Final coalgebras in categories with factorization systems

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short contribution

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What the talk is about

We will explore properties of the final sequence and give

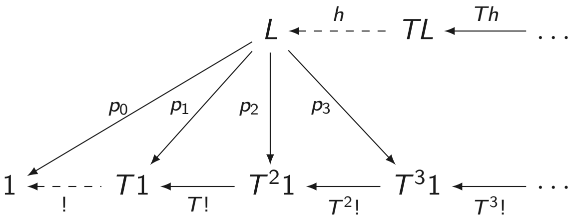
+ a minimization functor
  (constructively, still without a final coalgebra!)
+ characterizations of final coalgebras, given weakly final coalgebras

provided that the category admits a

factorization system $(\mathcal{L}, \mathcal{R})$ with $\mathcal{R} \subseteq$ **Monic**

Final sequence

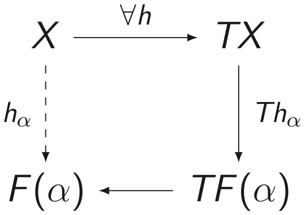
For an endofunctor $T\colon \mathcal{C} \to \mathcal{C}$ the final sequence is given as



. . . formally it is a limit preserving functor

$$F\colon \mathbf{Ord}^{op} \to \mathcal{C} \quad \text{such that} \quad \begin{cases} F(0) = 1 \\ F(\beta{+}1) = TF(\beta) \\ F(\beta{+}1 \to \gamma{+}1) = TF(\beta \to \gamma) \end{cases}$$

Final coalgebra from the final sequence



$\forall \alpha \in \textbf{Ord}$

Theorem (Adamek,Barr)

If the final sequence *stabilizes* at α then $F(\alpha{+}1 \to \alpha)^{-1}$ is a final T-coalgebra.

Corollary (Adamek,Worrell)

If $F(\alpha{+}1 \to \alpha)$ is monic and T preserves monos, then T has final coalgebra, provided that $\mathcal{C}$ is well-powered.

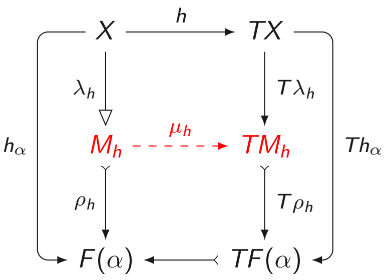
Assumptions (I)

- a factorization system $(\mathcal{L}, \mathcal{R})$ with $\mathcal{R} \subseteq \mathbf{Monic}$
- $F(\alpha{+}1 \to \alpha) \in \mathcal{R}$ for some $\alpha \in \mathbf{Ord}$
- T preserves $\mathcal{R}$-morphisms

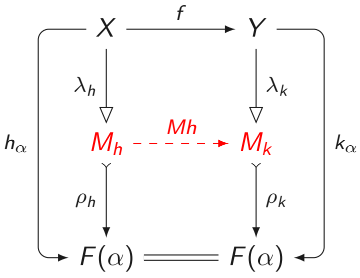
Minimization functor

$$M\colon T\text{-Coalg}_{\mathcal{C}} \to T\text{-Coalg}_{\mathcal{C}}$$

Objects



Arrows

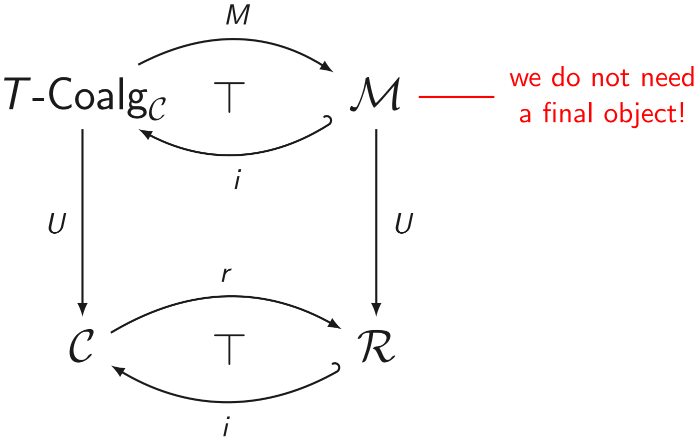


(*) We do not require the existence of a final T-coalgebra (ex. $\mathbf{Ord}^{op}$)

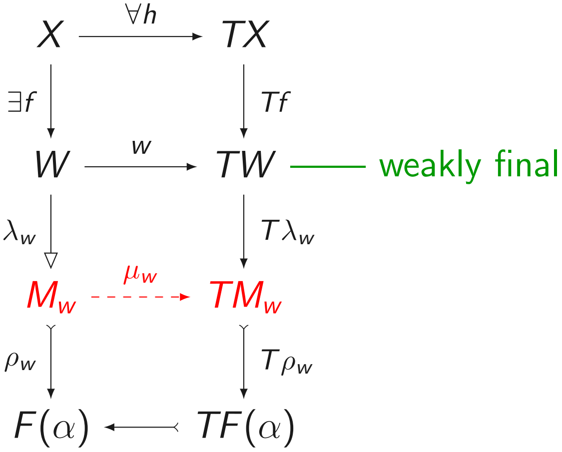
Reflective subcategory of minimized coalgebras

Factorization systems on $\mathcal{C}$ induces a reflective subcategory $\mathcal{R}$, provided that $\mathcal{C}$ has final object

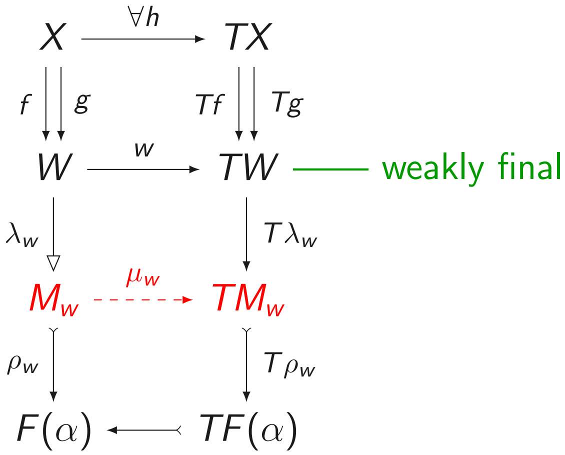
[Borceux, *Handbook 1*, Prop.5.5.5]



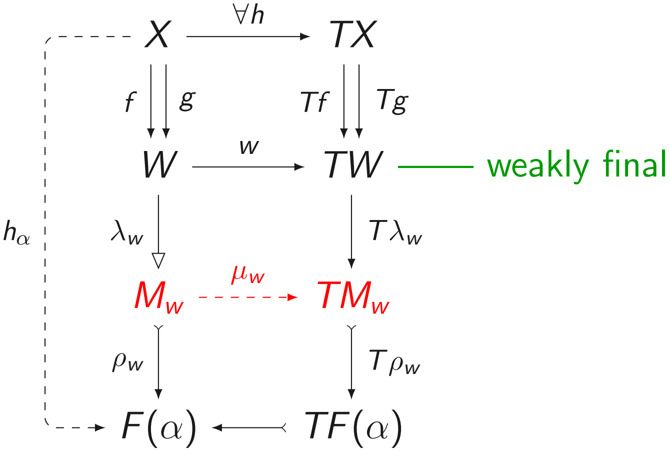
Final from weakly final



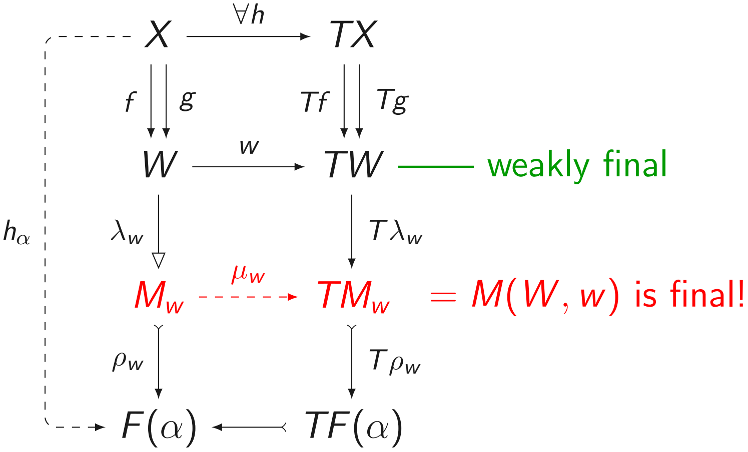
Final from weakly final



Final from weakly final



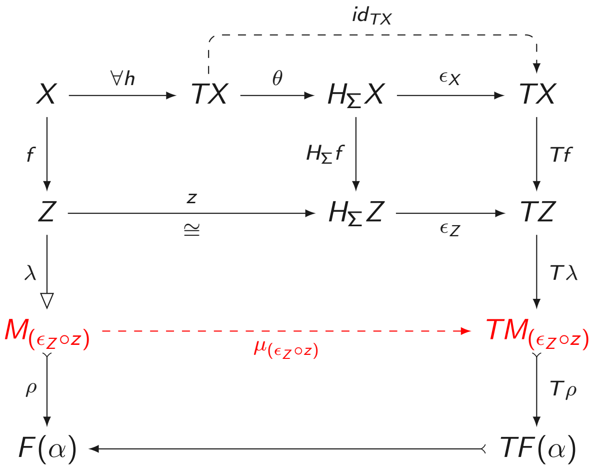
Final from weakly final



Results for finitary **Set** endofunctors T [Adamek/Worrell]

- the final sequence for T has monic arrow at $\omega{+}1 \to \omega$
- T is a quotient of some finitary polynomial endofunctor H_Σ

$$\epsilon\colon H_\Sigma \Rightarrow T \qquad \text{with epic componets}$$



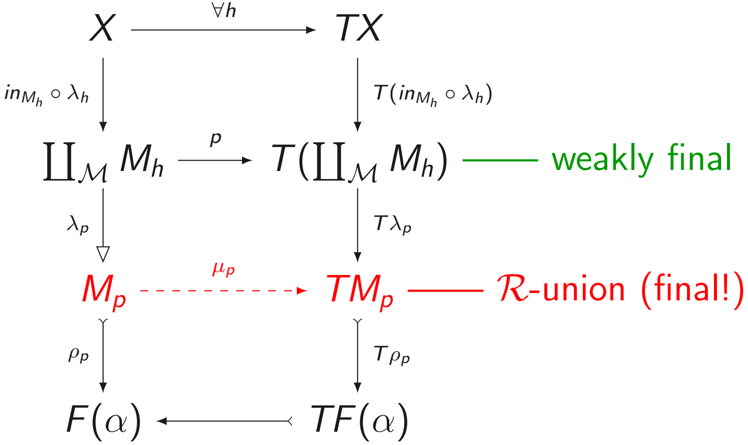
Assumption (2)

$\mathcal{C}$ is $\mathcal{R}$-well-powered

(each object has only a **set** of $\mathcal{R}$-subobjects)

$\mathcal{R}$-union of minimized coalgebras is final

Under this further assumption the category $\mathcal{M}$ is small...



Thanks