Structural Operational Semantics for continuous state probabilistic processes*

Giorgio Bacci Marino Miculan

Dept. of Mathematics and Computer Science
University of Udine, Italy

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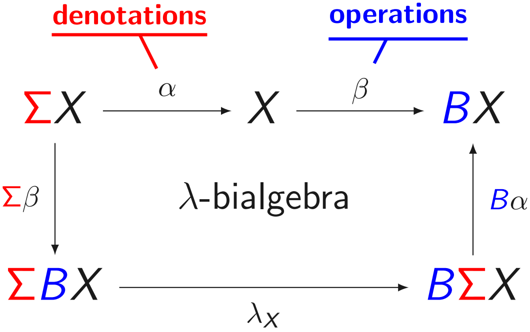
Structural Operational Semantics

(distributing syntax over behaviours: $\lambda\colon \Sigma B \Rightarrow B\Sigma$)



Benefits of the bialgebraic framework

[Turi-Plotkin'97]

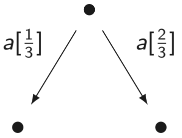
+ denotational model on the final B-coalgebra (by co-induction)
+ operational model on the initial Σ-algebra (by induction)

+ universal semantics (full-abstaction)

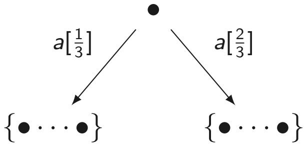
  initial algebra semantics = final coalgebra semantics
+ B-behavioural equivalence is a Σ-congruence
+ B-bisimilarity is a Σ-congruence (if B pres. weak pullbacks)

Probabilistic Systems

Discrete state
(labelled Markov chains)



Continuous state
(labelled Markov processes)



Probabilistic Systems

Discrete state
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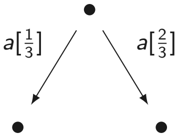
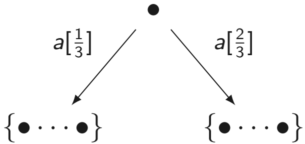


$X \to (\mathcal{D}_{\text{fin}} X)^L$ in **Set**

Continuous state
(labelled Markov processes)



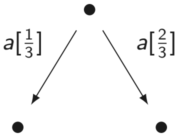
$\mathcal{D}_{\text{fin}} \colon \mathbf{Set} \to \mathbf{Set}$ (sub-probability distribution functor)

$$\mathcal{D}_{\text{fin}} X = \{\varphi \colon X \to [0,1] \mid \sum_{x \in X} \varphi(x) \leq 1, |supp(\varphi)| < \infty\}$$

Probabilistic Systems

Discrete state

(labelled Markov chains)



$X \to (\mathcal{D}_{\text{fin}} X)^L$ in **Set**

Continuous state

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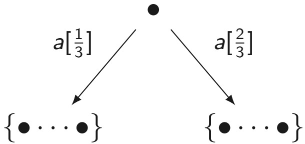

$X \to (\Delta X)^L$ in **Meas**

$\mathcal{D}_{\text{fin}} \colon \mathbf{Set} \to \mathbf{Set}$ (sub-probability distribution functor)

$$\mathcal{D}_{\text{fin}} X = \{\varphi \colon X \to [0,1] \mid \sum_{x \in X} \varphi(x) \leq 1, |supp(\varphi)| < \infty\}$$

$\Delta \colon \mathbf{Meas} \to \mathbf{Meas}$ (Giry functor)

$$\Delta X = \{\mu \colon \Sigma_X \to [0,1] \mid \mu \text{ sub-probability measure}\}$$

Rule Formats for Discrete Systems

Simpler way of specifying distributive laws [Bartels'04]

$$\left\{ \frac{\begin{array}{ll} x_i \xrightarrow{a} & a \in A_i, 1 \leq i \leq n \\ x_i \stackrel{b}{\not\rightarrow} & b \in B_i, 1 \leq i \leq n \\ x_{a_j} \xrightarrow{l_j[p_j]} y_j & 1 \leq i \leq J \end{array}}{f(x_1, \ldots, x_n) \xrightarrow{c[w \cdot p_1 \cdot \ldots \cdot p_J]} t} \right\}_{\text{image finite}}$$

corresponds to...

$$\lambda \colon \Sigma(Id \times (\mathcal{D}_{\text{fin}})^L) \Rightarrow (\mathcal{D}_{\text{fin}} T_\Sigma)^L$$

hence, a distributive law of the free monad (T_Σ, η, μ) over the copointed functor $(Id \times (\mathcal{D}_{\text{fin}})^L, \epsilon)$ [Lenisa-Power-Watanabe'04]

Aim: Congruential Rule Formats for Probabilistic Processes with Continuous State Spaces

...hence, inducing distributive laws of type

$$\lambda \colon \Sigma(Id \times \Delta^L) \Rightarrow (\Delta T_\Sigma)^L$$

The shape of transitions

The behaviour functor suggests the shape of transitions...

Discrete state

$$t \xrightarrow{a[p]} t'$$

Σ-term

Continuous state

$$t \xrightarrow{a} \mu$$

measure
on Σ-terms

Earlier attempts. . .

[Cardelli-Mardare'10]

The Measurable Space of Stochastic Processes

Luca Cardelli
Microsoft Research Cambridge, UK
Email: luca@microsoft.com

Radu Mardare
The Microsoft Research-University of Trento Centre for Computational and Systems Biology
Trento, Italy
Email: mardare@cosbi.eu

$(Null).\quad \dfrac{}{0 \to \overline{\omega}}$

$(Guard).\quad \dfrac{}{\varepsilon.P \to [^{\varepsilon}_{P}]}$

$(Sum).\quad \dfrac{P \to \mu' \qquad Q \to \mu''}{P + Q \to \mu' \oplus \mu''}$

$(Par).\quad \dfrac{P \to \mu' \qquad Q \to \mu''}{P|Q \to \mu' \;{}_{P}\!\otimes_{Q}\; \mu''}$

Table I
STRUCTURAL OPERATIONAL SEMANTICS

Earlier attempts. . .

[Cardelli-Mardare'10]

The Measurable Space of Stochastic Processes

Luca Cardelli
Microsoft Research Cambridge, UK
Email: luca@microsoft.com

Radu Mardare
The Microsoft Research-University of Trento
Centre for Computational and Systems Biology
Trento, Italy
Email: mardare@cosbi.eu

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Table I
STRUCTURAL OPERATIONAL SEMANTICS

operations on measures

Earlier attempts. . . [Cardelli-Mardare'10]

rather ad hoc. . .
(no general framework)

(*Null*). $$\frac{}{0 \to \overline{\omega}}$$

(*Guard*). $$\frac{}{\varepsilon.P \to [{}^{\varepsilon}_{P}]}$$

(*Sum*). $$\frac{P \to \mu' \qquad Q \to \mu''}{P + Q \to \mu' \oplus \mu''}$$

(*Par*). $$\frac{P \to \mu' \qquad Q \to \mu''}{P|Q \to \mu' \ {}_{P}\!\otimes_{Q}\ \mu''}$$

operations on measures

Table I
STRUCTURAL OPERATIONAL SEMANTICS

Measure terms

We adopt a new syntax to handle measures syntactically

$$\Sigma : \mathbf{Meas} \to \mathbf{Meas} \qquad \text{(process syntax)}$$

$$M : \mathbf{Meas} \to \mathbf{Meas} \qquad \text{(measure syntax)}$$

$$t \xrightarrow{a} \mu$$

it's a M-term!

Measure GSOS rule format

$$\frac{\left\{x_i \xrightarrow{a_{ij}} \mu_{ij}\right\}_{1\leq i\leq n,\, a_{ij}\in A_i}^{1\leq j\leq m_i} \quad \left\{x_i \stackrel{b}{\not\longrightarrow}\right\}_{1\leq i\leq n}^{b\in B_i}}{f(x_1,\ldots,x_n) \xrightarrow{c} \mu} \qquad \text{(MGSOS)}$$

where

- $f \in \Sigma$ with $ar(f) = n$;
- $\{x_1,\ldots,x_n\}$ and $\{\mu_{ij} \mid 1 \leq i \leq n,\, 1 \leq j \leq m_i\}$ are pairwise distinct process and measure *variables*;
- $A_i \cap B_i = \emptyset$ are disjoint subsets of labels in L, and $c \in L$;
- μ is a M-term with variables in $\{x_1,\ldots,x_n\}$ and $\{\mu_{ij} \mid 1 \leq i \leq n, 1 \leq j \leq m_i\}$.

An MGSOS specification system consists of

Set of MGSOS rules:

$$\mathcal{R} = \left\{ \frac{\left\{x_i \xrightarrow{a_{ij}} \mu_{ij}\right\}^{1\leq j\leq m_i}_{1\leq i\leq n,\, a_{ij}\in A_i} \quad \left\{x_i \stackrel{b}{\not\rightarrow}\right\}^{b\in B_i}_{1\leq i\leq n}}{f(x_1,\ldots,x_n) \xrightarrow{c} \mu} \right\}_{\text{image finte}}$$

Measure terms interpretation:

$$\langle\!|\cdot|\!\rangle : T_M\Delta \Rightarrow \Delta T_\Sigma$$

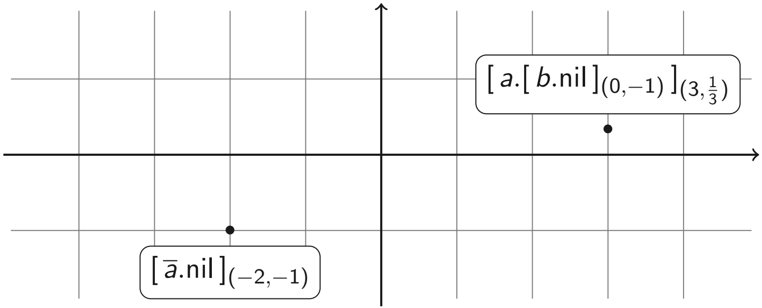
Example: FlatCCS

CCS-like processes living in the Euclidean space $\mathbb{R}^2$

frame $(z \in \mathbb{R}^2)$

$$p, q ::= \text{nil} \mid \alpha.p \mid p + q \mid p \parallel q \mid [\, p \,]_z$$

$$\alpha ::= a \mid \overline{a} \mid \tau$$



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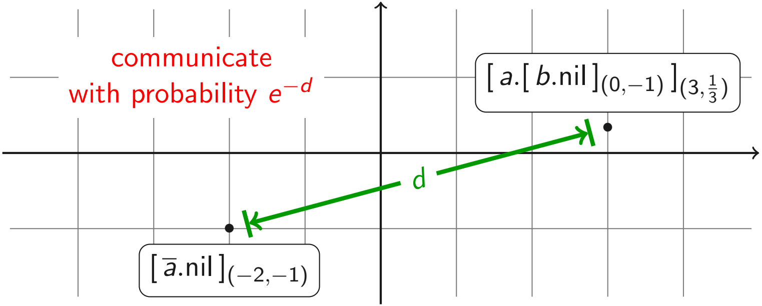

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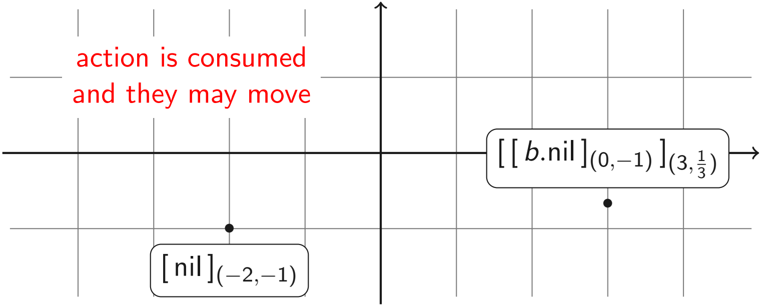


$$\Sigma X = \overbrace{1}^{\mathsf{nil}} + \overbrace{\underline{A} \times X}^{a.x} + \overbrace{\underline{A} \times X}^{\bar{a}.x} + \overbrace{X}^{\tau.x} + \overbrace{X^2}^{x+x} + \overbrace{X^2}^{x \| x} + \overbrace{\underline{\mathbb{R}^2} \times X}^{[\,x\,]_z},$$

$$MX = \overbrace{X^2}^{x \blacktriangleleft x} + \overbrace{X^2}^{x \blacktriangleright x} + \overbrace{X^2}^{x \blacktriangledown x} + \overbrace{\underline{\mathbb{R}^2} \times X}^{\langle\, x\,\rangle_z}.$$

$$\frac{}{\alpha.x \xrightarrow{\alpha} x} \qquad \frac{x \xrightarrow{\alpha} \mu}{x + x' \xrightarrow{\alpha} \mu} \qquad \frac{x' \xrightarrow{\alpha} \mu}{x + x' \xrightarrow{\alpha} \mu}$$

$$\frac{x \xrightarrow{\alpha} \mu}{[\,x\,]_z \xrightarrow{\alpha} \langle\, \mu\,\rangle_z} \qquad \frac{x \xrightarrow{\alpha} \mu}{x \parallel x' \xrightarrow{\alpha} \mu \blacktriangleleft x'} \qquad \frac{x' \xrightarrow{\alpha} \mu}{x \parallel x' \xrightarrow{\alpha} x \blacktriangleright \mu}$$

$$\frac{x \xrightarrow{a} \mu \qquad x' \xrightarrow{\bar{a}} \mu'}{x \parallel x' \xrightarrow{\tau} \mu \blacktriangledown \mu'} \qquad \frac{x \xrightarrow{\bar{a}} \mu \qquad x' \xrightarrow{a} \mu'}{x \parallel x' \xrightarrow{\tau} \mu \blacktriangledown \mu'}$$

$$\llparenthesis \cdot \rrparenthesis^{\mathsf{fl}} \colon T_M\Delta \Rightarrow \Delta T_\Sigma$$

$$\begin{aligned}
\llparenthesis \beta \rrparenthesis^{\mathsf{fl}}_X &= \beta \\
\llparenthesis \mu \blacktriangleleft \mu' \rrparenthesis^{\mathsf{fl}}_X &= (\llparenthesis \mu \rrparenthesis^{\mathsf{fl}}_X \times \llparenthesis \mu' \rrparenthesis^{\mathsf{fl}}_X) \circ (\lambda(x,x').\, x \parallel x')^{-1} \\
\llparenthesis \mu \blacktriangleright \mu' \rrparenthesis^{\mathsf{fl}}_X &= (\llparenthesis \mu \rrparenthesis^{\mathsf{fl}}_X \times \llparenthesis \mu' \rrparenthesis^{\mathsf{fl}}_X) \circ (\lambda(x,x').\, x \parallel x')^{-1} \\
\llparenthesis \mu \blacktriangledown \mu' \rrparenthesis^{\mathsf{fl}}_X &= \left(e^{-\|pos(\mu)-pos(\mu')\|} \cdot (\llparenthesis \mu \rrparenthesis^{\mathsf{fl}}_X \times \llparenthesis \mu' \rrparenthesis^{\mathsf{fl}}_X)\right) \circ (\lambda(x,x').\, x \parallel x')^{-1} \\
\llparenthesis \langle\, \mu \,\rangle_z \rrparenthesis^{\mathsf{fl}}_X &= \llparenthesis \mu \rrparenthesis^{\mathsf{fl}}_X \circ (\lambda x.\, [\, x \,]_z)^{-1}
\end{aligned}$$

where $pos \colon T_M\Delta X \to \mathbb{R}^2$ determines the position of an action

$$pos(\beta) = (0,0) \qquad pos(\mu \blacktriangleleft \mu') = pos(\mu) \qquad pos(\mu \blacktriangleright \mu') = pos(\mu')$$

$$pos(\mu \blacktriangledown \mu') = \tfrac{1}{2}(pos(\mu) + pos(\mu')) \qquad pos(\langle\, \mu \,\rangle_z) = z + pos(\mu)$$

From MGSOS to distributive laws

$$\Sigma(Id \times \Delta^L)$$

$$\Big\Downarrow \lambda$$

$$(\Delta T_\Sigma)^L$$

how do we get the distributive law λ
out of an MGSOS specification systems?

From MGSOS to distributive laws

$$\Sigma(Id \times \Delta^L)$$

$$\Big\Downarrow [\![\mathcal{R}]\!]$$

$$(\mathcal{P}_{\text{fin}} T_M \Delta)^L$$

$$(\Delta T_\Sigma)^L$$

1. define the natural transformation $[\![\mathcal{R}]\!]$ from the image finite set MGSOS rules

From MGSOS to distributive laws

$$\Sigma(Id \times \Delta^L)$$

$$\Big\Downarrow [\![\mathcal{R}]\!]$$

$$(\mathcal{P}_{\mathsf{fin}} T_M \Delta)^L$$

$$\Big\Downarrow (\mathcal{P}_{\mathsf{fin}} \langle\!| \cdot |\!\rangle)^L$$

$$(\mathcal{P}_{\mathsf{fin}} \Delta T_\Sigma)^L$$

$$(\Delta T_\Sigma)^L$$

1. define the natural transformation $[\![\mathcal{R}]\!]$ from the image finite set MGSOS rules
2. apply the measure terms interpretation

$$\langle\!| \cdot |\!\rangle \colon T_M \Delta \Rightarrow \Delta T_\Sigma$$

From MGSOS to distributive laws

$$\Sigma(Id \times \Delta^L) \overset{[\![\mathcal{R}]\!]}{\Longrightarrow} (\mathcal{P}_{\text{fin}} T_M \Delta)^L \overset{(\mathcal{P}_{\text{fin}} \langle\!|\cdot|\!\rangle)^L}{\Longrightarrow} (\mathcal{P}_{\text{fin}} \Delta T_\Sigma)^L \overset{(\oplus T_\Sigma)^L}{\Longrightarrow} (\Delta T_\Sigma)^L$$

1. define the natural transformation $[\![\mathcal{R}]\!]$ from the image finite set MGSOS rules
2. apply the measure terms interpretation

$$\langle\!|\cdot|\!\rangle \colon T_M \Delta \Rightarrow \Delta T_\Sigma$$

3. obtain the actual measure by averaging

$$\oplus_X(\mu_1, \ldots, \mu_n)(E) = \frac{\mu_1(E) + \cdots + \mu_n(E)}{\mu_1(X) + \cdots + \mu_n(X)}$$

Benefits from the bialgebraic framework

For continuous state probabilistic processes described by means of MGSOS specification systems we have:

- + denotational model on the final Δ^L-coalgebra
- + operational model on the initial Σ-algebra

- + universal semantics (full-abstaction)

  initial algebra semantics = final coalgebra semantics
- + Δ^L-behavioural equivalence is a Σ-congruence
- + is Δ^L-bisimilarity a Σ-congruence?

  (Δ^L does not preserves weak pullbacks! [Viglizzo'05])

From MGSOS to distributive laws

$$\Sigma(Id \times \Delta^L)$$
$$\Big\Downarrow [\![\mathcal{R}]\!]$$
$$(\mathcal{P}_{\text{fin}} T_M \Delta)^L$$
$$\Big\Downarrow (\mathcal{P}_{\text{fin}} \llparenthesis \cdot \rrparenthesis)^L$$
$$(\mathcal{P}_{\text{fin}} \Delta T_\Sigma)^L$$
$$\Big\Downarrow (\oplus T_\Sigma)^L$$
$$(\Delta T_\Sigma)^L$$

Naturality of the distributive laws depends on naturality of $\llparenthesis \cdot \rrparenthesis \colon T_M \Delta \Rightarrow \Delta T_\Sigma$

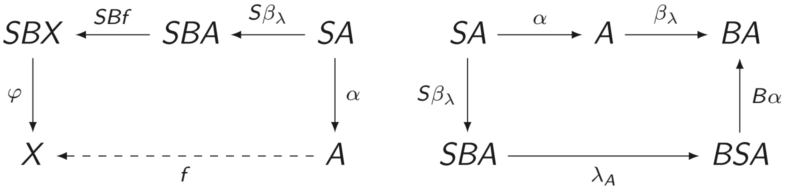
we need (general) techniques in order to derive natural transformations of type

$$T_M \Delta \Rightarrow \Delta T_\Sigma$$

λ-iterative recursion [Bartels'03]

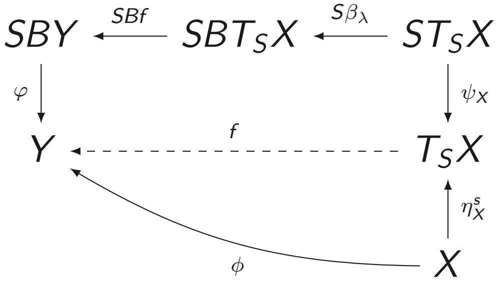
We adopt a generalized induction proof principle...

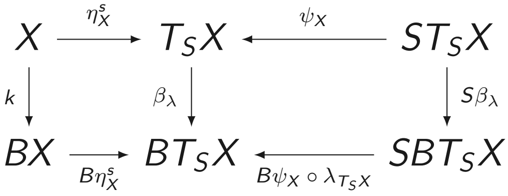
For any distributive law $\lambda\colon SB \Rightarrow BS$ and SB-algebra (X, φ) there exists a unique $f\colon A \to X$ making the following commute



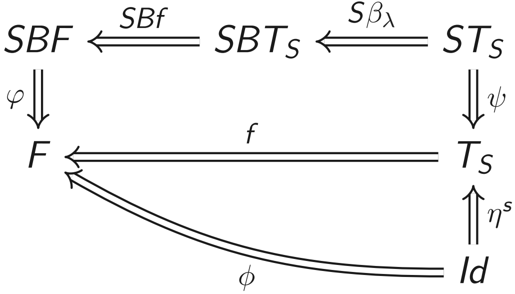
Structural λ-iterative recursion

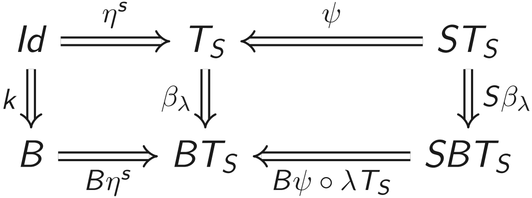
... can be extended on the free monad (T_S, η^s, μ^s)



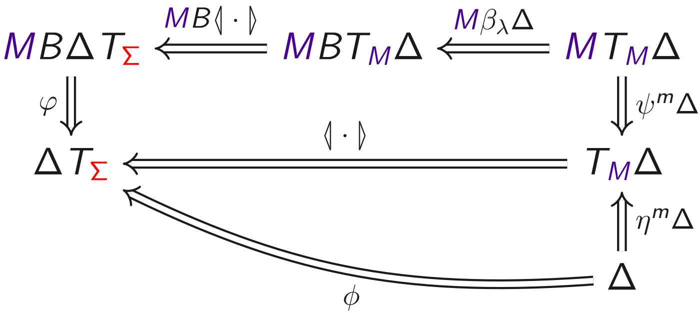


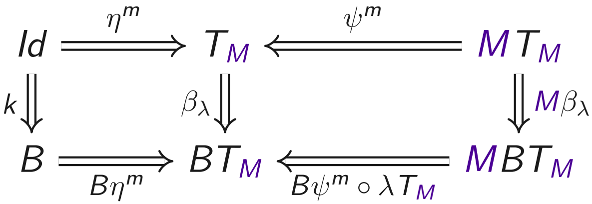
. . . and can be turned to a proof principle on natural transformations





. . . to be used to derive measure terms interpretations





Example: FlatCCS $B = (\mathbb{R}^2 \times Id)$

$$\lambda \colon M(\mathbb{R}^2 \times Id) \Rightarrow (\mathbb{R}^2 \times Id)M$$

(distributes M over position)

$$k \colon Id \Rightarrow (\mathbb{R}^2 \times Id)$$

(initialize at the origin)

$$\lambda_X((z,x) \blacktriangleleft (z',x')) = (z, x \blacktriangleleft x')$$
$$\lambda_X((z,x) \blacktriangleright (z',x')) = (z', x \blacktriangleright x')$$
$$\lambda_X((z,x) \blacktriangledown (z',x')) = (\tfrac{1}{2}(z+z'), x \blacktriangleleft x')$$
$$\lambda_X(\langle\, (z,x)\,\rangle_{z'}) = (z+z', \langle\, x \,\rangle_{z'})\,.$$

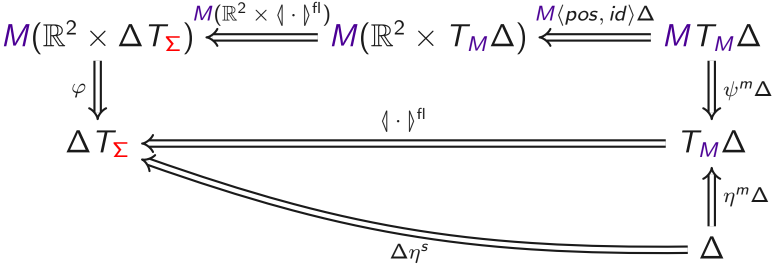
$$k_X(x) = ((0,0), x)$$

$$\begin{array}{ccccc}
Id & \overset{\eta^m}{\Longrightarrow} & T_M & \overset{\psi^m}{\Longleftarrow} & M T_M \\
{\scriptstyle k}\Downarrow & & \Downarrow{\scriptstyle \langle pos, id\rangle} & & \Downarrow{\scriptstyle M\langle pos, id\rangle} \\
\mathbb{R}^2 \times Id & \underset{\mathbb{R}^2 \times \eta^m}{\Longrightarrow} & \mathbb{R}^2 \times T_M & \underset{(\mathbb{R}^2 \times \psi^m) \circ \lambda T_M}{\Longleftarrow} & M(\mathbb{R}^2 \times T_M)
\end{array}$$

Example: FlatCCS $B = (\mathbb{R}^2 \times Id)$

$\varphi\colon M(\underline{\mathbb{R}^2} \times \Delta T_\Sigma) \Rightarrow \Delta T_\Sigma$ **(probabilities w.r.t. positions)**

$$\varphi_X((z,\beta) \blacktriangleleft (z',\beta')) = (\beta \times \beta') \circ (\lambda(x,x').\, x \parallel x')^{-1}$$
$$\varphi_X((z,\beta) \blacktriangleright (z',\beta')) = (\beta \times \beta') \circ (\lambda(x,x').\, x \parallel x')^{-1}$$
$$\varphi_X((z,\beta) \blacktriangledown (z',\beta')) = \left(e^{-\|z-z'\|} \cdot (\beta \times \beta')\right) \circ (\lambda(x,x').\, x \parallel x')^{-1}$$
$$\varphi_X(\langle\, (z,\beta)\, \rangle_{z'}) = \beta \circ (\lambda x.\, [\, x\,]_{z'})^{-1}\,.$$



Conclusions and future work

Done:

- rule format for continuous state probabilistic processes
- syntactical treatment of measures via M-terms
- general techniques for defining interpretations
- initial algebra for polynomial functors in **Meas** (not in this talk)

To do:

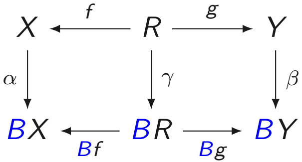
- move from probabilistic to general measures (bounded?)
- find a rule format that coincides with the distributive law
- formal expressivity analysis of the
  intermediate syntax + interpretation method

Thanks

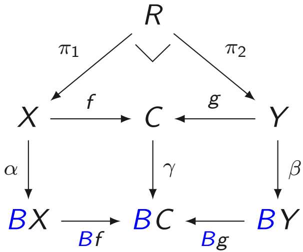
Appendix

Bisimulation vs Kernel-bisimulation

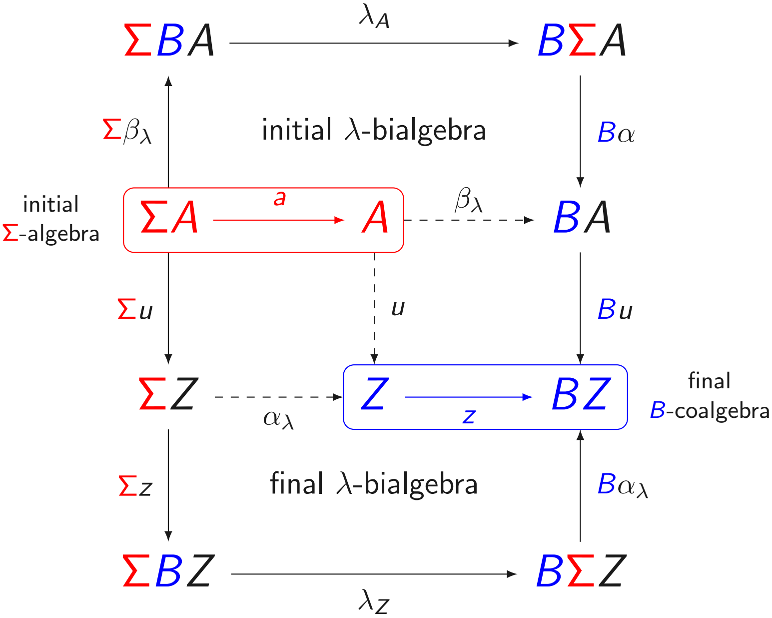
Bisimulation
(a span)



Kernel-bisimulation
(pullback of a cospan)



if B preserves weak-pullbacks, bisimulation and kernel-bisimulation coincide (provided that **C** has pullbacks and pushouts) [Staton'11]


ΣBA
λ_A
BΣA
Σβ_λ
initial λ-bialgebra
Bα
initial
Σ-algebra
ΣA
a
A
β_λ
BA
Σu
u
Bu
ΣZ
α_λ
Z
z
BZ
final
B-coalgebra
Σz
final λ-bialgebra
Bα_λ
ΣBZ
λ_Z
BΣZ