Complete Axiomatization for the Bisimilarity Distance on MCs

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Introduction

- **Kleene's Theorem**: fundamental correspondence between regular expressions and DFAs
- **Salomaa'66**, **Kozen'91**: complete axiomatization for proving equivalence of regular expressions
- **Milner'84**: applied the above program on process behaviors and LTSs
- Many variations of the above schema

Example: Markov chains

Expressions: $t, s := X \mid \mathbf{a}.t \mid t +_e s \mid \mathrm{rec}\ X.t$

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$X \in \mathbb{X}$

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probabilistic choice $e \in [0,1]$

Example: Markov chains

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Expressions: $t, s := \; X \;|\; \mathbf{a}.t \;|\; t +_e s \;|\; \text{rec } X.t$

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recursion

Example: Markov chains

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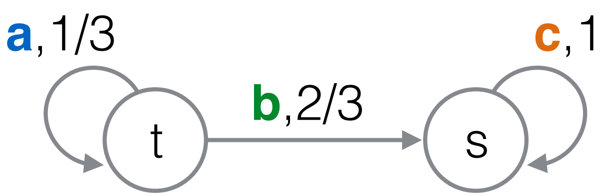
- names: $X \in \mathbb{X}$
- action-prefix: $a \in \mathbb{A}$
- probabilistic choice: $e \in [0,1]$
- recursion

Kleene's theorem for MCs



$\Longleftrightarrow$

$t = \text{rec } X.(a.X +_{1/3} b.s)$

$s = \text{rec } Y.(c.Y)$

Example: Markov chains

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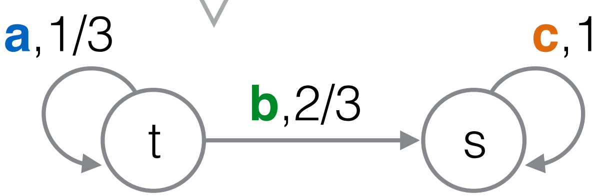
Kleene's theorem for MCs

finite MCs



⟺

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Example: Markov chains

(B1) $\vdash t +_1 s = t$

(B2) $\vdash t +_e t = t$

(SC) $\vdash t +_e s = s +_{1-e} t$

(SA) $\vdash (t +_e s) +_{e'} u = t +_{ee'} (s +_{\frac{e'-ee'}{1-ee'}} u)$ — for $e,e' \in [0,1)$

(Unfold) $\vdash$ rec X.t = t[rec X.t / X]

(Fix) {t = s[t / X]} $\vdash$ t = rec X.s — for X guarded in t

(Unguard) $\vdash$ rec X.(t $+_e$ X) = rec X.t

Example: Markov chains

Stark-Smolka axiomatization

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Milner's recursion axioms

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…for probabilistic systems

- **Generative Markov chains:**
  Baeten-Bergstra-Smolka'95 & Stark-Smolka'00
- **Simple Probabilistic Automata:**
  Bandini-Segala'01
- **(fully) Probabilistic Automata:**
  Mislove-Ouaknine-Worrell'04 (strong-bisimulation)
  Deng-Palamidessi'07 (weak-bisimulation & behavioral eq.)
- **Quantitative Kleene Coalgebras:**
  Silva-Bonchi-Bonsangue-Rutten'11 (coagebraic bisim.)

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  By using **Quantitative Equational Theories*** of **Mardare-Panangaden-Plotkin** (LICS'16)

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- How do we do it?
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$$s = t \quad \Longrightarrow \quad s =_{\varepsilon} t$$

Equational Theories

$$\{t_i = s_i \mid i \in I\} \vdash t = s$$

inference

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$$\{t_i = s_i \mid i \in I\} \vdash t = s$$

inference

(Refl) $\vdash t = t$

(Symm) $\{t = s\} \vdash s = t$

(Trans) $\{t = u,\ u = s\} \vdash t = s$

(Cong) $\{t_1 = s_1, \ldots, t_n = s_n\} \vdash f(t_1, \ldots, t_n) = f(s_1, \ldots s_n)$ — for $f \in \Sigma$

Quantitative Theories

Mardare-Panangaden-Plotkin (LICS'16)

$$\{t_i =_{\varepsilon_i} s_i \mid i \in I\} \vdash t =_{\varepsilon} s$$

quantitative inference

(Refl) $\vdash t =_0 t$

(Symm) $\{t =_\varepsilon s\} \vdash s =_\varepsilon t$

(Triang) $\{t =_\varepsilon u, u =_\delta s\} \vdash t =_{\varepsilon+\delta} s$

(NExp) $\{t_1 =_\varepsilon s_1,\ldots,t_n =_\varepsilon s_n\} \vdash f(t_1,\ldots,t_n) =_\varepsilon f(s_1,\ldots s_n)$ — for $f \in \Sigma$

(Max) $\{t =_\varepsilon s\} \vdash t =_{\varepsilon+\delta} s$ — for $\delta>0$

(Arch) $\{t =_\delta s \mid \delta>\varepsilon \} \vdash t =_\varepsilon s$

Quantitative Semantics

Quantitative Algebra

$\mathcal{A} = (A, \Sigma_A, d_A)$ → (A, Σ_A) — Universal algebra
→ (A, d_A) — metric space

Satisfiability

$$\mathcal{A} \models \left(\{ t_i =_{\varepsilon_i} s_i \mid i \in I \} \vdash t =_{\varepsilon} s \right)$$

iff

for all $i \in I$. $d_A([\![t_i]\!], [\![s_i]\!]) \leq \varepsilon_i$ implies $d_A([\![t]\!], [\![s]\!]) \leq \varepsilon$

completeness

quantitative algebra

quantitative theory

$\mathscr{A} \models (\vdash t =_\varepsilon s)$ $\quad$ $(\vdash t =_\varepsilon s) \in \mathscr{U}$

soundness

completeness

quantitative algebra

$\mathscr{A}_{MC} \models (\vdash t =_{\varepsilon} s)$

quantitative theory

$(\vdash t =_{\varepsilon} s) \in \mathscr{U}_{MC}$

soundness

The Quantitative Universal Algebra

Universal Algebra of MCs

Signature: $X : 0 \mid$ **a**$.\text{-} : 1 \mid +_e : 2 \mid \mathrm{rec}\ X : 1$

Universal Algebra of MCs

Signature: X : 0 | **a**.- : 1 | $+_e$: 2 | rec X : 1

$(X)_{MC}$ = X

Universal Algebra of MCs

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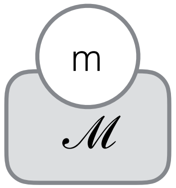
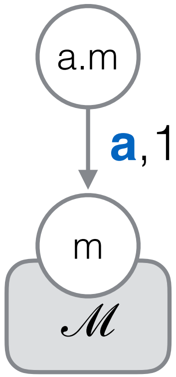
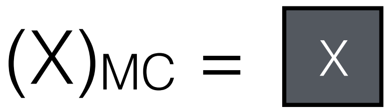
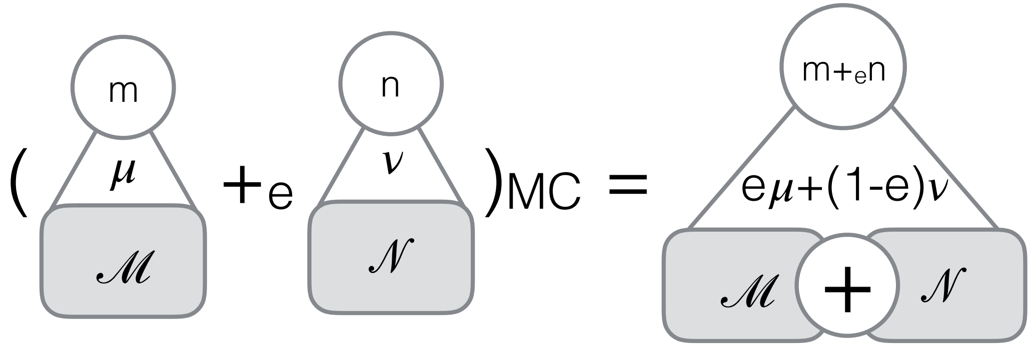
(**a**. 

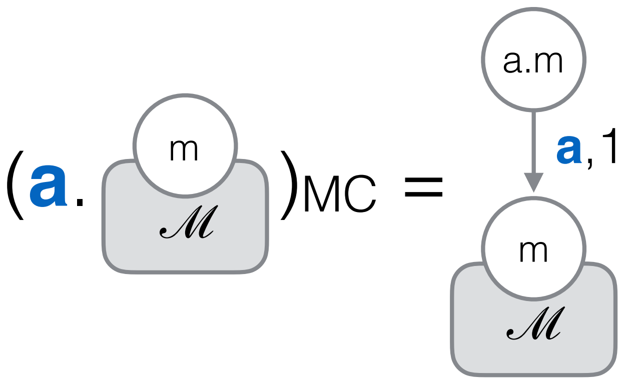
$)_{MC}$ = 

Universal Algebra of MCs

Signature: X : 0 | **a**.- : 1 | $+_e$: 2 | rec X : 1







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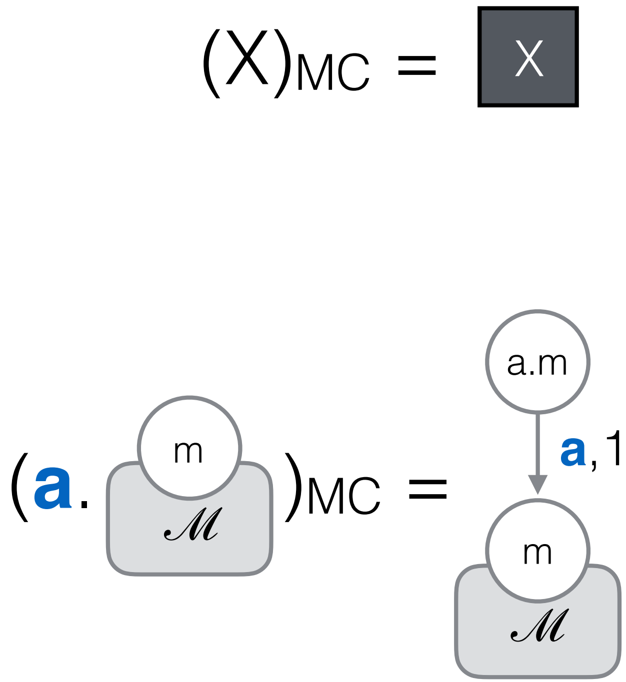
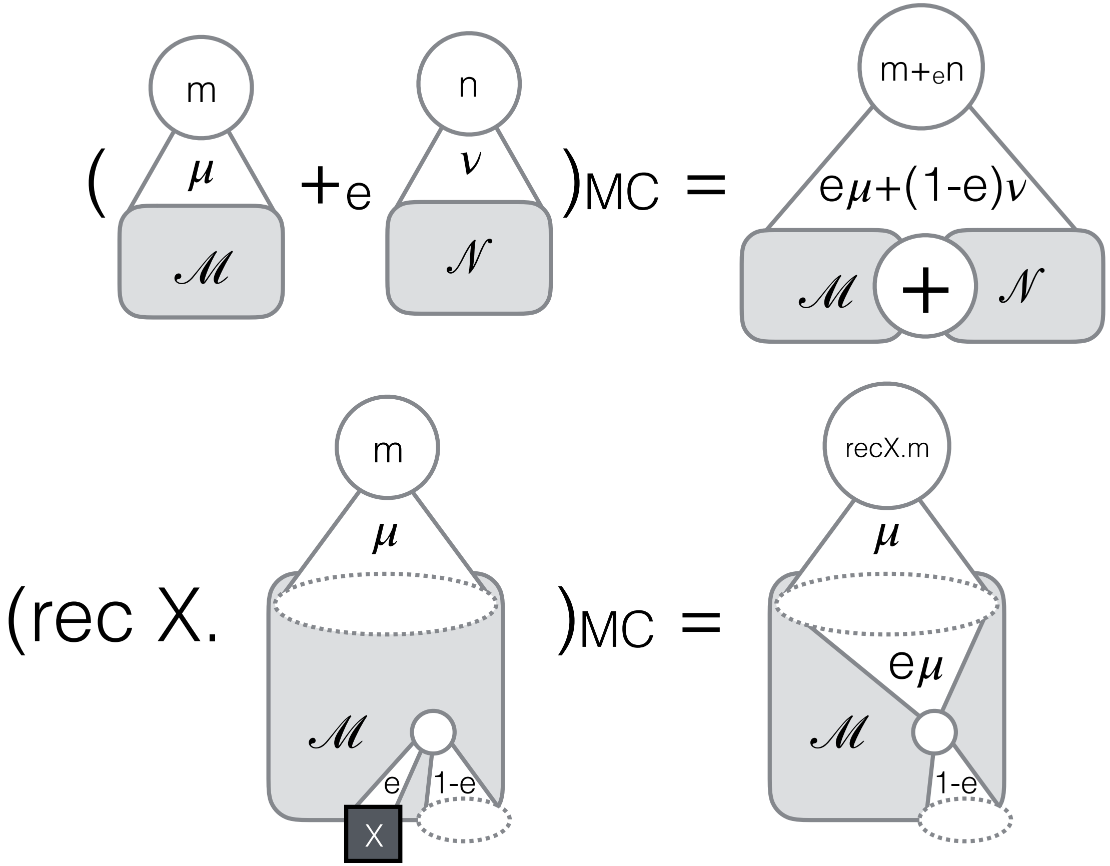
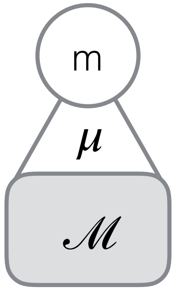


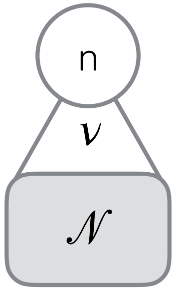


Bisimilarity distance for MCs

(Desharnais et al. TCS'04)

it is the least 1-bounded pseudometric satisfying

$d_{MC}($ 

, 

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$$d_{MC}(\, m\ \mu\ \mathcal{M} \,,\, n\ \nu\ \mathcal{N} \,) = \min \{ \int \Lambda(d_{MC})\, d\omega \mid \omega \in \Omega(\mu,\nu) \}$$

$\Lambda(d_{MC})$ — greatest 1-bounded pseudometric on $(\mathbb{A}\times MC)\cup\mathbb{X}$

s.t, for all $\mathbf{a}\in\mathbb{A}$, $\Lambda(d_{MC})((\mathbf{a}, m\ \mathcal{M}),(\mathbf{a}, n\ \mathcal{N})) = d_{MC}(m\ \mathcal{M}, n\ \mathcal{N})$

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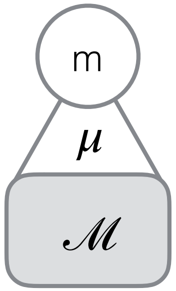
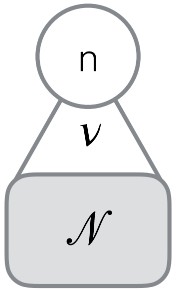
$d_{MC}($ 

, 

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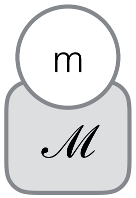
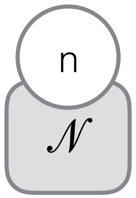
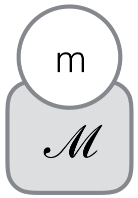
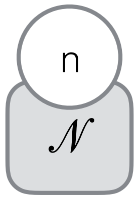
s.t, for all $\mathbf{a} \in \mathbb{A}$, $\Lambda(d_{MC})((\mathbf{a},$ 

$),(\mathbf{a},$ 

$)) = d_{MC}($ 

, 

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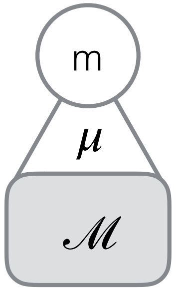
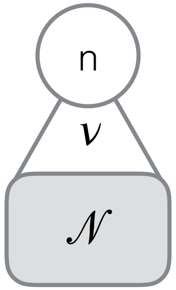
$d_{MC}($ 

$,$ 

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Kantorovich lifting

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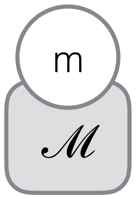
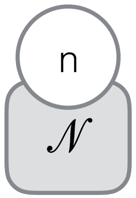
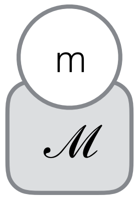
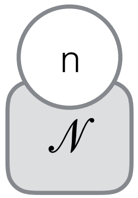
s.t, for all $\mathbf{a} \in \mathbb{A}$, $\Lambda(d_{MC})((\mathbf{a},$ 

$),(\mathbf{a},$ 

$)) = d_{MC}($ 

$,$ 

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Bisimilarity distance for MCs

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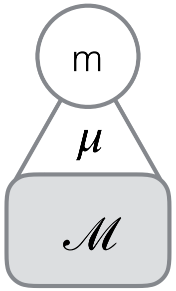
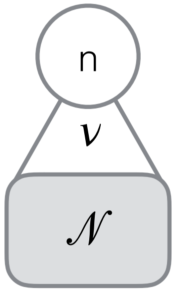
couplings
=
probabilistic "relations"

$d_{MC}($  ,  $) = \min \{ \int \Lambda(d_{MC})\, d\omega \mid \omega \in \Omega(\mu,\nu) \}$

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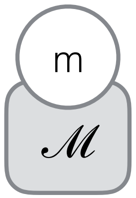
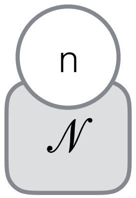
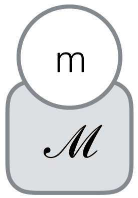
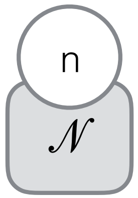
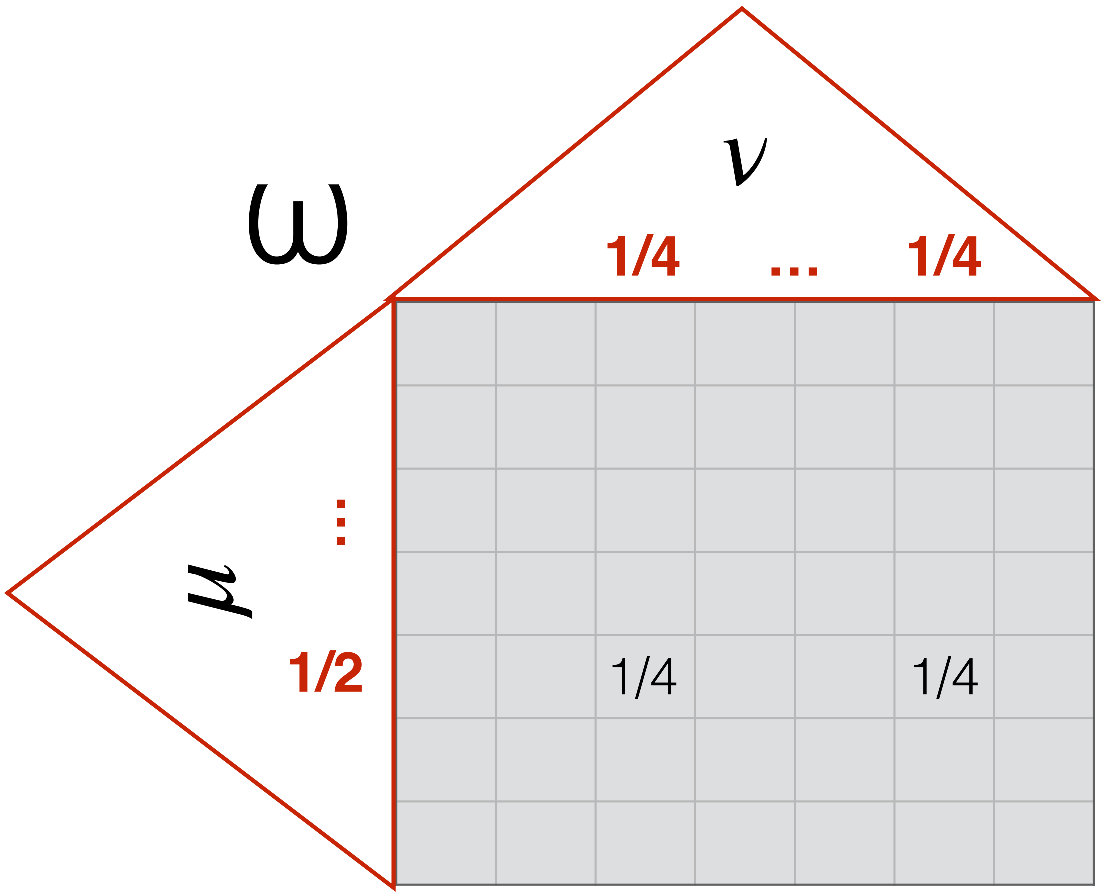
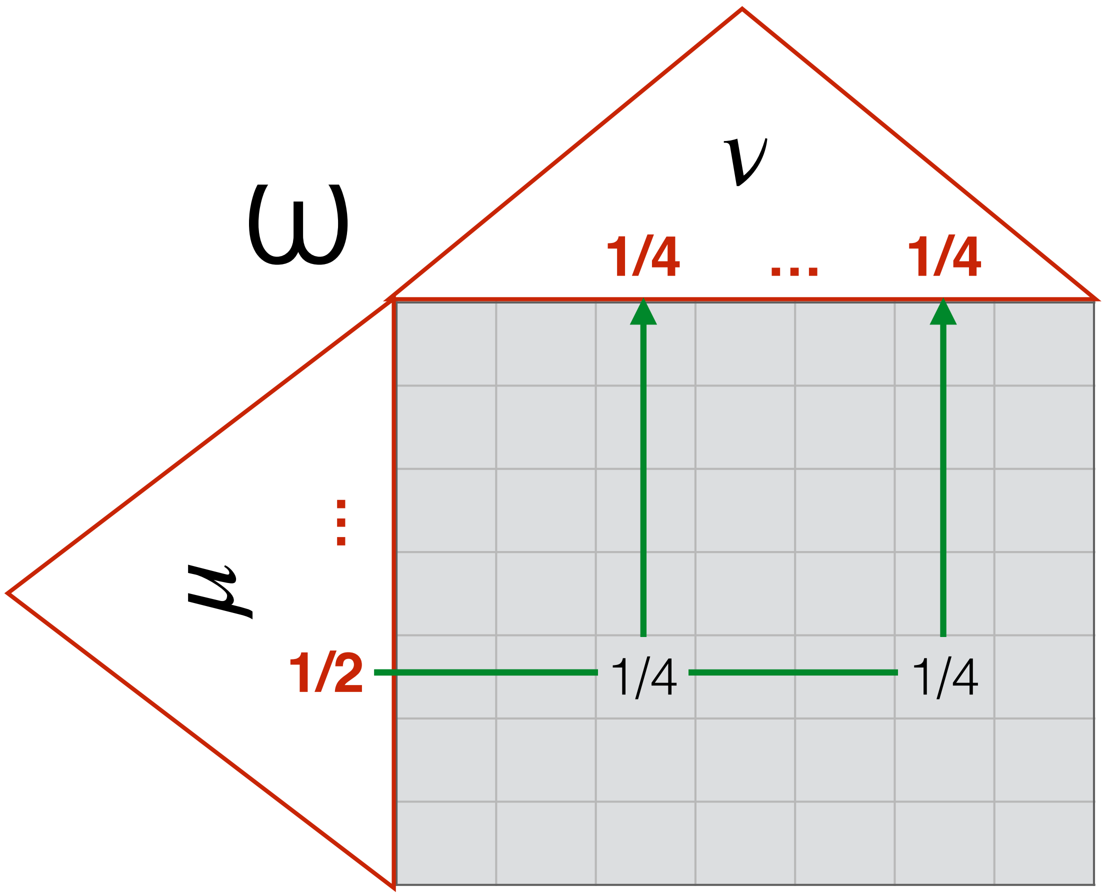
s.t, for all $\mathbf{a} \in \mathbb{A}$, $\Lambda(d_{MC})((\mathbf{a},$  $),(\mathbf{a},$  $)) = d_{MC}($  ,  $)$

Couplings: the meaning



Couplings: the meaning



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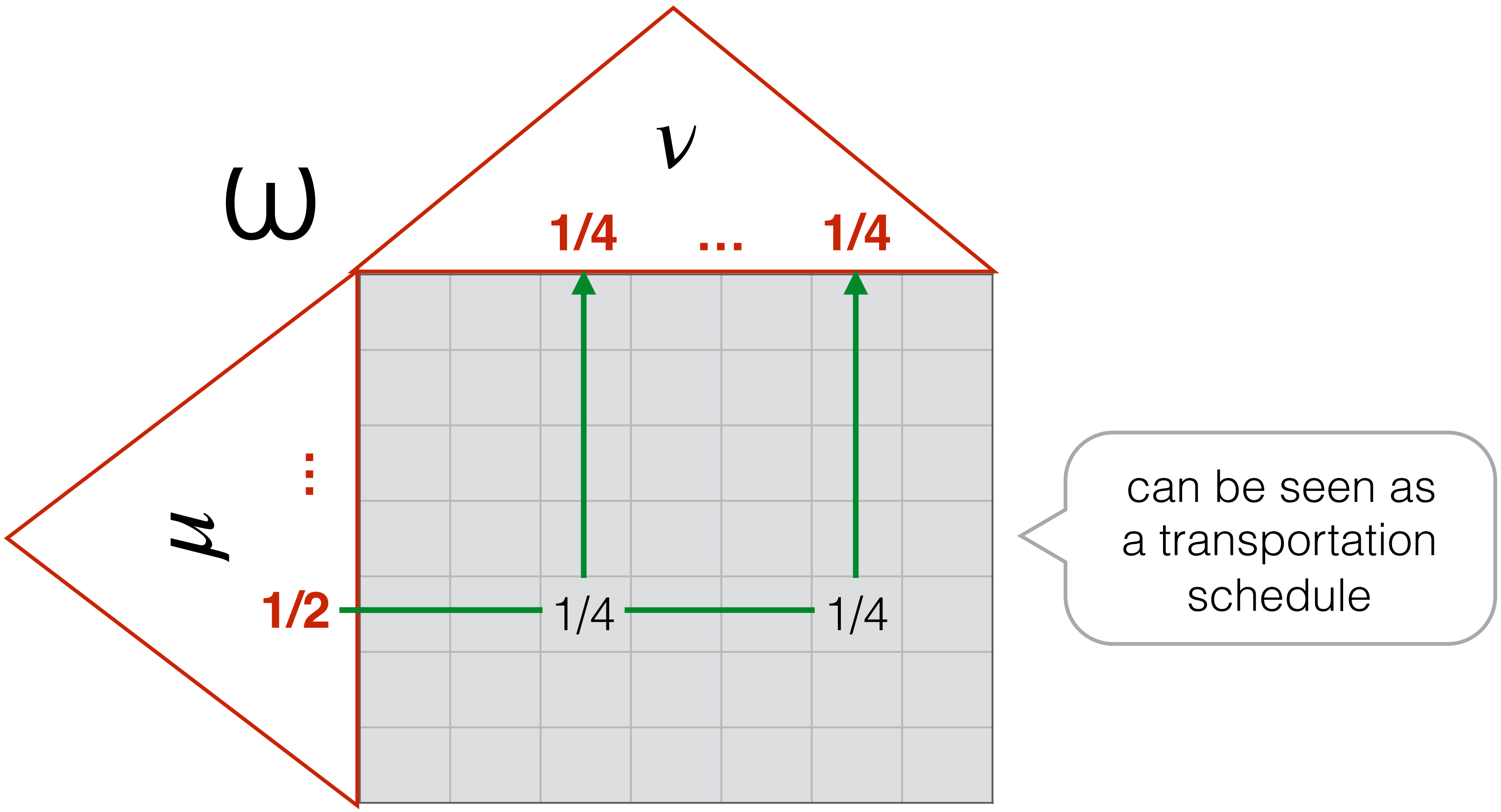
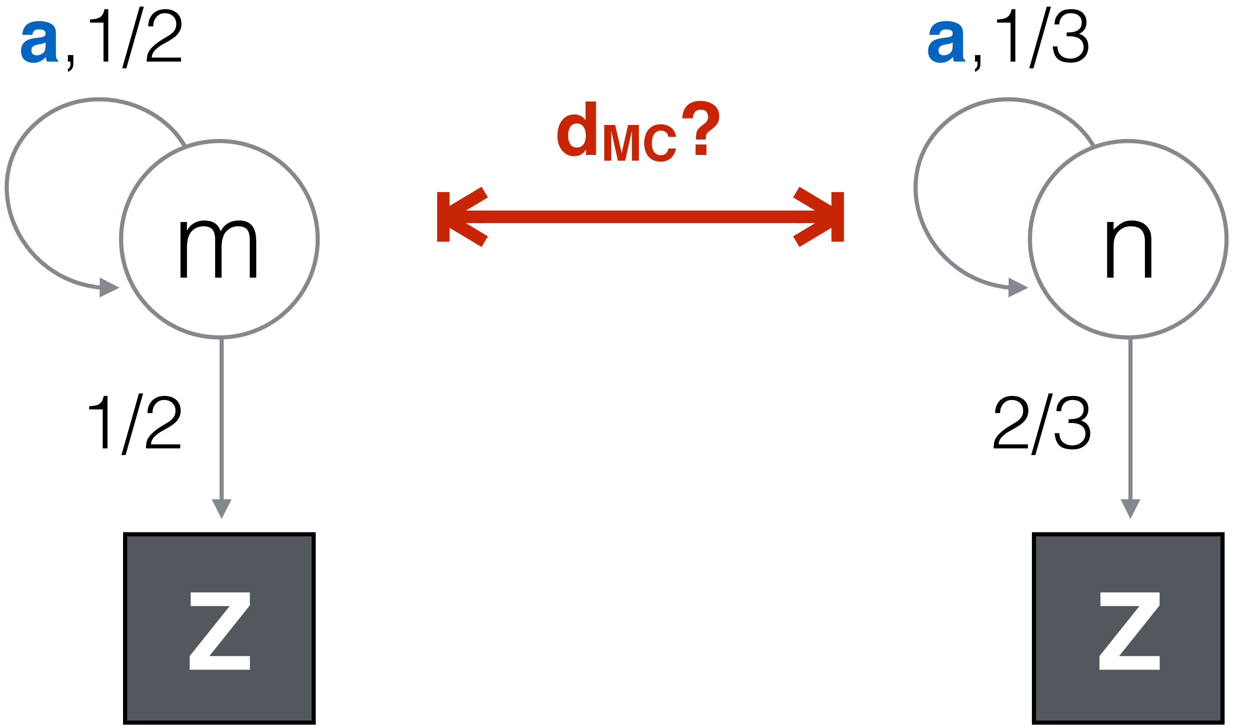


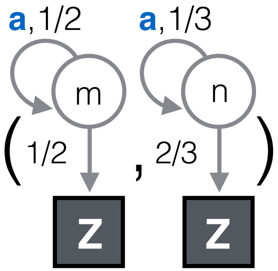
Running example



m = rec X. (**a**.X $+_{1/2}$ Z) n = rec Y. (**a**.Y $+_{1/3}$ Z)

optimal coupling between transition probabilities of m and n

ω^*	$\nu((\mathbf{a},n))$ = **1/3**	$\nu(Z)$ = **2/3**
$\mu((\mathbf{a},m))$=**1/2**	1/3	1/6
$\mu(Z)$=**1/2**		1/2

d() =

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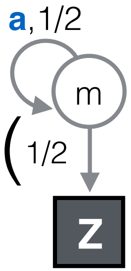
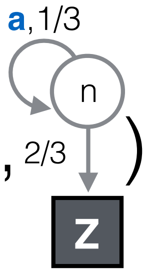
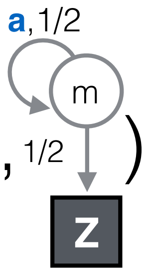
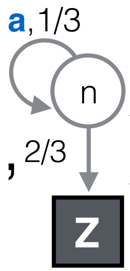
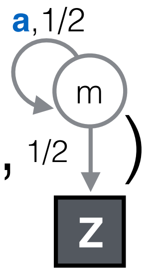
d(

, 

) =

$= \frac{1}{3}\Lambda(d)((\mathbf{a},$ 

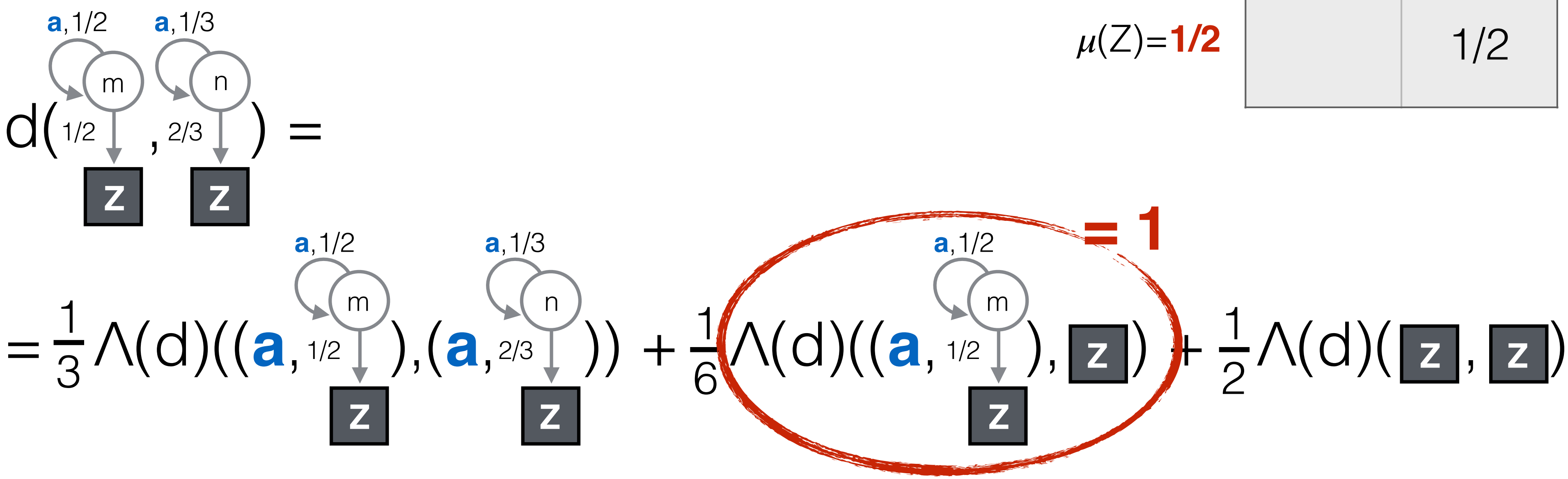
$),(\mathbf{a},$ 

$)) + \frac{1}{6}\Lambda(d)((\mathbf{a},$ 

$),$ Z $) + \frac{1}{2}\Lambda(d)($ Z, Z $)$

optimal coupling between
transition probabilities
of m and n

ω^*	$\nu((\mathbf{a},n)) = \mathbf{1/3}$	$\nu(Z) = \mathbf{2/3}$
$\mu((\mathbf{a},m))=\mathbf{1/2}$	1/3	1/6
$\mu(Z)=\mathbf{1/2}$		1/2



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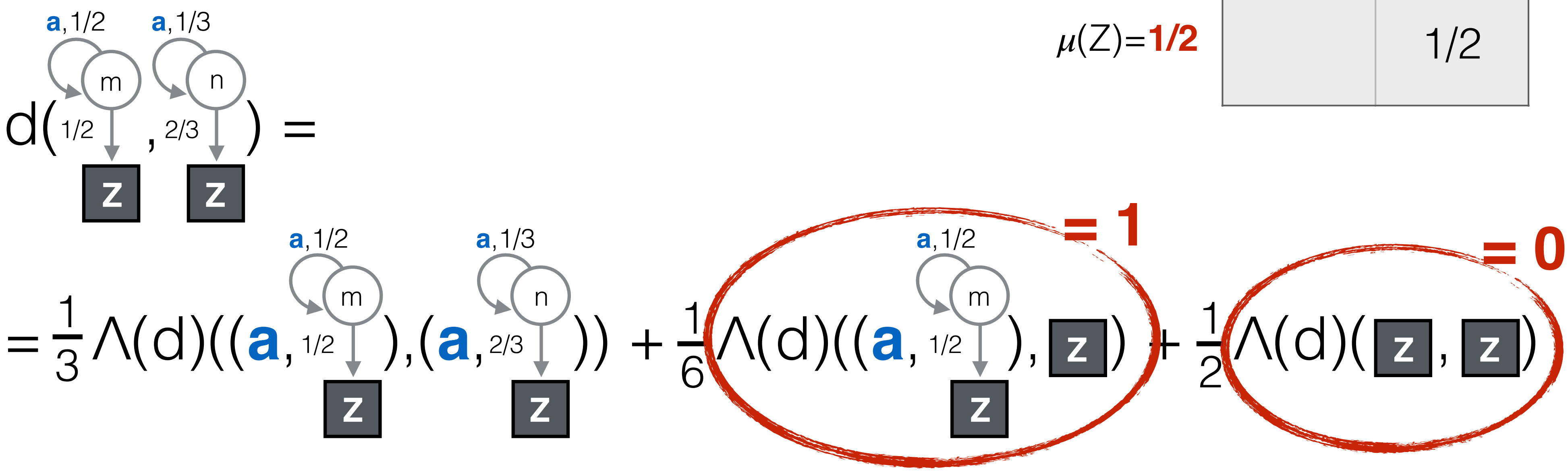

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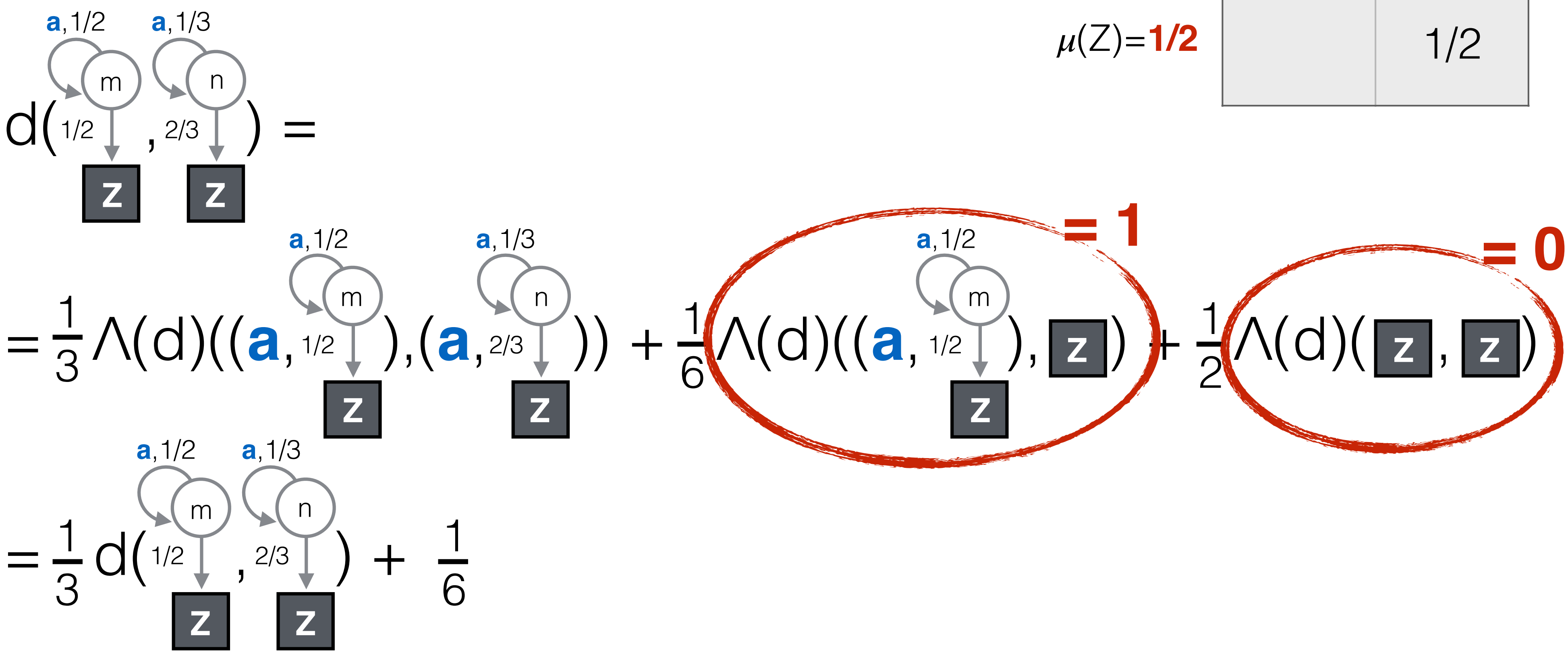

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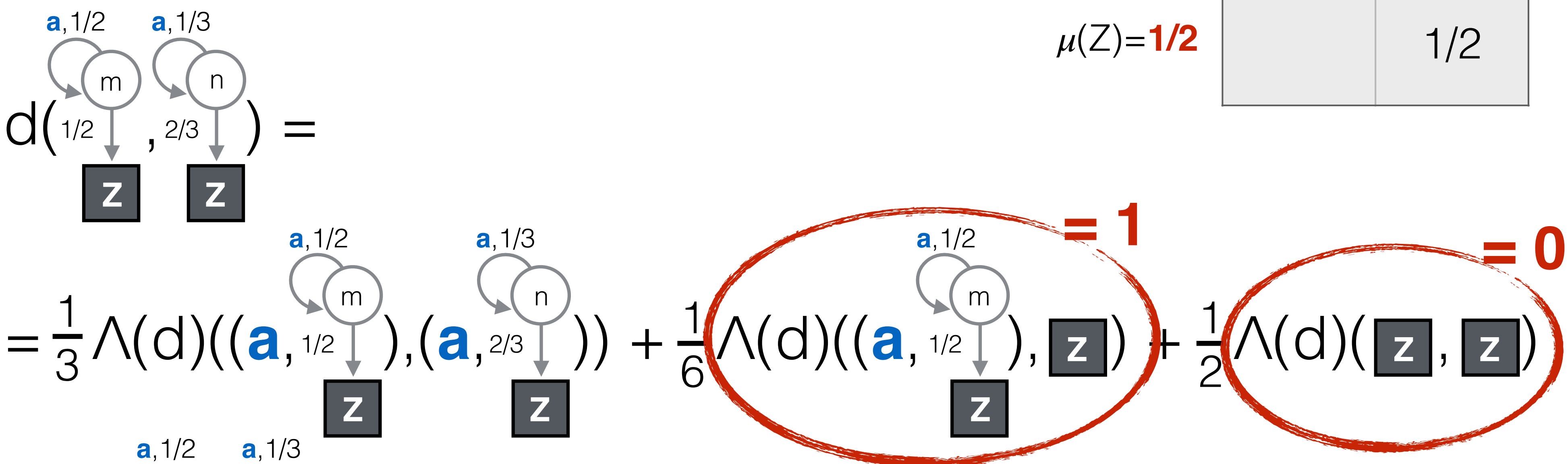
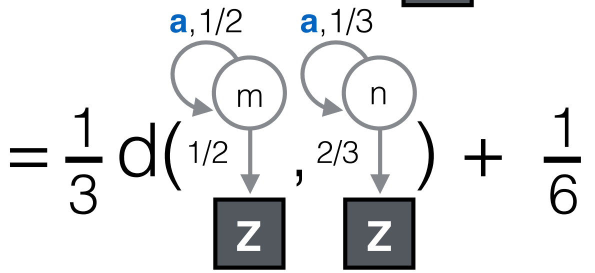
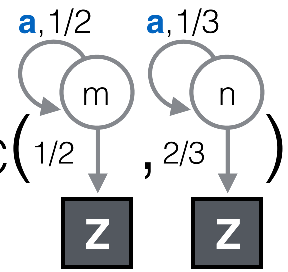




Solution: $d_{MC}($  $) = \frac{1}{4}$

The Quantitative Equational Theory

Axiomatization (first attempt)

(B1) $\vdash t +_1 s =_0 t$

(B2) $\vdash t +_e t =_0 t$

(SC) $\vdash t +_e s =_0 s +_{1-e} t$

(SA) $\vdash (t +_e s) +_{e'} u =_0 t +_{ee'} (s +_{\frac{e'-ee'}{1-ee'}} u)$ — for $e,e' \in [0,1)$

(IB) $\{t =_\varepsilon s, t' =_{\varepsilon'} s'\} \vdash t +_e t' =_\delta s +_e s'$ — for $\delta \leq e\varepsilon+(1-e)\varepsilon'$

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the terms from the example…

$m = \text{rec } X.\ (\mathbf{a}.X +_{1/2} Z)$ $n = \text{rec } Y.\ (\mathbf{a}.Y +_{1/3} Z)$

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The quantitative equational framework
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all operators to be **non-expansive**

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… but the NExp axiom is not sound for recursion

$$\mathcal{A}_{MC} \not\models \left(\{t =_\varepsilon s\} \vdash \mathrm{rec}\ X.t =_\varepsilon \mathrm{rec}\ X.s\right)$$

(see Gebler-Larsen-Tini FoSSaCS'15)

Relaxing non-expansivity

we keep all the axioms of quantitative algebras
but the NExp axiom

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(Symm) $\{t =_\varepsilon s\} \vdash s =_\varepsilon t$

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the Archimedian axiom will be used to recover completeness

from what we have seen in the example before and (Fix)+(Unfold)+(Top)+(IB) we obtain

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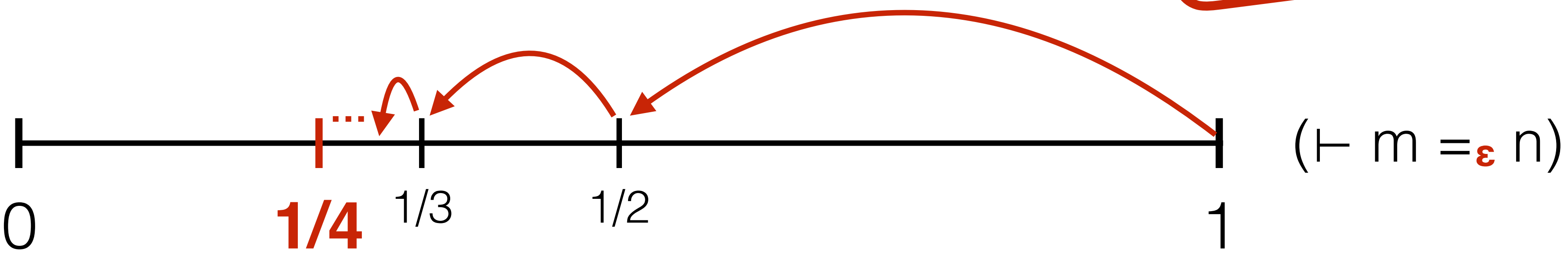

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⇒ $\vdash m =_{1/4} n$

Sound & Complete Axiomatization

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A quantitative Kleene's theorem

$$(MC/\sim, d_{MC})$$

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$d_{\vdash}([t],[s]) = \inf\{ \varepsilon \mid \vdash t =_{\varepsilon} s \}$

A quantitative Kleene's theorem

$$(MC/\sim,\ d_{MC})$$

$$\cong$$ isometric isomorphism

$$(Exp/{=},\ d_{\vdash})$$

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Conclusions

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- What about different models? (e.g., non-determinism)
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- Beyond non-expansive operators

Thank you
for your attention