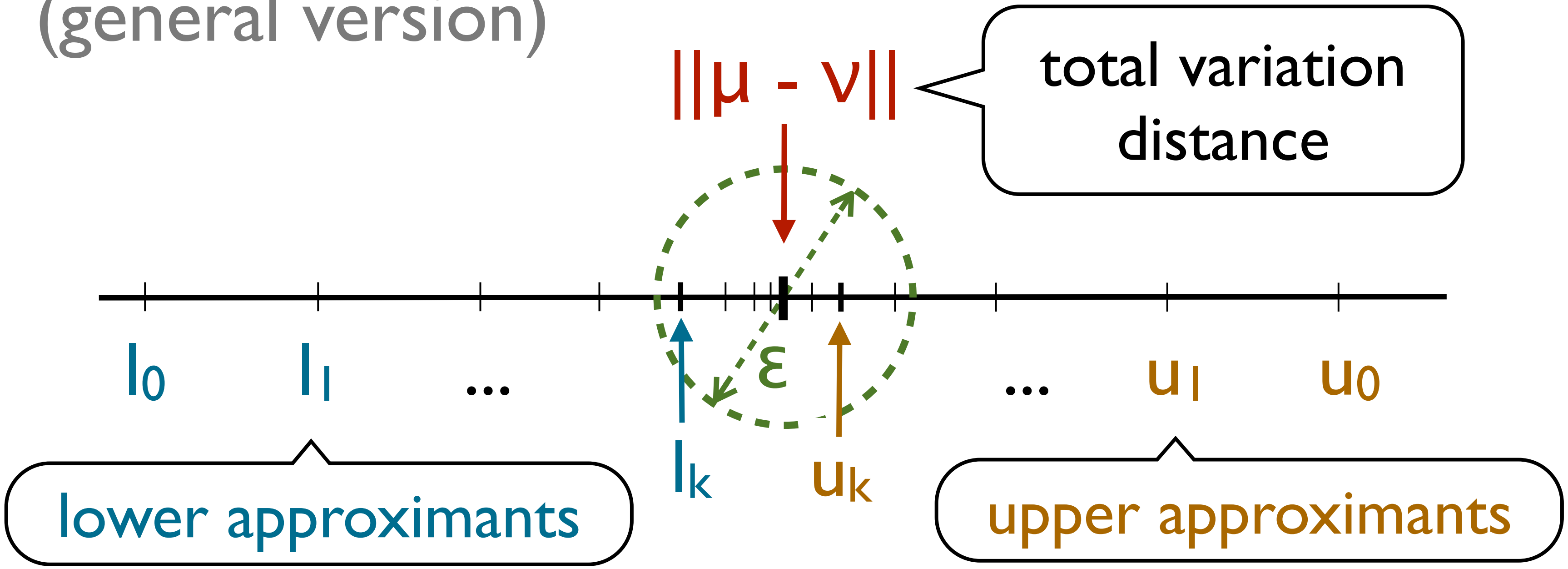


- l_i and u_i must converge to d(s,s')
- For all $i \in \mathbb{N}$, l_i and u_i must be *computable*.

Approximation Algorithm

(general version)



- l_i and u_i must converge to $\|\mu - \nu\|$
- For all $i \in \mathbb{N}$, l_i and u_i must be *computable*.

... from below

... from below

Representation Theorem

$$\|\mu - \nu\| = \sup_{E \in F} |\mu(E) - \nu(E)|$$

recall that...

... from below

Representation Theorem

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recall that...

F field that generates Σ

... from below

Representation Theorem

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$$||\mu - \nu|| = \sup_{E \in F} |\mu(E) - \nu(E)|$$

F field that generates Σ

We need $F_0 \subseteq F_1 \subseteq F_2 \subseteq \ldots$ such that $\bigcup_i F_i = F$

$$l_i = \sup_{E \in F_i} |\mu(E) - \nu(E)|$$

... from below

Representation Theorem

recall that...

$$\|\mu - \nu\| = \sup_{E \in F} |\mu(E) - \nu(E)|$$

F field that generates Σ

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$$l_i = \sup_{E \in F_i} |\mu(E) - \nu(E)|$$

so that $\forall i \geq 0, l_i \leq l_{i+1}$ & $\sup_i l_i = \|\mu - \nu\|$

increasing

limiting

Trace dist. (from below)

...seen before

Provide $F_0 \subseteq F_1 \subseteq F_2 \subseteq \ldots$ such that

$\bigcup_i F_i$ is a field for $\sigma(\mathcal{T})$

Trace dist. (from below)

...seen before

Provide $F_0 \subseteq F_1 \subseteq F_2 \subseteq \ldots$ such that
$\cup_i F_i$ is a field for $\sigma(\mathcal{T})$

Take F_i to be the collection of finite unions of cylinders

$$\mathfrak{C}(L_0, R_0, \ldots, R_{i-1}, L_i) \in \mathcal{T}$$

where $R_j \in \{[\frac{n}{2^i}, \frac{n+1}{2^i}) \mid 0 \leq n \leq i2^i\} \cup \{[i, \infty)\}$

Trace dist. (from below)

...seen before

Provide $F_0 \subseteq F_1 \subseteq F_2 \subseteq \ldots$ such that

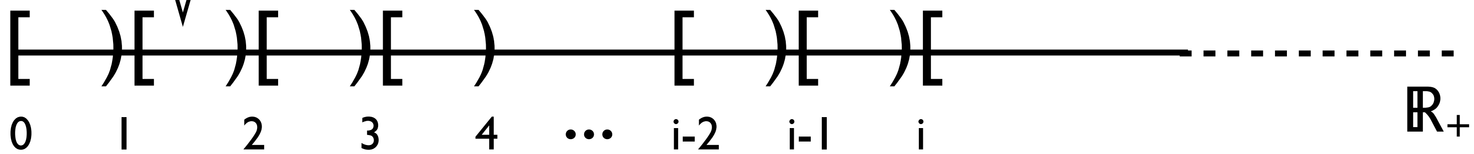
$\cup_i F_i$ is a field for $\sigma(\mathcal{T})$

Take F_i to be the collection of finite unions of cylinders

$$\mathfrak{C}(L_0, R_0, \ldots, R_{i-1}, L_i) \in \mathcal{T}$$

$$\text{where } R_j \in \{[\tfrac{n}{2^i}, \tfrac{n+1}{2^i}) \mid 0 \leq n \leq i2^i\} \cup \{[i, \infty)\}$$

each repartitioned in 2^i [closed-open) intervals



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Coupling Characterization

$$||\mu - \nu|| = \min \{w(\neq) \mid w \in \Omega(\mu,\nu)\}$$

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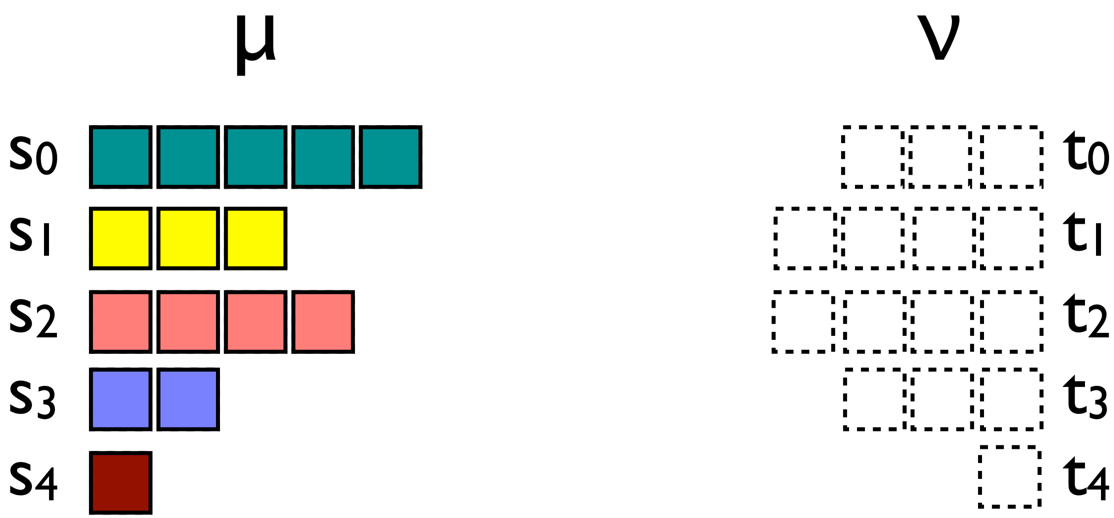

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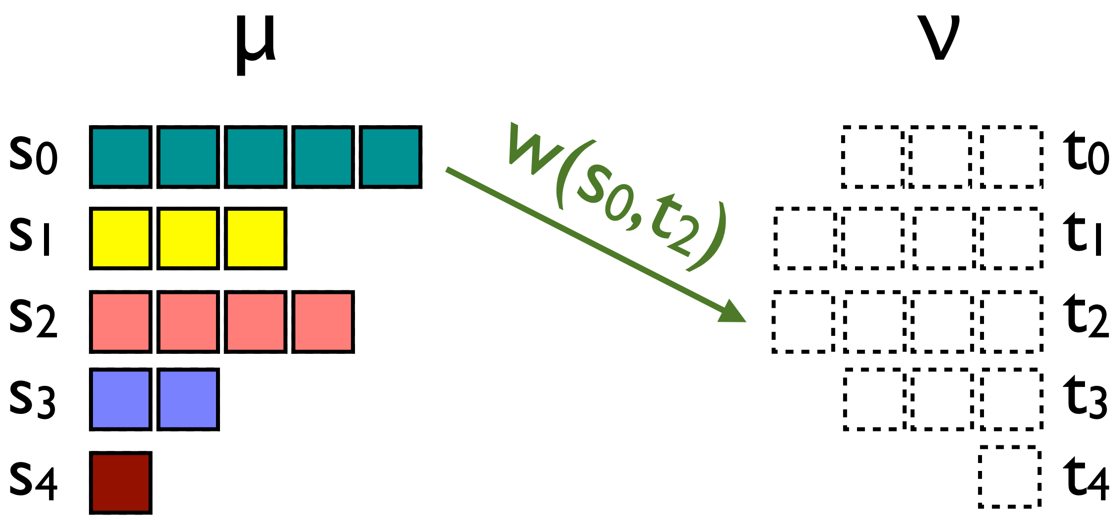

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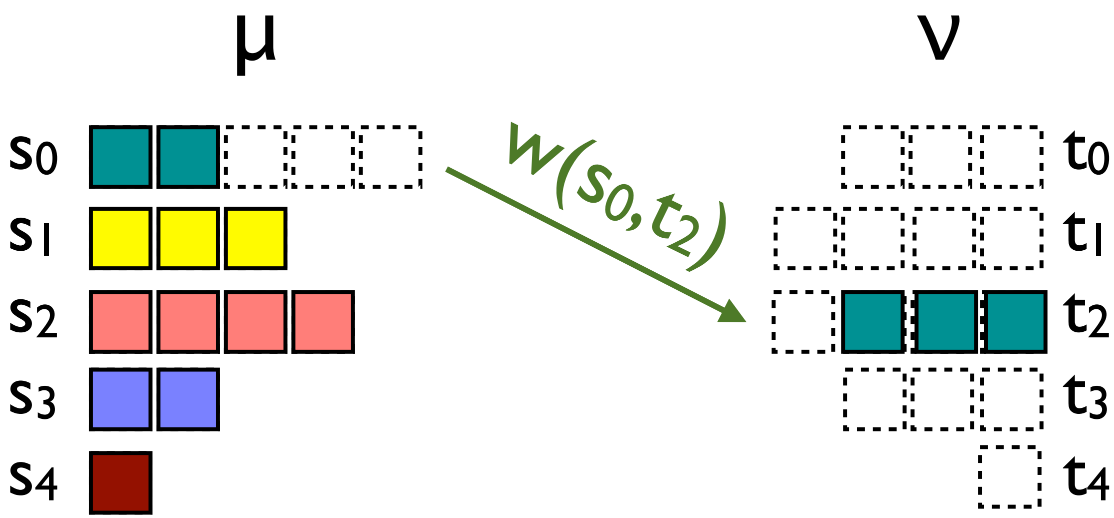

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so that $\forall i \geq 0,\ u_i \geq u_{i+1}$ & $\inf_i u_i = ||\mu - \nu||$

decreasing

limiting

Trace dist. (from above)

...seen before

Provide $\Omega_0 \subseteq \Omega_1 \subseteq \Omega_2 \subseteq \ldots$ such that $\bigcup_i \Omega_i$ is dense in $\Omega(P[s],P[s'])$

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$\mathcal{C}: S{\times}S \to \Delta(S^k \times S^k)$

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Stochastic process generating pairs of timed paths divided in multisteps of length k

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Take $\Omega_i = \{P_{\mathcal{C}}[s,s'] \in \Omega(P[s],P[s']) \mid \mathcal{C}$ of rank $2^i\}$

where $P_{\mathcal{C}}[s,s']$ is the probability generated by $\mathcal{C}$

Decidability

- A1: rational transition probabilities & residence-time distributions are computable on [q,q') with $q,q' \in \mathbb{Q}_+$
- A2: total variation between residence-time distributions is computable

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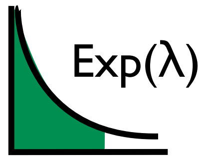
- A2: total variation between residence-time distributions is computable 

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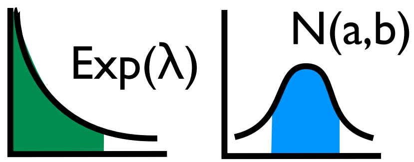

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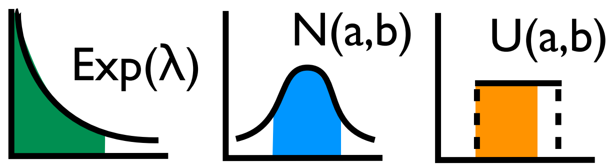

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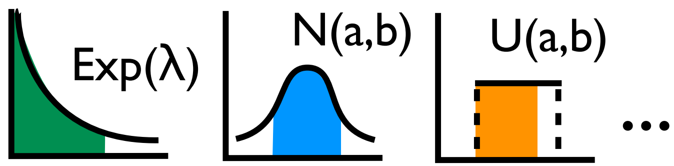

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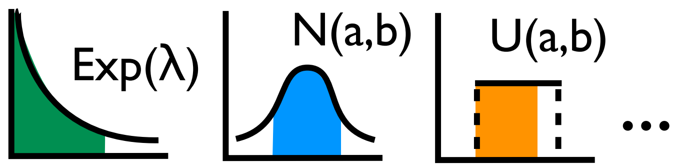

For any ε>0, the approximation procedure for the trace distance is decidable.

Concluding Remarks

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Thank you
for the attention