Quantitative Equational Reasoning

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Behavioural Metrics and Quantitative Logics

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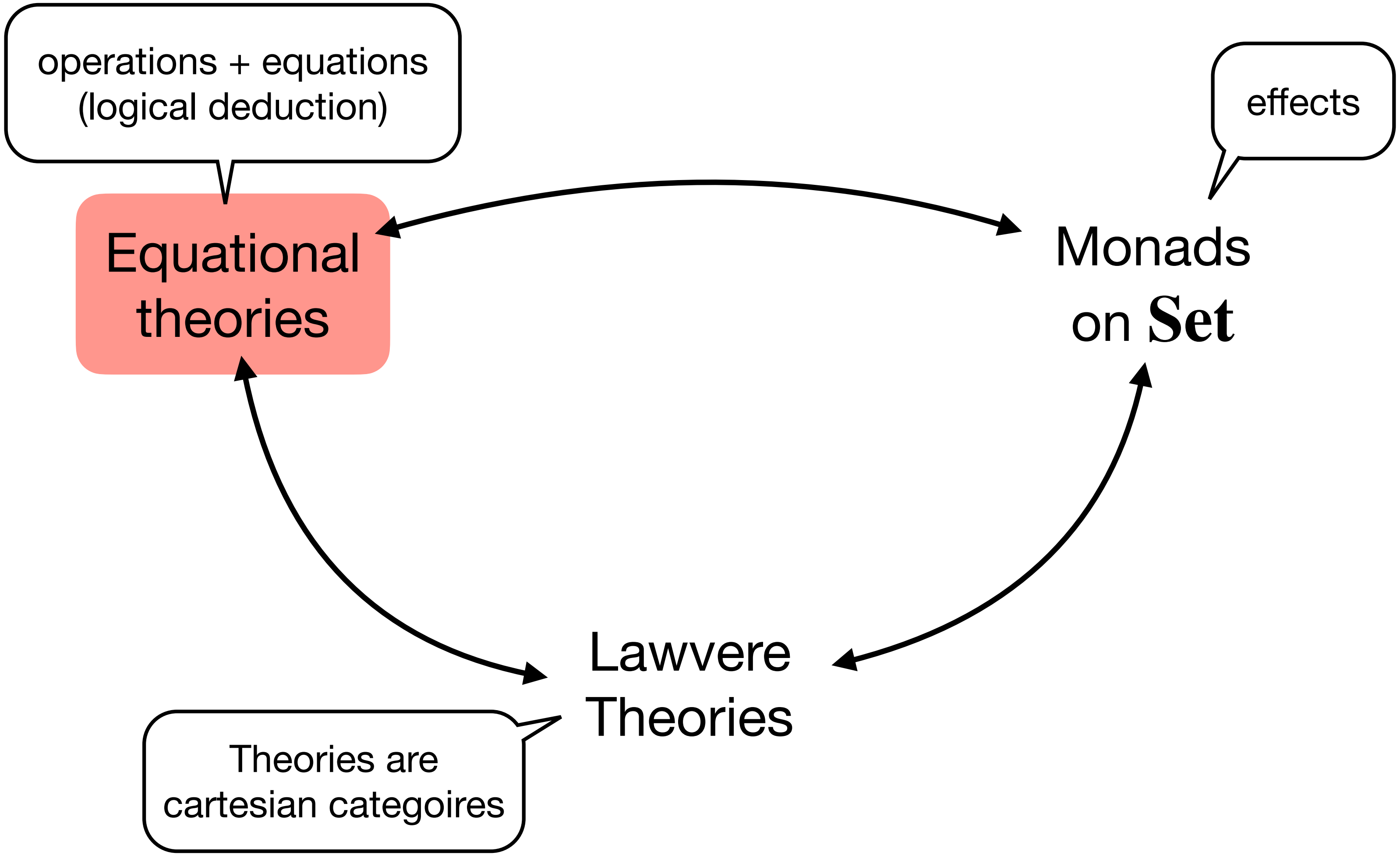
Motivations*

- Equations are at the heart of mathematical reasoning
- Reasoning about programs is also based on program equivalences
- This is the dawning of the age of quantitative reasoning
- We want quantitative analogues of algebraic reasoning
- (Pseudo)metrics instead of equivalence relations
- Quantitative Effects: monads on categories of metric spaces

(*) from P. Panangaden (Category Theory Seminar)

A Trinity of Ideas



operations + equations
(logical deduction)

Equational theories

effects

Monads on **Set**

Lawvere Theories

Theories are
cartesian categoires

Equational Theories by Example

(Barycentric Algebras - *M. Stone 1949*)

Signature: $\{ +_e : 2 \mid e \in [0,1] \}$

convex sum

Equational theory is obtained by closing under reflexivity, symmetry, transitivity of =, and congruence

(B1) $s +_1 t = s$

(B2) $t +_e t = t$

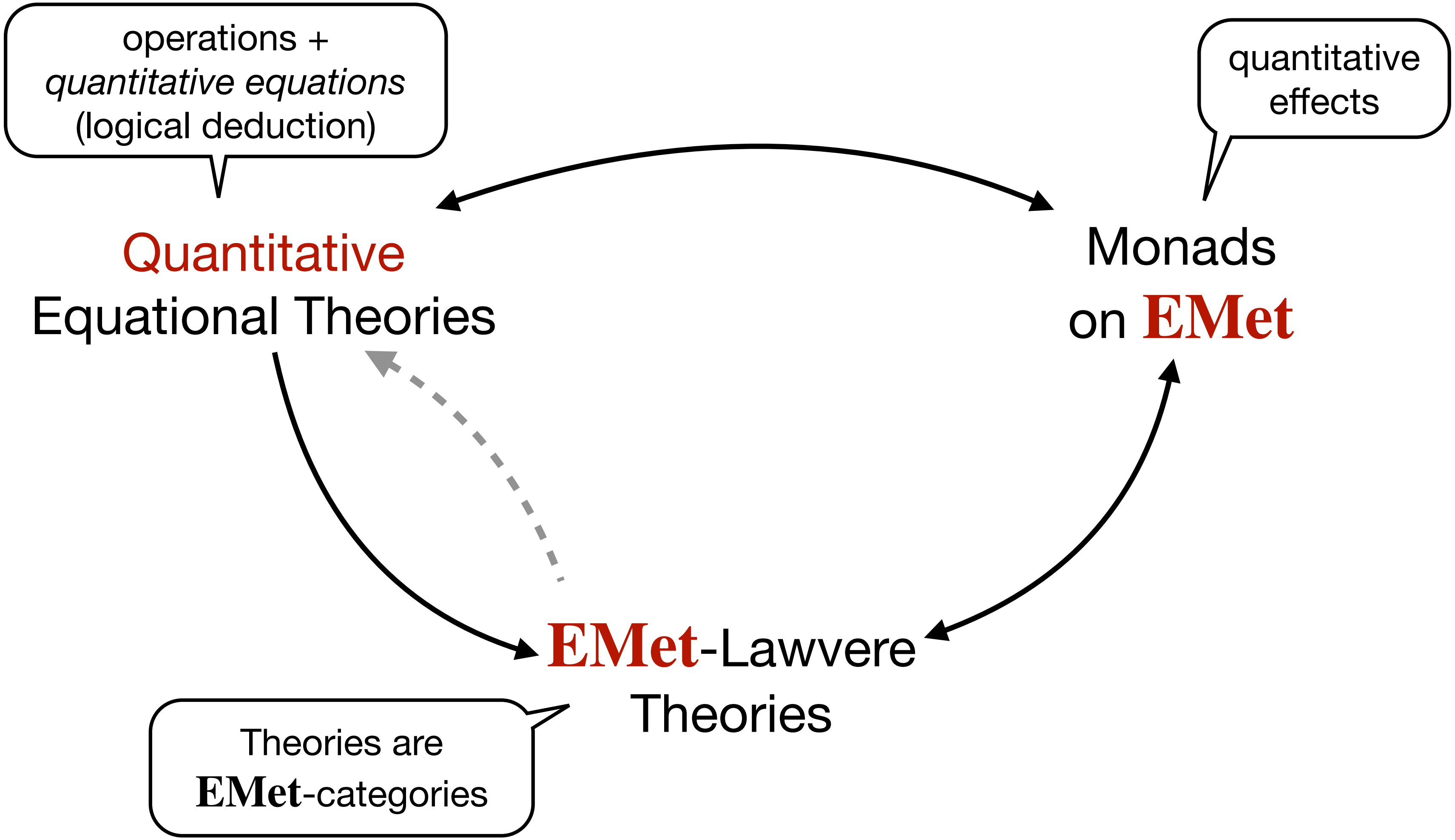
(SC) $s +_e t = t +_{1-e} s$

(SA) $(s +_e t) +_d u = s +_{ed} (t +_{\frac{(1-e)d}{1-ed}} u)$, **for** $e, d \in (0,1)$

The models interpret $+_e$ while respecting the axioms

The free algebra for this equational theory corresponds to the set of *finitely supported distributions*

the *Quantitative* Picture



operations +
quantitative equations
(logical deduction)

Quantitative
Equational Theories

quantitative
effects

Monads
on **EMet**

EMet-Lawvere
Theories

Theories are
EMet-categories

The Basic Idea

replace standard equations with quantitative equations

quantitative equation

$$s =_{\varepsilon} t$$

"s is within ε of t"

Goals: completeness theorem, universal free algebras, algebraic effects, and Birkhoff-like variety theorem…

Quantitative Equational Theories

Mardare, Panangaden, Plotkin (LICS'16)

finite arity

- **Signature of operations**: $\Sigma = \{(f_0 \colon n_0), \dots, (f_k \colon n_k), \dots\}$
- **Terms**: $s, t ::= x \mid f(t_1, \dots, t_n)$ ($\mathbb{T}_\Sigma X$ set of terms over X)
- **Quantitative equations**: $s =_\varepsilon t$ where $\varepsilon \in \mathbb{Q}_{\geq 0}$
- **Quantitative inferences**: $\{s_1 =_{\varepsilon_1} t_1, \dots, s_n =_{\varepsilon_n} t_n\} \vdash s =_\varepsilon t$
- **Quantitative equational theories**: sets $\mathscr{U}$ of quantitative inferences satisfying certain closure properties telling us what can be deduced...

Closure Properties

typically, we describe
a quantitative theory
using a set of **axioms**

(Refl) $\emptyset \vdash t =_0 t \in \mathcal{U}$

(Symm) $\{s =_\varepsilon t\} \vdash t =_\varepsilon s \in \mathcal{U}$

(Triang) $\{s =_\varepsilon t, t =_\delta u\} \vdash s =_{\varepsilon+\delta} u \in \mathcal{U}$

(Max) $\{s =_\varepsilon t\} \vdash s =_{\varepsilon+\delta} t \in \mathcal{U}$ (for $\delta > 0$)

(NExp) for $f\colon n \in \Sigma$,
$\{s_1 =_\varepsilon t_1, \ldots, s_n =_\varepsilon t_n\} \vdash f(s_1, \ldots, s_n) =_\varepsilon f(t_1, \ldots, t_n) \in \mathcal{U}$

(Cont) $\{\Gamma \vdash s =_\delta t \mid \delta > \varepsilon\} \subseteq \mathcal{U}$ implies $\Gamma \vdash s =_\varepsilon t \in \mathcal{U}$

(Subst) $\Gamma \vdash s =_\varepsilon t \in \mathcal{U}$ implies $\Gamma[u/x] \vdash s[u/x] =_\varepsilon t[u/x] \in \mathcal{U}$

(Cut) $\Gamma \vdash \Theta \subseteq \mathcal{U}$ and $\Theta \vdash s =_\varepsilon t \in \mathcal{U}$ implies $\Gamma \vdash s =_\varepsilon t \in \mathcal{U}$

(Assum) $s =_\varepsilon t \in \Gamma$ implies $\Gamma \vdash s =_\varepsilon t \in \mathcal{U}$

Quantitative Algebras

$$\mathscr{A} = \left(A, d_A, \{f_{\mathscr{A}} \colon A^n \to A \mid f \colon n \in \Sigma\}\right)$$

- (A, d_A) is an *extended* metric space (carrier)
- $f_{\mathscr{A}} \colon A^n \to A$ interpretations of operations are non-expansive

$$\max_i d_A(a_i, b_i) \geq d(f_{\mathscr{A}}(a_1, \ldots, a_n), f_{\mathscr{A}}(b_1, \ldots, b_n))$$

Morphisms: $h \colon \mathscr{A} \to \mathscr{B}$

- Σ-homomorphisms

  $h(f_{\mathscr{A}}(a_1, \ldots, a_n)) = f_{\mathscr{B}}(h(a_1), \ldots, h(a_n))$ for all $f \colon n \in \Sigma$
- non-expansive $d_A(a, a') \geq d_B(h(a), h(a'))$

Models of a Theory

Satisfiability

$$\mathscr{A} \models \left(\{ t_i =_{\varepsilon_i} s_i \mid i = 1, \ldots, n \} \vdash t =_{\varepsilon} s \right)$$

iff

for any Σ-homomorphism $\imath \colon \mathbb{T}_\Sigma X \to A$

$d_A(\imath(t_i), \imath(s_i)) \leq \varepsilon_i$, for $i = 1, \ldots, n$ implies $d_A(\imath(t), \imath(s)) \leq \varepsilon$

A quantitative algebra $\mathscr{A}$ *is a model for a quantitative theory* $\mathscr{U}$ if it satisfies all quantitative inferences in it

Interpolative Barycentric Algebras

Mardare, Panangaden, Plotkin (LICS'16)

Signature: $\{ +_e : 2 \mid e \in [0,1] \}$

convex sum

(B1) $\vdash s +_1 t =_0 s$

(B2) $\vdash t +_e t =_0 t$

(SC) $\vdash s +_e t =_0 t +_{1-e} s$

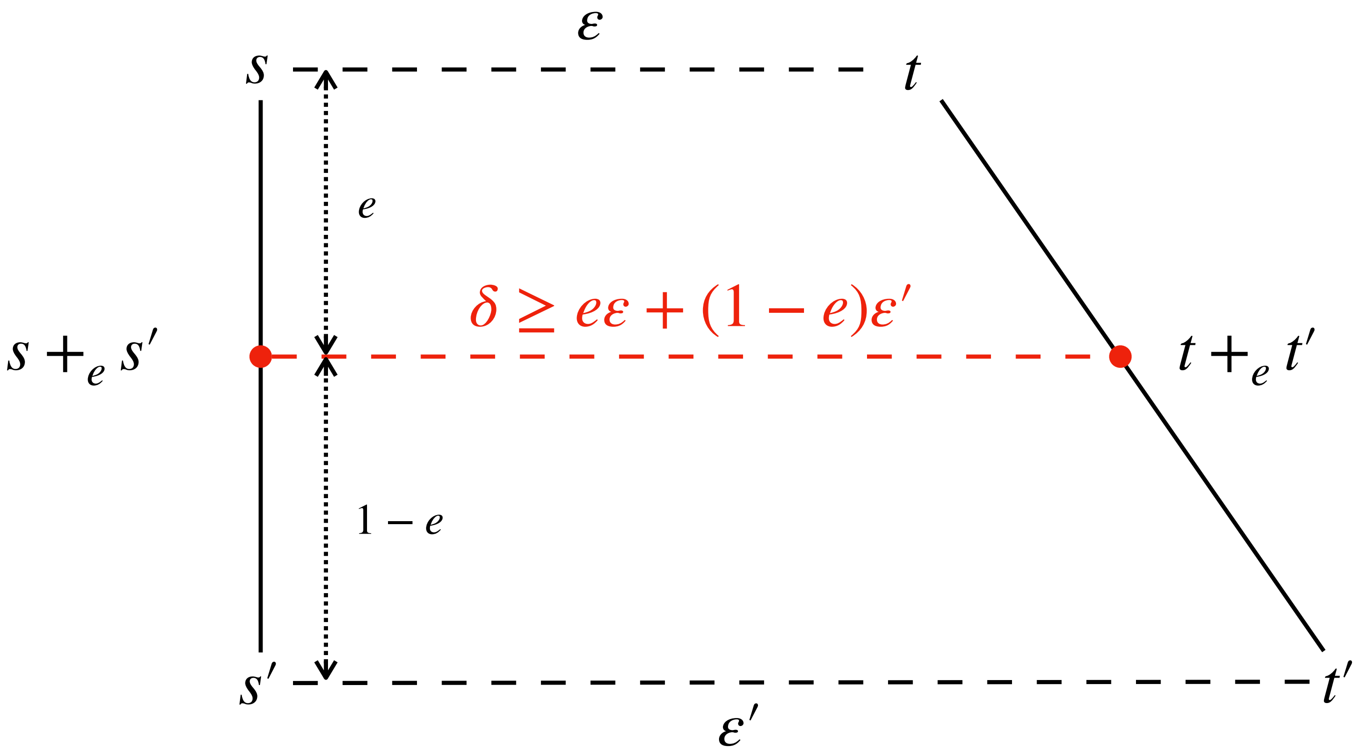
(SA) $\vdash (s +_e t) +_d u =_0 s +_{ed} (t +_{\frac{(1-e)d}{1-ed}} u)$, **for** $e, d \in (0,1)$

(IB) $\{s =_\varepsilon t, s' =_{\varepsilon'} t'\} \vdash s +_e s' =_\delta t +_e t'$,

where $\delta \geq e\varepsilon + (1-e)\varepsilon'$

A geometric intuition

(IB) $\{s =_\varepsilon t, s' =_{\varepsilon'} t'\} \vdash s +_e s' =_\delta t +_e t'$, **where** $\delta \geq e\varepsilon + (1-e)\varepsilon'$



..some of models

Unit interval with Euclidian distance and convex combinators

$([0,1], d_{[0,1]}) \qquad (+_e)^{[0,1]}(a,b) = ea + (1-e)b$

Finitely supported distributions with **Kantorovich distance**

$(\mathscr{D}(M), \mathscr{K}(d_M)) \qquad (+_e)^{\mathscr{D}}(\mu,\nu) = e\mu + (1-e)\nu$

Radon probability measures with **Kantorovich distance**

$(\Delta(M), \mathscr{K}(d_M)) \qquad (+_e)^{\Delta}(\mu,\nu) = e\mu + (1-e)\nu$

Main general results

1. Completeness
2. Free-universal algebras
3. Quantitative effects (monads)

Completeness

Mardare, Panangaden, Plotkin (LICS'16)

For quantitative the equational logic we have an analogue of the usual completeness theorem

Theorem (Birkhoff completeness)

$$\forall \mathscr{A} \in \mathbf{Mod}(\mathscr{U}) \,.\, \mathscr{A} \models (\Gamma \vdash t =_{\varepsilon} s)$$

iff

$$(\Gamma \vdash t =_{\varepsilon} s) \in \mathscr{U}$$

Free Models

- Given $\mathscr{U}$ quantitative theory for the signature Σ
- and (M, d_M) an *extended* metric space,
- we define $\mathscr{U}_M$ as the quantitative theory for the signature $\Sigma + M$ with $\mathscr{U}$ and $\{ \vdash m =_\varepsilon n \mid d_M(m,n) \leq \varepsilon \}$ as set of axioms

An extended (pseudo)metric on $\mathbb{T}_\Sigma M$

$$d_{\mathscr{U}}(s,t) = \inf\{\varepsilon \mid \emptyset \vdash t =_\varepsilon s \in \mathscr{U}_M\}$$

Free model of $\mathscr{U}$ over (M, d_M)

$$\mathbf{T}_{\mathscr{U}} M = \left(\mathbb{T}_\Sigma M, d_{\mathscr{U}}, \{f_{\mathscr{U}} \mid f \colon n \in \Sigma\}\right)$$

quotentied wrt $=_0$-provability

Universal Property

The free-model enjoys the expected universal property of free-algebras

$\mathbf{EMet}$

$$\begin{tikzcd}
M \arrow[r, "\eta_M"] \arrow[dr, "\alpha"'] & \mathbb{T}_\Sigma M \arrow[d, "h"] \\
& A
\end{tikzcd}$$

non-expansive

$\mathbf{Mod}(\mathscr{U})$

$$\begin{tikzcd}
T_{\mathscr{U}} \arrow[d, dashed, "h"] \\
\mathscr{A}
\end{tikzcd}$$

Free model of $\mathscr{U}$

Algebraic Effects on $\mathbf{EMet}$

$$\begin{tikzcd}
\mathbf{Mod}(\mathscr{U}) \arrow[d, bend left] \arrow[d, phantom, "\dashv"] \\
\mathbf{EMet} \arrow[u, bend left] \arrow[d, Rightarrow] \\
T_{\mathscr{U}}
\end{tikzcd}$$

Free Model functor
(maps an extended metric space to the free model of $\mathscr{U}$)

Forgetful functor
(maps a quantitative algebra to its carrier)

This is how we get algebraic **quantitative effects**!

Monad induced by $\mathscr{U}$ on $\mathbf{EMet}$

Representation Theorem

Bacci, Mardare, Panangaden, Plotkin (LICS'18)

Theorem

For any ***basic*** quantitative equational theory $\mathscr{U}$

$$\mathbf{Mod}(\mathscr{U}) \cong T_{\mathscr{U}}\textbf{-Alg}$$

EM algebras for the monad $T_{\mathscr{U}}$

A quantitative equational theory $\mathscr{U}$ is *basic* if it can be axiomatised by a set of quantitative inferences of the form

$$\{x_1 =_{\varepsilon_1} y_1, \ldots, x_n =_{\varepsilon_n} y_n\} \vdash s =_{\varepsilon} t$$

only quantitative equations between variables

Examples

Ex1: Barycentric Algebras

Mardare, Panangaden, Plotkin (LICS’16)

Signature: $\{ +_e : 2 \mid e \in [0,1] \}$

(B1) $\vdash s +_1 t =_0 s$

(B2) $\vdash t +_e t =_0 t$

(SC) $\vdash s +_e t =_0 t +_{1-e} s$

(SA) $\vdash (s +_e t) +_d u =_0 s +_{ed} (t +_{\frac{(1-e)d}{1-ed}} u)$, **for** $e, d \in (0,1)$

(IB) $\{s =_\varepsilon t, s' =_{\varepsilon'} t'\} \vdash s +_e s' =_\delta t +_e t'$,

where $\delta \geq e\varepsilon + (1-e)\varepsilon'$

$$\begin{tikzcd}
\mathbf{Mod}(\mathscr{B}) \arrow[r, bend right] \arrow[r, phantom, "\perp"] & \mathbf{EMet} \arrow[l, bend right]
\end{tikzcd}$$

➡ $T_{\mathscr{B}} \cong \mathscr{D}$

Finitely supported distributions with **Kantorovic distance**

Ex2: Quantitative Semilattices

Mardare, Panangaden, Plotkin (LICS'16)

Signature: $\{\mathbf{0} : 0, \oplus : 2\}$

bottom

join

(S0) $\vdash 0 \oplus t =_0 t$

(S1) $\vdash t \oplus t =_0 t$

(S2) $\vdash s \oplus t =_0 t \oplus s$

(S3) $\vdash (s \oplus t) \oplus u =_0 s \oplus (t \oplus u)$

(S4) $\{s =_\varepsilon t, s' =_{\varepsilon'} t'\} \vdash s \oplus s' =_\delta t \oplus t'$, **where** $\delta \geq \max\{\varepsilon, \varepsilon'\}$

$$\begin{tikzcd}
\mathbf{Mod}(\mathcal{S}) \arrow[r, bend right] \arrow[r, phantom, "\perp"] & \mathbf{EMet} \arrow[l, bend right] \arrow[r, Rightarrow] & T_{\mathcal{S}} \cong \mathscr{P}_f
\end{tikzcd}$$

Finite powerset monad with **Hausdorff distance**

Ex3: Quantitative Exceptions

Bacci, Mardare, Panangaden, Plotkin (LICS'18)

Signature: $\{e : 0 \mid e \in E\}$

A metric space (E, d_E) of exceptions

(E0) $\vdash e_1 =_\varepsilon e_2\,,$ **where** $\varepsilon \geq d_E(e_1, e_2)$

$$\begin{tikzcd} \mathbf{Mod}(\mathscr{E}_E) \arrow[r, bend right, ""] \arrow[r, phantom, "\perp"] & \mathbf{EMet} \arrow[l, bend right, ""] \end{tikzcd} \quad ➡ \quad T_{\mathscr{E}_E} \cong (- + E)$$

Quantitative Exception Monad

Ex4: Quantitative Reader

Bacci, Mardare, Panangaden, Plotkin (CALCO'21)

Signature: $\{\mathbf{r} : |A|\}$

reads from a finite set of input actions $A = \{a_1, \dots, a_n\}$ and proceeds

(Idem) $\vdash x =_0 \mathbf{r}(x, \dots, x)$

(Diag) $\vdash \mathbf{r}(x_{1,1}, \dots, x_{n,n}) =_0 \mathbf{r}(\mathbf{r}(x_{1,1}, \dots, x_{1,n}), \dots, \mathbf{r}(x_{n,1}, \dots, x_{n,n}))$

Monad in $\mathbf{EMet}$ only for *discrete* spaces of inputs!

$$\begin{tikzcd}\mathbf{Mod}(\mathscr{R}) \arrow[r, bend right] \arrow[r, phantom, "\perp"] & \mathbf{EMet} \arrow[l, bend right]\end{tikzcd}$$

➡ $T_{\mathscr{R}} \cong (-)^{\underline{A}}$

Reader monad for the discrete space $\underline{A}$

Ex5: Quantitative Writer

Bacci, Mardare, Panangaden, Plotkin (CALCO’21)

metric space

Let $(\Lambda, \star, 0)$ be a monoid with non-expansive multiplication

Signature: $\{\mathbf{w}_a : 1 \mid a \in \Lambda\}$

writes the output symbol a and proceeds

(Zero) $\vdash x =_0 \mathbf{w}_0(x)$

(Mult) $\vdash \mathbf{w}_a(\mathbf{w}_b(x)) =_0 \mathbf{w}_{a\star b}(x)$

(Diff) $\{x =_\varepsilon x'\} \vdash \mathbf{w}_a(x) =_\delta \mathbf{w}_b(x')\,,$ **for** $\delta \geq d_\Lambda(a,b) + \varepsilon$

$$\begin{tikzcd}
\mathbf{Mod}(\mathscr{W}) \arrow[r, bend right] \arrow[r, phantom, "\perp"] & \mathbf{EMet} \arrow[l, bend right] \arrow[r, Rightarrow] & T_{\mathscr{W}} \cong (\Lambda \oplus -\,)
\end{tikzcd}$$

Writer monad for the metric space Λ

all Cauchy
sequences have limit

…on Complete metric Spaces

Representation Theorem

Bacci, Mardare, Panangaden, Plotkin (LICS'18)

Theorem

For any *continuous* quantitative equational theory $\mathscr{U}$

$$\mathbf{CMod}(\mathscr{U}) \cong \mathbb{C}T_{\mathscr{U}}\textbf{-Alg}$$

EM algebras for the monad $\mathbb{C}T_{\mathscr{U}}$

A quantitative equational theory is *continuous* if it can be axiomatised by a collection of *continuous schemata* of quantitative inferences

$$\{x_1 =_{\varepsilon_1} y_1, \ldots, x_n =_{\varepsilon_n} y_n\} \vdash t =_\varepsilon s \quad - \textbf{for } \varepsilon \geq f(\varepsilon_1, \ldots, \varepsilon_n)$$

continuous real-valued function

Algebraic Effects on **CEMet**

… this happens because, for *continuous* equational theories the completion functor lifts to the models of a theory

$$\begin{tikzcd}
\mathbf{Mod}(\mathscr{U}) \arrow[r, "\hat{\mathbb{C}}"] \arrow[d, bend right] \arrow[d, phantom, "\vdash"] & \mathbf{CMod}(\mathscr{U}) \arrow[d, bend right] \arrow[d, phantom, "\vdash"] \\
\mathbf{EMet} \arrow[r, "\mathbb{C}"] \arrow[u, bend right] \arrow[d, thick] & \mathbf{CEMet} \arrow[u, bend right] \arrow[d, thick] \\
T_{\mathscr{U}} & \mathbb{C}T_{\mathscr{U}}
\end{tikzcd}$$

Models of $\mathscr{U}$ over *complete* metric spaces

effects on complete metric spaces

Barycentric Algebras (again!)

Mardare, Panangaden, Plotkin (LICS'16)

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where $\delta \geq e\varepsilon + (1-e)\varepsilon'$

$$\begin{tikzcd} \mathbf{CMod}(\mathscr{B}) \arrow[r, bend right] \arrow[r, phantom, "\perp"] & \mathbf{CEMet} \arrow[l, bend right] \end{tikzcd}$$

➡ $\mathbb{C}T_{\mathscr{B}} \cong \Delta$

Radon probability measures with
Kantorovic distance

Quantitative Semilattices (again)

Mardare, Panangaden, Plotkin (LICS'16)

Signature: $\{\mathbf{0}\colon 0, \oplus : 2\}$

bottom

join

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(S1) $\vdash t \oplus t =_0 t$

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$$\begin{tikzcd}
\mathbf{CMod}(\mathcal{S}) \arrow[r, bend right] \arrow[r, phantom, "\perp"] & \mathbf{CEMet} \arrow[l, bend right] \arrow[r, Rightarrow] & \mathbb{C}T_{\mathcal{S}} \cong \mathscr{C}
\end{tikzcd}$$

Compact powerset monad
with Hausdorff distance

There’s much more!

1. Birkhoff-like variety theorem
2. Expressiveness of Quantitative Effects
3. Compositionality results
4. Fixed points

Birkhoff Variety Theorem

Mardare, Panangaden, Plotkin (LICS'17)
Milius and Urbat (FoSSaCS'19)

It is stated for signatures of operators of possibly infinite arity

Axiomatised by a set of basic inferences of the form
$\Gamma \vdash s =_\varepsilon t$, where $|\mathsf{Vars}(\Gamma)| < \lambda$ (regular cardinal)

Theorem

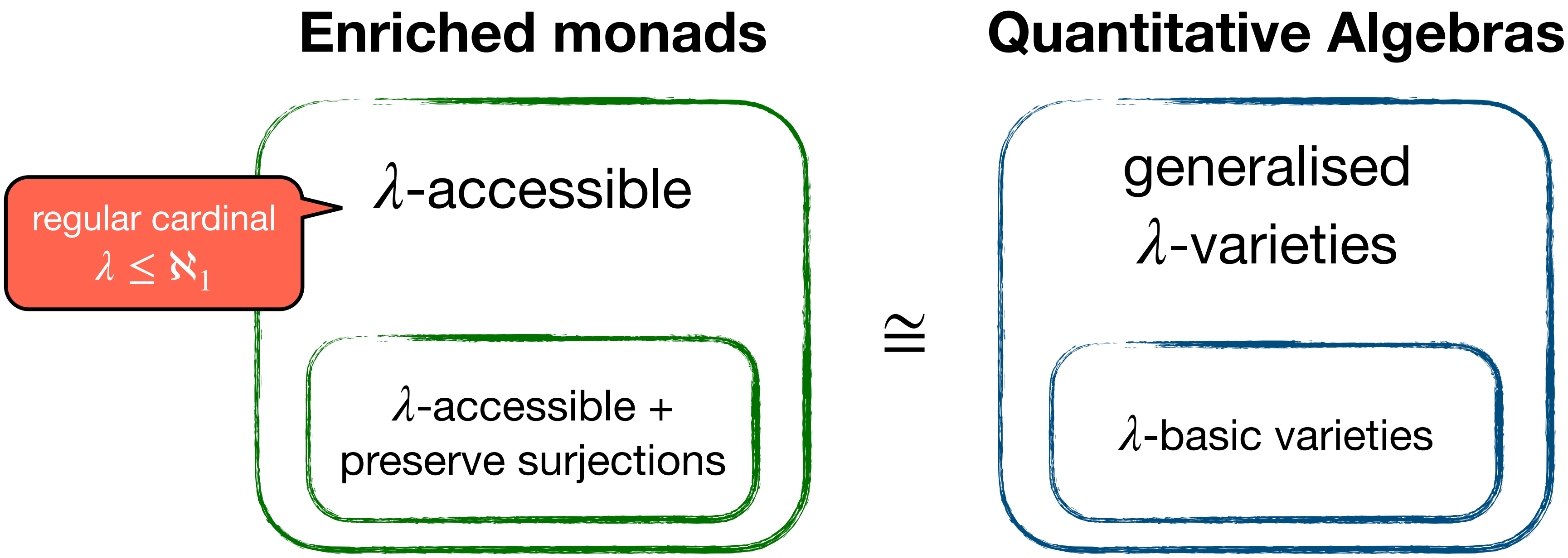
A full subcategory of $\mathbf{QA}$ is a λ-variety iff it is closed under products, subalgebras, and λ-*reflexive* homomorphic images

surjections $e \colon M \to N$ s.t., for all $N' \subseteq N$ with $|N'| < \lambda$,
there exists $M' \subseteq M$ such that e restricts to $M' \xrightarrow{\cong} N'$

Expressiveness of Effects

Ford, Milius, Schröder (CALCO’21)
Adamek (LICS’22)+(CALCO’23)

The format of the equations determines the class of monads that quantitative equational theories characterise



Enriched monads

regular cardinal
$\lambda \leq \aleph_1$

λ-accessible

λ-accessible +
preserve surjections

$\cong$

Quantitative Algebras

generalised
λ-varieties

λ-basic varieties

The cases for $\lambda = 1$ and $\lambda = \aleph_0$ are still **open problems**!

Compositionality Results

Bacci, Mardare, Panangaden, Plotkin (LICS’18)+(CALCO’21)

Following Hyland, Plotkin, and Power (TCS’06) we considered the combinations of theories via *sum* and *tensor*

Theorem

Given $\mathscr{U}$ and $\mathscr{U}'$ over disjoint signatures it holds that

$$T_{\mathscr{U}} + T_{\mathscr{U}'} \cong T_{\mathscr{U}+\mathscr{U}'} \qquad T_{\mathscr{U}} \otimes T_{\mathscr{U}'} \cong T_{\mathscr{U}\otimes\mathscr{U}'}$$

disjoint union of their axioms

the operations of the theories commutes over each other

We get compositional axiomatization of bahavioral metrics for (labelled) Markov Processes, MDPs, Mealy machines…

Quantitative Theory Transformers

We can obtain quantitative analogues of Cenciarelli and Moggi's monad transformers at the level of theories via sum & tensor

Exception transformer

$\mathscr{U} \mapsto \mathscr{U} + \mathscr{E}_E$ ➡ $T_{\mathscr{U}} + \mathscr{E}_E \cong T_{\mathscr{U}}(-+E)$

Reader transformer

$\mathscr{U} \mapsto \mathscr{U} \otimes \mathscr{R}$ ➡ $T_{\mathscr{U}} \otimes \mathscr{R} \cong (T_{\mathscr{U}}-)^{\underline{A}}$

Writer transformer

$\mathscr{U} \mapsto \mathscr{U} \otimes \mathscr{W}$ ➡ $T_{\mathscr{U}} \otimes \mathscr{W} \cong (\Lambda \oplus T_{\mathscr{U}}-)$

Fixed points

Mardare, Panangaden, Plotkin (LICS’21)

Quantitative equational logic has been extended with “Banach” fixed points: reasoning about the distance of recursively defined terms

Key idea

The fixed point operators $f\colon A^n \to A$ should admit a finite *Banach pattern* $\theta \subseteq \left\{(\alpha_1, \ldots, \alpha_n) \mid \sum_i \alpha_i \leq 1\right\}$, that is:

$$d_A(f(a_1, \ldots, a_n), f(b_1, \ldots, n_n)) \leq \max_{\alpha \in \theta} \sum_i \alpha_i \cdot d_A(a_i, b_i)$$

The resulting theory is the *quantitative analogue of an iteration theory (Bloom-Ésik // Hasegawa)*

Conclusions

- Quantitative theories are *the right tool* to algebraically describe **quantitative effects** ("effects with a metric twist")
- Plenty of *non-trivial examples*: Kantorovich metric, Hausdorff metric, Total variation, p-Wasserstein metric, etc.
- Non-trivial generalisations of results holding in $\mathbf{Set}$ (Birkhoff variety theorem, enriched accessible monads, combination of theories via sum & tensor, fixed points)
- Still many interesting (unexpected) open problems