Tensor of Quantitative Equational Theories

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Historical Perspective

- **Moggi'88:** *How to incorporate effects into denotational semantics?* -**Monads** as notions of computations

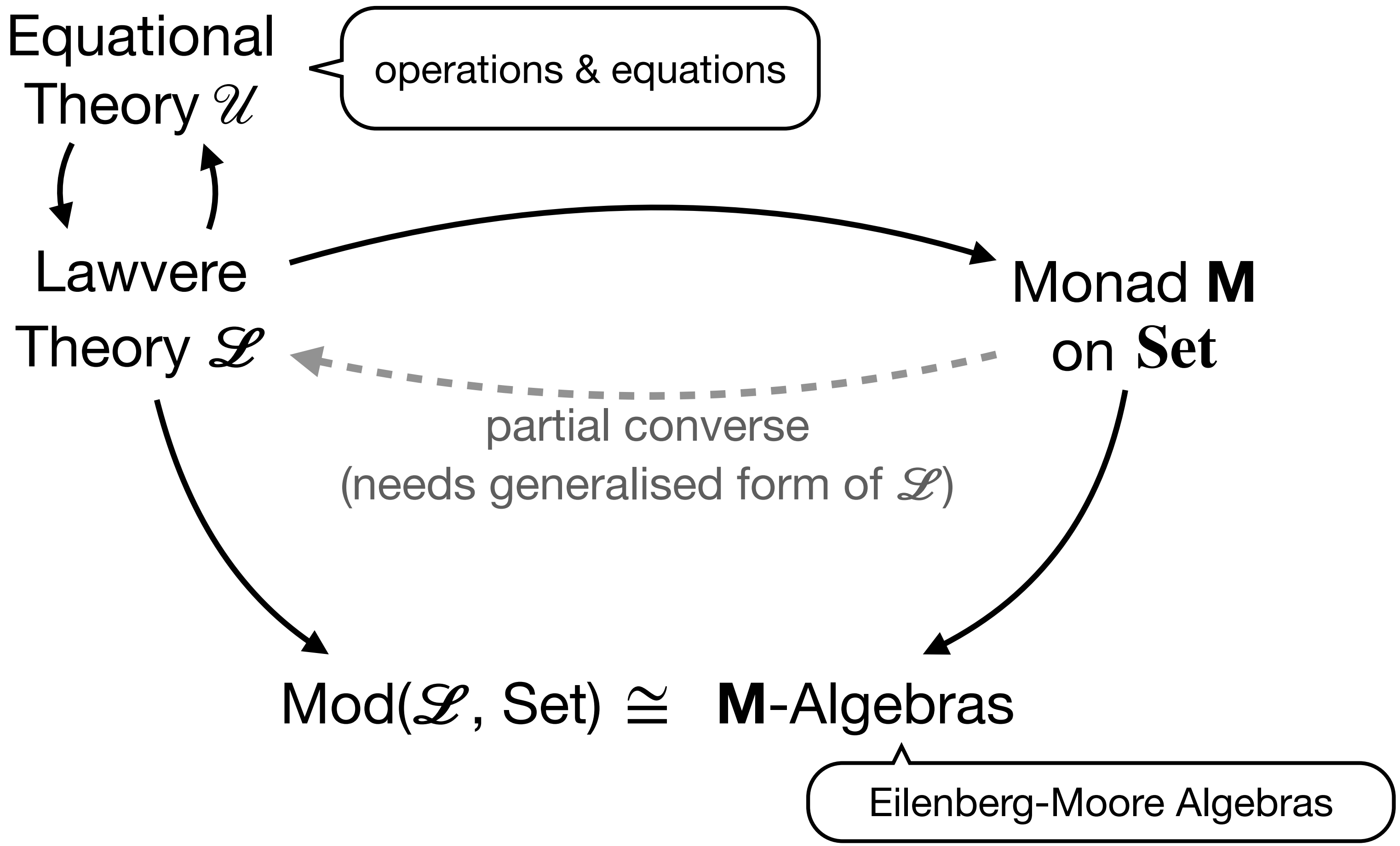
- **Plotkin & Power'01:** *(most of the) Monads are given by operations and equations* -**Algebraic Effects**

- **Hyland, Plotkin, Power'06:** *sum and tensor of theories* -**Combining Algebraic Effects**

- **Mardare, Panangaden, Plotkin (LICS'16):** *Theory of effects in a metric setting* -**Quantitative Algebraic Effects** (operations & *quantitative equations* give monads on **EMet**)

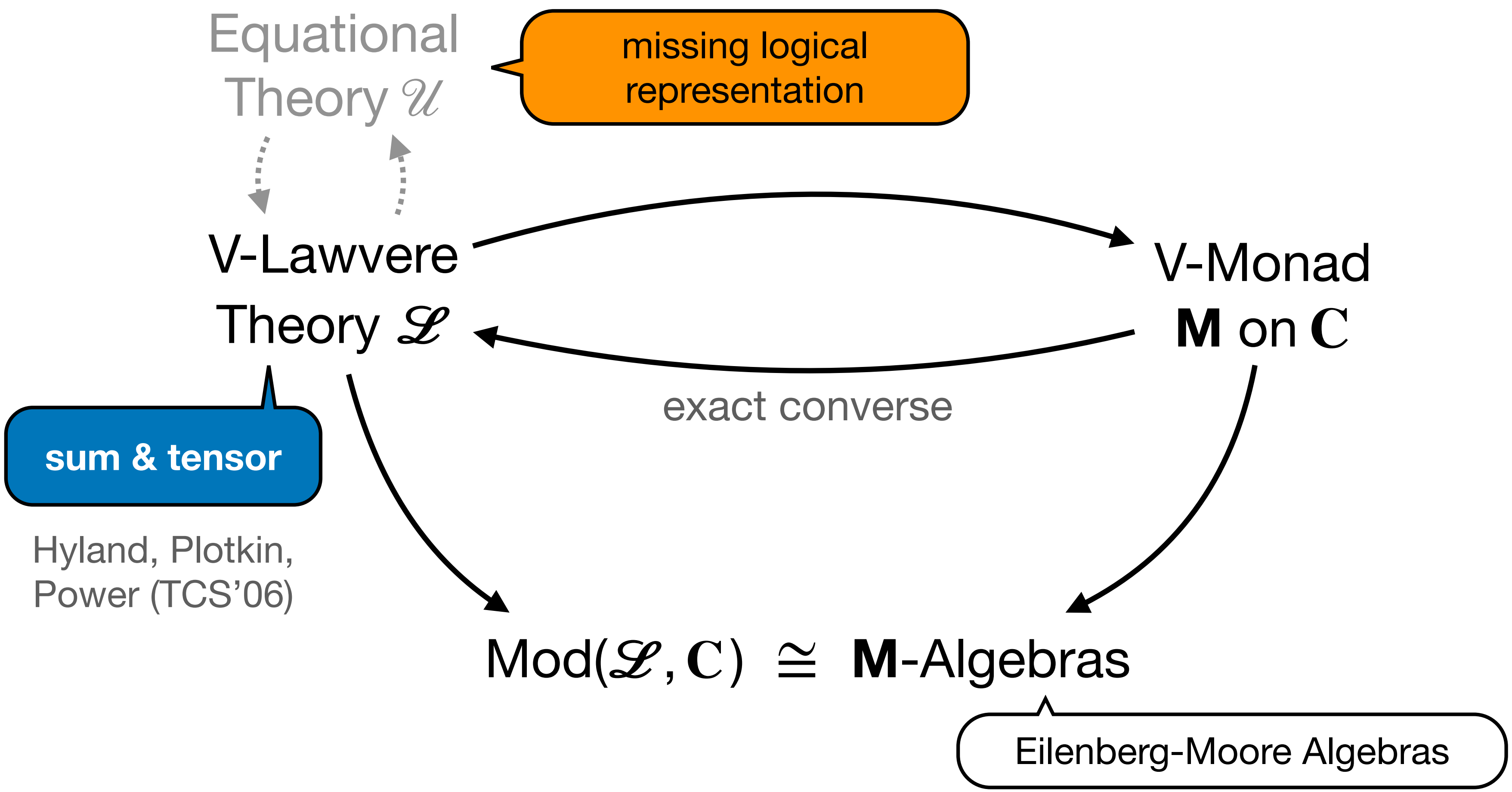
The Standard Picture

Lawvere'64, Linton'66



The Enriched Picture*

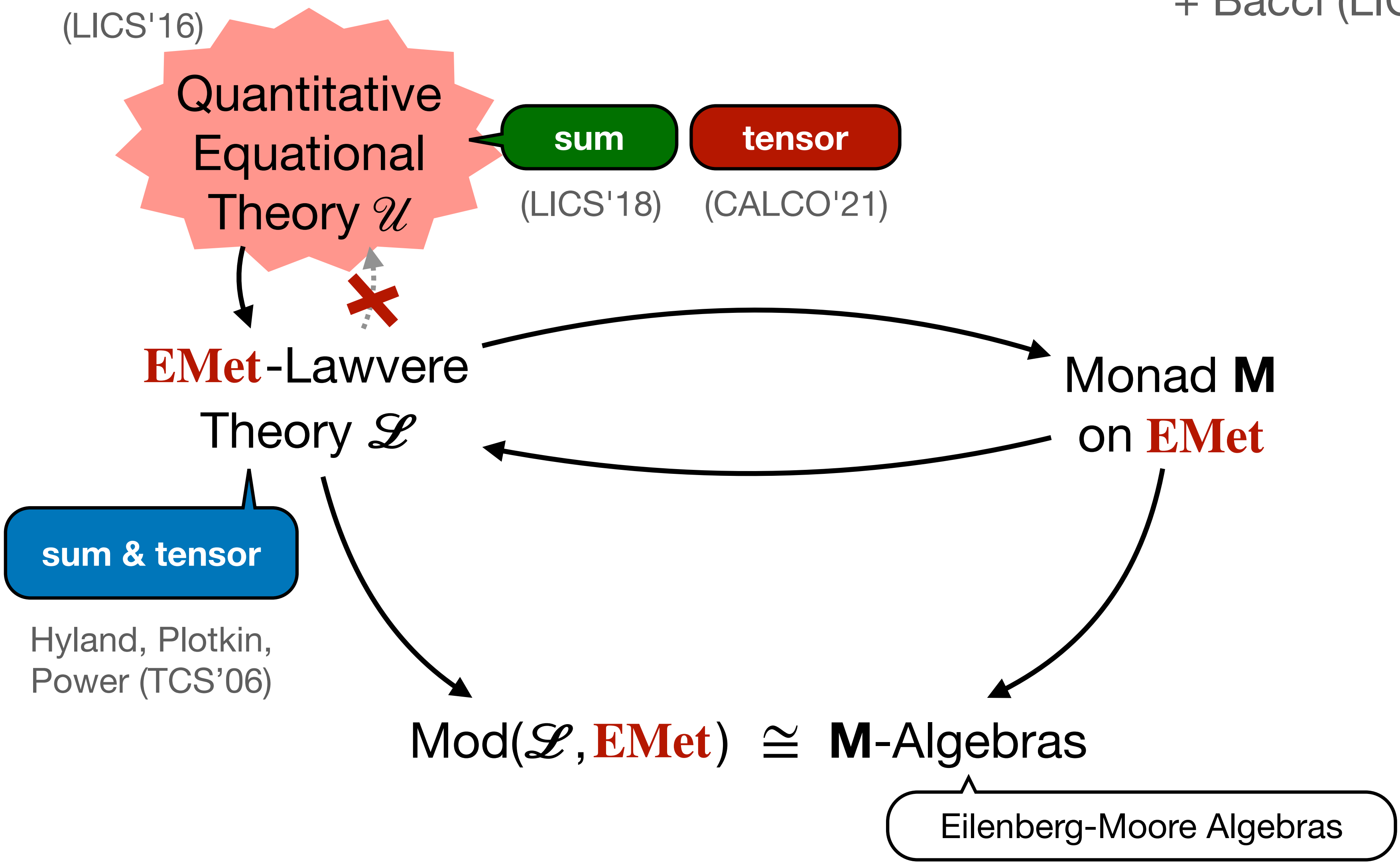
Power (TAC'99)



(*) enriched over a locally finitely presentable monoidal category V

The Quantitative Picture

Mardare, Panangaden, Plotkin (LICS'16)
+ Bacci (LICS'18)



What have we done

- Shown that the **tensor of quantitative theories** corresponds to the tensor of their quantitative effects as monads
- Given *quantitative analogues of* ***Moggi's reader and writer monad transformers*** **at the level of theories** using tensor
- Shown how to combine -by sum and tensor- different theories to produce new interesting examples
- Specifically, **equational axiomatization of LMPs and MDPs** with their discounted *bisimilarity metrics*

Quantitative Equations

Mardare, Panangaden, Plotkin (LICS'16)

$$s =_{\varepsilon} t$$

"s is approximately equal to t up to an error ε"

Example: Barycentric Algebras

Are the quantitative algebras over the signature

$$\Sigma_{\mathscr{B}} = \{ +_e : 2 \mid e \in [0,1] \}$$

convex sum

satisfying the following conditional quantitative equations

(B1) $\vdash x +_1 y =_0 x$

(B2) $\vdash x +_e x =_0 x$

(B3) $\vdash x + y =_0 y + x$

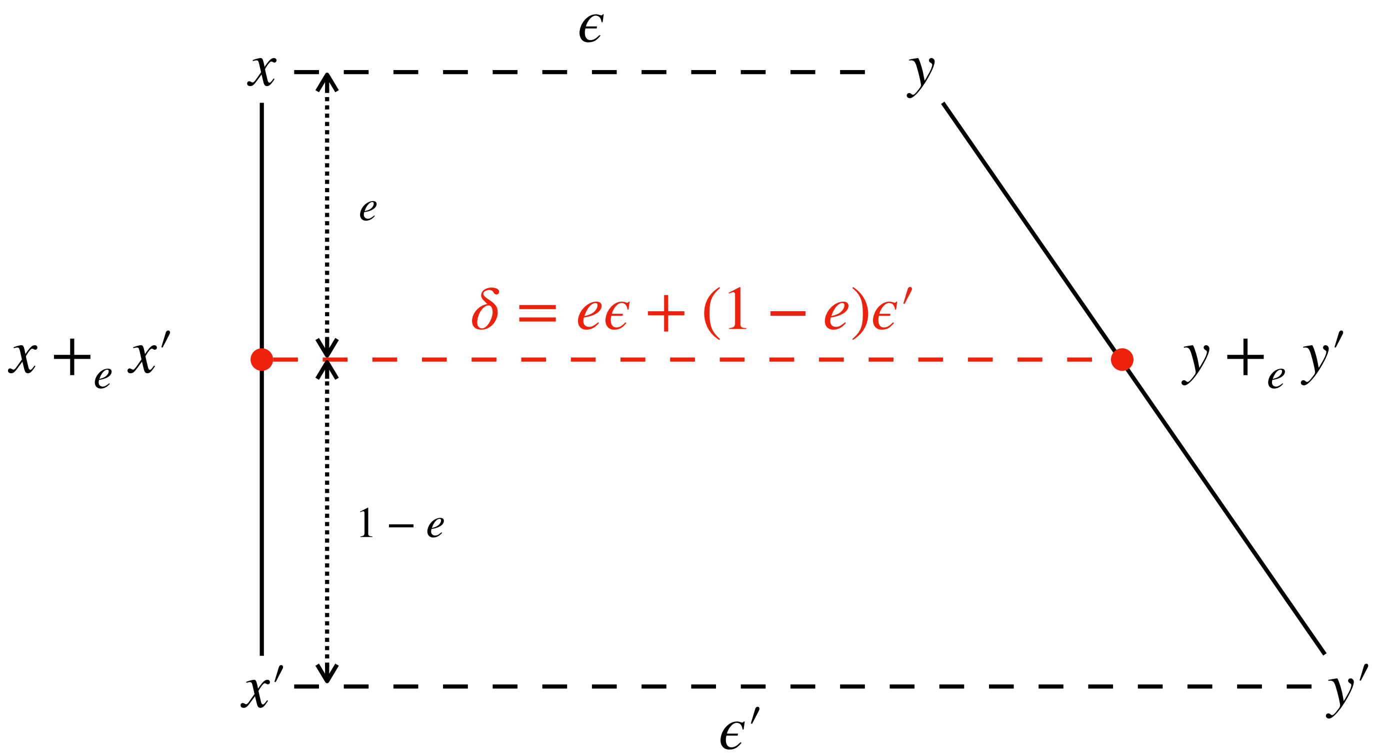
(SC) $\vdash x +_e y =_0 y +_{1-e} x$

(SA) $\vdash (x +_e y) +_{e'} z =_0 x +_{ee'} (y +_{\frac{(1-e)e'}{1-ee'}} z)$, **for** $e, e' \in (0,1)$

(IB) $x =_\epsilon y, x' =_{\epsilon'} y' \vdash x +_e x' =_\delta y +_e y'$, **where** $\delta = e\epsilon + (1-e)\epsilon'$

A geometric intuition

(IB) $x =_\epsilon y, x' =_{\epsilon'} y' \vdash x +_e x' =_\delta y +_e y'$, **where** $\delta = e\epsilon + (1-e)\epsilon'$



Example of models

Unit interval with Euclidian distance and convex combinators

$([0,1], d_{[0,1]})$ $\quad$ $(\ +_e\ )^{[0,1]}(a, b) = ea + (1 - e)b$

Finitely supported distributions with **Kantorovich distance**

$(\mathscr{D}(X), \mathscr{K}(d_X))$ $\quad$ $(\ +_e\ )^{\mathscr{D}}(\mu, \nu) = e\mu + (1 - e)\nu$

Borel probability measures with **Kantorovich distance**

$(\Delta(X), \mathscr{K}(d_X))$ $\quad$ $(\ +\ )^{\Delta}(\mu, \nu) = e\mu + (1 - e)\nu$

Quantitative Equational Theory

Mardare, Panangaden, Plotkin (LICS'16)

A quantitative equational theory $\mathscr{U}$ of type Σ is a set of

$$\{s_i =_{\varepsilon_i} t_i \mid i \in I\} \vdash s =_\varepsilon t$$

conditional quantitative equations

closed under substitution of variables, logical inference, and the following "meta axioms"

(Refl) $\vdash x =_0 x$

(Symm) $x =_\varepsilon y \vdash y =_\varepsilon x$

(Triang) $x =_\varepsilon y, y =_\delta z \vdash x =_{\varepsilon+\delta} y$

(NExp) $x_1 =_\varepsilon y_1, \ldots, y_n =_\varepsilon y_n \vdash f(x_1, \ldots, x_n) =_\varepsilon f(y_1, \ldots, y_n)$ – **for** $f \in \Sigma$

(Max) $x =_\varepsilon y \vdash x =_{\varepsilon+\delta} y$ – **for** $\delta > 0$

(Inf) $\{x =_\delta y \mid \delta > \varepsilon\} \vdash x =_\varepsilon y$

Quantitative Algebras

Mardare, Panangaden, Plotkin (LICS'16)

The models of a quantitative equational theory $\mathscr{U}$ of type Σ are

Quantitative Σ-Algebras:

$\mathscr{A} = (A, \alpha\colon \Sigma A \to A)$ – **Universal Σ-algebras on EMet**

Satisfying the all the conditional quantitative equations in $\mathscr{U}$

Satisfiability

$$\mathscr{A} \models \left(\{t_i =_{\varepsilon_i} s_i \mid i \in I\} \vdash t =_{\varepsilon} s \right)$$

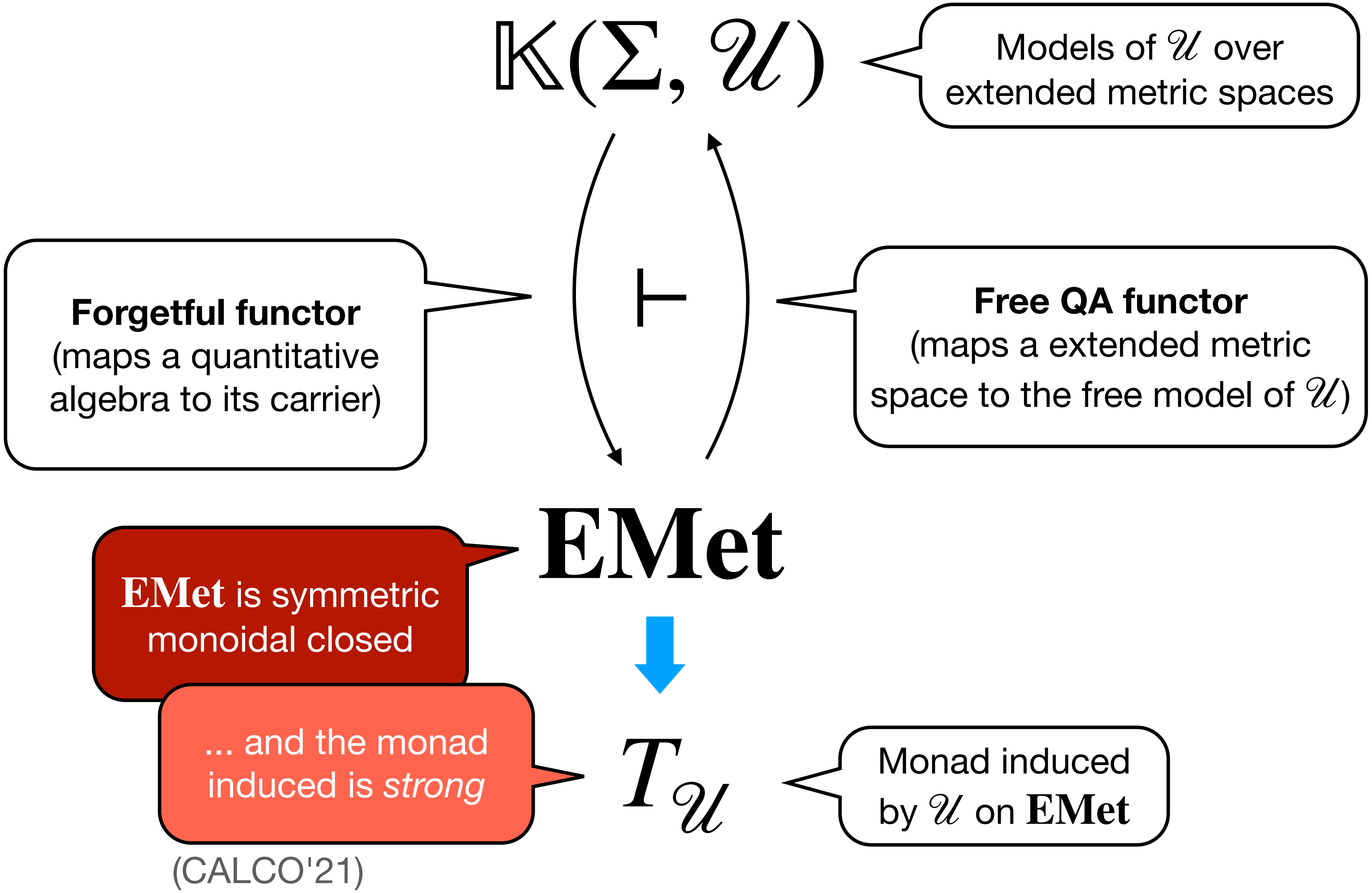
iff

for any assignment $\iota\colon X \to A$

$\left(\forall i \in I \,.\, d_A(\iota(t_i), \iota(s_i)) \leq \varepsilon_i\right)$ **implies** $d_A(\iota(t), \iota(s)) \leq \varepsilon$

Free Monad on $\mathbf{EMet}$

Mardare, Panangaden, Plotkin (LICS'16)



Example: Barycentric Algebras

$$\Sigma_{\mathscr{B}} = \{ +_e : 2 \mid e \in [0,1] \}$$

convex sum

(B1) $\vdash x +_1 y =_0 x$

(B2) $\vdash x +_e x =_0 x$

(B3) $\vdash x + y =_0 y + x$

(SC) $\vdash x +_e y =_0 y +_{1-e} x$

(SA) $\vdash (x +_e y) +_{e'} z =_0 x +_{ee'} (y +_{\frac{(1-e)e'}{1-ee'}} z)$, **for** $e, e' \in (0,1)$

(IB) $x =_\epsilon y, x' =_{\epsilon'} y' \vdash x +_e x' =_\delta y +_e y'$, **where** $\delta = e\epsilon + (1-e)\epsilon'$

$$\mathbb{K}(\Sigma_{\mathscr{B}}, \mathscr{B}) \underset{}{\overset{}{\rightleftarrows}} \mathbf{EMet} \quad (\bot) \quad \Rightarrow \quad T_{\mathscr{B}} \cong \mathscr{D}$$

Interpolative barycentric algebras

Finitely supported probability distributions with **Kantorovich metric**

Compositional Reasoning via Tensor

Tensor of Quantitative Theories

It's the operation that combines two theories by imposing the *commutation* of the operations of the theories over each other

- **Freyd'66:** on equational theories
- **Hyland, Plotkin, Power'06:** on enriched Lawvere theories

we follow Freyd'66

Our Definition

Let $\mathcal{U}$, $\mathcal{U}'$ be *quantitative theories* with disjoint signatures Σ, Σ'.
The *tensor* $\mathcal{U} \otimes \mathcal{U}'$ is the smallest theory containing the two theories and such that for all $f \colon n \in \Sigma$ and $g : m \in \Sigma'$

$$\vdash f(g(x_{1,1}, \ldots, x_{1,m}), \ldots, g(x_{n,1}, \ldots, x_{n,m})) \equiv_0 g(f(x_{1,1}, \ldots, x_{n,1}), \ldots, f(x_{1,m}, \ldots, x_{n,m}))$$

Main contribution

Theorem

The tensor of quantitative theories corresponds to the categorical tensor of their quantitative effects as monads

$$\mathbb{K}(\Sigma, \mathscr{U}) \quad \mathbb{K}(\Sigma', \mathscr{U}') \quad \mathbb{K}(\Sigma + \Sigma', \mathscr{U} \otimes \mathscr{U}')$$

$$\vdash \quad \vdash \quad \vdash$$

$$\mathbf{EMet} \quad \mathbf{EMet} \quad \mathbf{EMet}$$

$$T_{\mathscr{U}} \otimes T_{\mathscr{U}'} \cong T_{\mathscr{U} \otimes \mathscr{U}'}$$

Models of $\mathscr{U} \otimes \mathscr{U}'$

Tensor product of monads

Monad Transformers

- **Moggi, Cenciarelli'93:** *Combination of effects as strong monad transformers on cartesian closed categories*
- **Hyland, Plotkin, Power'06:** explained *many of Moggi's monad transformers as sum and tensors.*

In particular...

Reader monad transformer

$$T \mapsto (T-)^A \cong T \otimes (\,-\,)^A$$

Writer monad transformer

$$T \mapsto (A \times T-) \cong T \otimes (A \times -\,)$$

Quantitative Reader Algebras

$$\Sigma_{\mathscr{R}} = \{\mathbf{r} : |E|\}$$

reads from a finite set of inputs $E = \{e_1, \ldots, e_n\}$ and proceeds

(Idem) $\vdash x \equiv_0 \mathbf{r}(x, \ldots, x)$

(Diag) $\vdash \mathbf{r}(x_{1,1}, \ldots, x_{n,n}) \equiv_0 \mathbf{r}(\mathbf{r}(x_{1,1}, \ldots, x_{1,n}), \ldots, \mathbf{r}(x_{n,1}, \ldots, x_{n,n}))$

Monad in **EMet** only for *discrete* spaces of inputs!

$$\mathbb{K}(\Sigma_{\mathscr{R}}, \mathscr{R}) \perp \mathbf{EMet} \Rightarrow T_{\mathscr{R}} \cong (-)^{\underline{E}}$$

quantitative reader algebras

Reader monad for the discrete space $\underline{E}$

Quantitative Writer Algebras

Let $(\Lambda, \star, 0)$ be a monoid with non-expansive multiplication

metric space

$$\Sigma_{\mathscr{W}} = \{\mathbf{w}_a : 1 \mid a \in \Lambda\}$$

writes the output symbol a and proceeds

(Zero) $\vdash x \equiv_0 \mathbf{w}_0(x)$

(Mult) $\vdash \mathbf{w}_a(\mathbf{w}_b(x)) \equiv_0 \mathbf{w}_{a\star b}(x)$

(Diff) $\{x \equiv_\epsilon x'\} \vdash \mathbf{w}_a(x) \equiv_\delta \mathbf{w}_b(x')$, **for** $\delta \geq d_\Lambda(a,b) + \epsilon$

$$\mathbb{K}(\Sigma_{\mathscr{W}}, \mathscr{W}) \underset{\perp}{\rightleftarrows} \mathbf{EMet} \quad \Rightarrow \quad T_{\mathscr{W}} \cong (\Lambda \,\square\, -)$$

quantitative writer algebras

Writer monad for the metric space Λ

Quantitative Theory Transformers

We can obtain quantitative analogues of Moggi's reader and writer monad transformers at the level of theories using tensor

Reader transformer

$$\mathscr{U} \mapsto \mathscr{U} \otimes \mathscr{R}$$

$$\mathbb{K}(\Sigma + \Sigma_{\mathscr{R}}, \mathscr{U} \otimes \mathscr{R})$$

$$\vdash$$

$$\mathbf{EMet}$$

$$T_{\mathscr{U}} \otimes \mathscr{R} \cong (T_{\mathscr{U}} -)^{\underline{E}}$$

Writer transformer

$$\mathscr{U} \mapsto \mathscr{U} \otimes \mathscr{W}$$

$$\mathbb{K}(\Sigma + \Sigma_{\mathscr{W}}, \mathscr{U} \otimes \mathscr{W})$$

$$\vdash$$

$$\mathbf{EMet}$$

$$T_{\mathscr{U}} \otimes \mathscr{W} \cong (\Lambda \,\square\, T_{\mathscr{U}} -)$$

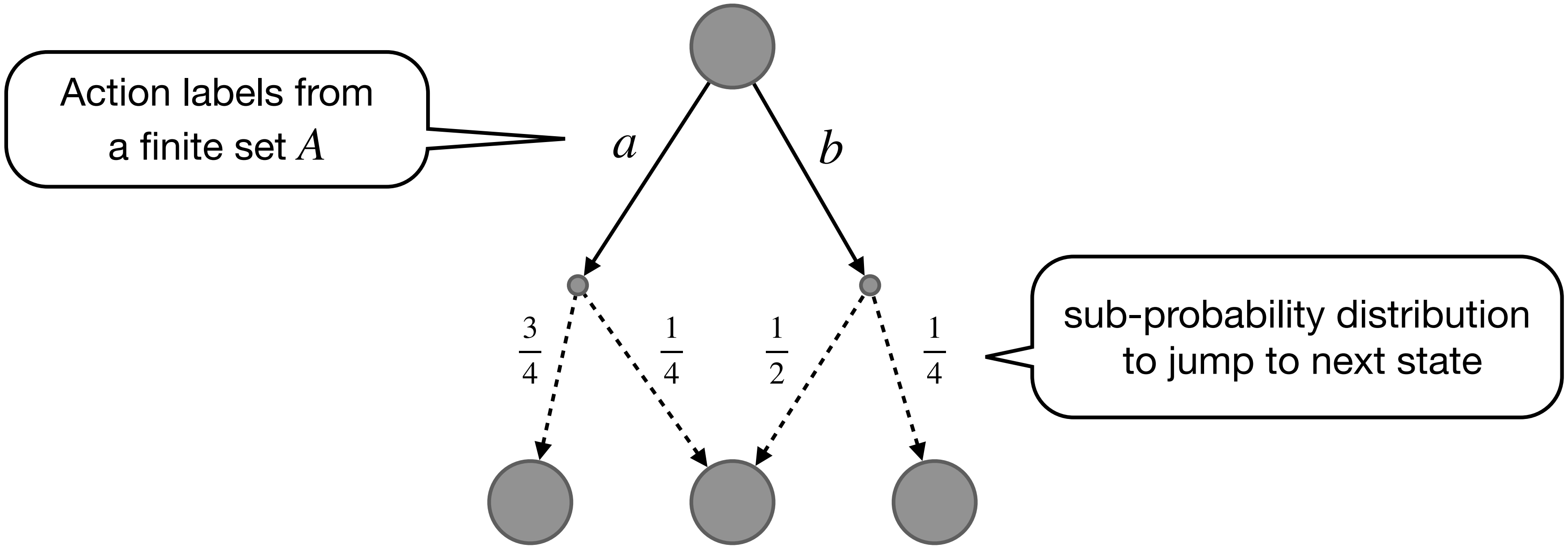
as the combination of simpler theories, via sum & tensor

Quantitative Axiomatizations of LMPs (and MDPs)

(with discounted bisimilarity metrics)

Labelled Markov Processes

and their c-discounted bisimilarity metric



As in *van Breugel et al. (TCS'03)*, we regard LMPs over metric spaces as the coalgebras for the functor $(\mathscr{D}(1 + c \cdot -))^{\underline{A}}$ in $\mathbf{EMet}$

The step-by-step recipe

STEP 1: *We axiomatize sub-probability distributions as the disjoint sum of the barycentric and pointed theory*

$$\mathscr{U}_1 = \mathscr{B} + \mathscr{Z} \;\Rightarrow\; T_{\mathscr{U}_1} \cong \mathscr{D}(1 + -)$$

sub-probability distributions monad

Interpolative barycentric theory

free-theory over $\Sigma_0 = \{\mathbf{0}\colon 0\}$

STEP 2: *apply the quantitative reader theory transformer*

adds reaction to action labels

$$\mathscr{U}_2 = \mathscr{U}_1 \otimes \mathscr{R} \;\Rightarrow\; T_{\mathscr{U}_2} \cong (\mathscr{D}(1 + -))^{\underline{A}}$$

STEP 3: *add a unary c-Lipschitz transition step operator* $\diamond\colon 1$

$$\mathscr{U}_{LMP} = \mathscr{U}_2 + \mathscr{T} \;\Rightarrow\; T_{\mathscr{U}_{LMP}} \cong \mu y.((\mathscr{D}(1 + c \cdot y + -))^{\underline{A}}$$

axiom for c-Lipschitz contractivity

LMPs with c-discounted bisimilarity metric

The resulting theory $\mathcal{U}_{\mathsf{LMP}}$

(B1) $\vdash x +_1 y =_0 x$

(B2) $\vdash x +_e x =_0 x$

(B3) $\vdash x + y =_0 y + x$

(SC) $\vdash x +_e y =_0 y +_{1-e} x$

(SA) $\vdash (x +_e y) +_{e'} z =_0 x +_{ee'} (y +_{\frac{(1-e)e'}{1-ee'}} z)$, **for** $e, e' \in (0,1)$

(IB) $x =_\epsilon y, x' =_{\epsilon'} y' \vdash x +_e x' =_\delta y +_e y'$, **where** $\delta = e\epsilon + (1-e)\epsilon'$

(Idem) $\vdash x \equiv_0 \mathbf{r}(x, x)$

(Diag) $\vdash \mathbf{r}(x, y) \equiv_0 \mathbf{r}(\mathbf{r}(x, z), \mathbf{r}(w, y))$

(Comm) $\vdash \mathbf{r}(x +_e y, x' +_e y') \equiv_0 \mathbf{r}(x, x') +_e \mathbf{r}(y, y')$

($\diamond$-Lip) $x =_\epsilon y \vdash \diamond x =_{c\epsilon} \diamond y$

Conclusions

- We developed the theory for the commutative combination of quantitative algebraic effects (equational theory + monads)
- We illustrated the applicability of our theory by showing how to produce novel interesting quantitative axiomatizations
- Introduced the concept of *pre-operation of a* functor
- Given an algebraic representation of the final coalgebra of LMPs and MDPs over extended metric spaces
- Probability + non-determinism (*distributive tensor?*)

Thank you
for the attention