Structural Operational Semantics for continuous state probabilistic processes*

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Breakfast Talk

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Structural Operational Semantics for CCS

Syntax: $P ::= \text{nil} \mid a.P \mid \bar{a}.P \mid \tau.P \mid P + P \mid P \parallel P \qquad (a \in A)$

$$\Sigma X = \overbrace{1}^{\text{nil}} + \overbrace{\underline{A} \times X}^{a.x} + \overbrace{\underline{A} \times X}^{\bar{a}.x} + \overbrace{X}^{\tau.x} + \overbrace{X \times X}^{x+x} + \overbrace{X \times X}^{x \parallel x}$$

$$\frac{}{\alpha.x \xrightarrow{\alpha} x} \qquad \frac{x \xrightarrow{\alpha} x'}{x + y \xrightarrow{\alpha} x'} \qquad \frac{y \xrightarrow{\alpha} y'}{x + y \xrightarrow{\alpha} y'}$$

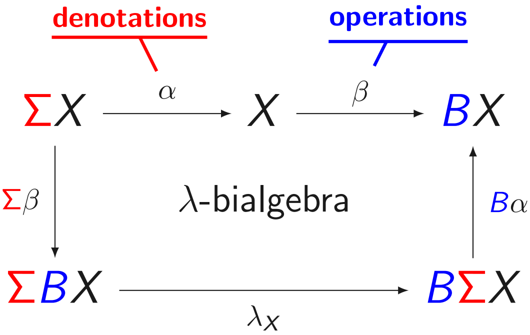
$$\frac{x \xrightarrow{\alpha} x'}{x \parallel y \xrightarrow{\alpha} x' \parallel y} \qquad \frac{y \xrightarrow{\alpha} y'}{x \parallel y \xrightarrow{\alpha} x \parallel y'}$$

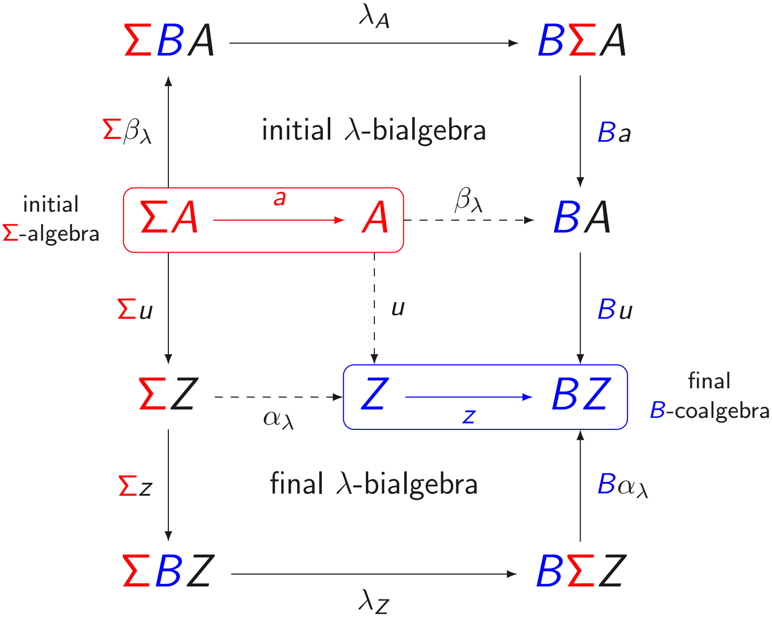
$$\frac{x \xrightarrow{a} x' \qquad y \xrightarrow{\bar{a}} y'}{x \parallel y \xrightarrow{\tau} x' \parallel y'} \qquad \frac{x \xrightarrow{\bar{a}} x' \qquad y \xrightarrow{a} y'}{x \parallel y \xrightarrow{\tau} x' \parallel y'}$$

This corresponds to: $\lambda \colon \Sigma(Id \times (\mathcal{P}_{\text{fin}})^L) \Rightarrow (\mathcal{P}_{\text{fin}} T_\Sigma)^L$

Abstract Structural Operational Semantics

(distributing syntax over behaviours: $\lambda \colon \Sigma B \Rightarrow B\Sigma$)





ΣBA
λ_A
BΣA
Σβ_λ
initial λ-bialgebra
Ba
initial
Σ-algebra
ΣA
a
A
β_λ
BA
Σu
u
Bu
ΣZ
α_λ
Z
z
BZ
final
B-coalgebra
Σz
final λ-bialgebra
Bα_λ
ΣBZ
λ_Z
BΣZ

Benefits of the bialgebraic framework

[Turi-Plotkin'97]

+ denotational model on the final B-coalgebra (by co-induction)
+ operational model on the initial Σ-algebra (by induction)

+ universal semantics (full-abstaction)

  initial algebra semantics = final coalgebra semantics
+ B-behavioural equivalence is a Σ-congruence
+ B-bisimilarity is a Σ-congruence (if B pres. weak pullbacks)

Congruential Rule Formats [Turi-Plotkin'97]

Distributive laws can be specified as sets of derivation rules

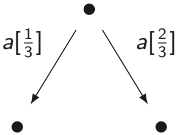
$$\left\{ \frac{\left\{x_i \xrightarrow{a} y_{ij}^a\right\}_{1\leq i\leq n,\, a\in A_i}^{1\leq j\leq m_i^a} \quad \left\{x_i \stackrel{b}{\not\rightarrow}\right\}_{1\leq i\leq n}^{b\in B_i}}{f(x_1,\ldots,x_n) \xrightarrow{c} t} \right\}_{\text{image finite}} \qquad \text{(GSOS)}$$

corresponds to. . .

$$\lambda\colon \Sigma(Id \times (\mathcal{P}_{\text{fin}})^L) \Rightarrow (\mathcal{P}_{\text{fin}} T_{\Sigma})^L$$

Discrete state sub-Probabilistic Systems

. . . hence labelled sub-probabilistic Markov chains



$$X \to (\mathcal{D}_{\mathsf{fin}} X)^L \quad \text{in } \mathbf{Set}$$

where

$$\mathcal{D}_{\mathsf{fin}} : \mathbf{Set} \to \mathbf{Set} \qquad \text{(sub-probability distribution functor)}$$

$$\mathcal{D}_{\mathsf{fin}} X = \{\varphi \colon X \to [0,1] \mid \sum_{x \in X} \varphi(x) \leq 1, |supp(\varphi)| < \infty\}$$

Rule Formats for Probabilistic Systems

[Bartels'04]

$$\left\{ \frac{\begin{array}{ll} x_i \xrightarrow{a} & a \in A_i, 1 \leq i \leq n \\ x_i \stackrel{b}{\not\rightarrow} & b \in B_i, 1 \leq i \leq n \\ x_{a_j} \xrightarrow{l_j[p_j]} y_j & 1 \leq i \leq J \end{array}}{f(x_1, \ldots, x_n) \xrightarrow{c[w \cdot p_1 \cdot \ldots \cdot p_J]} t} \right\}_{\text{image finite}}$$

corresponds to...

$$\lambda \colon \Sigma(Id \times (\mathcal{D}_{\text{fin}})^L) \Rightarrow (\mathcal{D}_{\text{fin}} T_\Sigma)^L$$

where

$\mathcal{D}_{\text{fin}} : \mathbf{Set} \to \mathbf{Set}$ (sub-probability distribution functor)

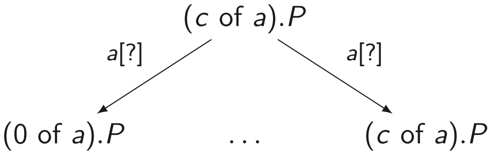
$$\mathcal{D}_{\text{fin}} X = \{\varphi \colon X \to [0,1] \mid \sum_{x \in X} \varphi(x) \leq 1, |supp(\varphi)| < \infty\}$$

What if the state space is continuous? (example)

Let us extend CCS with a quantitative operator

$$P ::= \mathsf{nil} \mid (c \text{ of } \alpha).P \mid P + P \mid P \parallel P \qquad (c \in \mathbb{R}_{\geq 0})$$

$$\alpha ::= a \mid \overline{a} \mid \tau \qquad (a \in A)$$

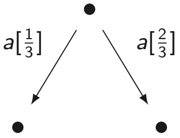


ideally we want that the outcomes are uniformly distributed...

$$U\big((c \text{ of } a).P\big)\big(\{(i \text{ of } a).P \mid i \in [a,b]\}\big) = \int_a^b \frac{1}{c}\,dx \quad (0 \leq a \leq b \leq c)$$

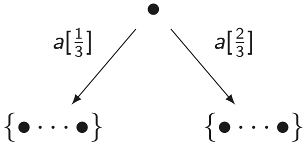
Discrete state

(labelled Markov chains)



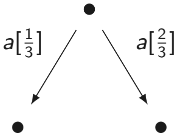
Continuous state

(labelled Markov processes)



Discrete state

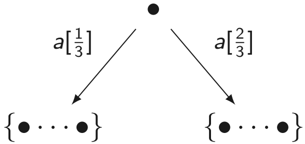
(labelled Markov chains)



$X \to (\mathcal{D}_{\text{fin}} X)^L$ in **Set**

Continuous state

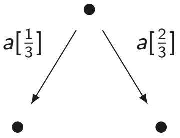
(labelled Markov processes)



$\mathcal{D}_{\text{fin}} : \textbf{Set} \to \textbf{Set}$ (sub-probability distribution functor)

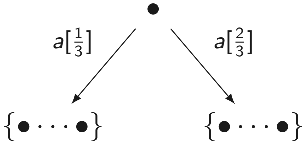
$$\mathcal{D}_{\text{fin}} X = \{\varphi \colon X \to [0,1] \mid \sum_{x \in X} \varphi(x) \leq 1, |supp(\varphi)| < \infty\}$$

Discrete state
(labelled Markov chains)



$X \to (\mathcal{D}_{\text{fin}} X)^L$ in **Set**

Continuous state
(labelled Markov processes)



$X \to (\Delta X)^L$ in **Meas**

$\mathcal{D}_{\text{fin}} \colon \mathbf{Set} \to \mathbf{Set}$ (sub-probability distribution functor)

$$\mathcal{D}_{\text{fin}} X = \{\varphi \colon X \to [0,1] \mid \sum_{x \in X} \varphi(x) \leq 1, |supp(\varphi)| < \infty\}$$

$\Delta \colon \mathbf{Meas} \to \mathbf{Meas}$ (Giry functor)

$$\Delta X = \{\mu \colon \Sigma_X \to [0,1] \mid \mu \text{ sub-probability measure}\}$$

Aim: Congruential Rule Formats
for Probabilistic Processes
with Continuous State Spaces

…hence, inducing distributive laws of type

$$\lambda\colon \Sigma(Id \times \Delta^L) \Rightarrow (\Delta T_\Sigma)^L$$

The shape of transitions

The behaviour functor suggests the shape of transitions...

Discrete state

$$t \xrightarrow{a[p]} t'$$

Σ-term

Continuous state

$$t \xrightarrow{a} \mu$$

measure
on Σ-terms

Earlier attempts. . . [Cardelli-Mardare'10]

The Measurable Space of Stochastic Processes

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$$(Null).\qquad \frac{}{0 \to \overline{\omega}}$$

$$(Guard).\qquad \frac{}{\varepsilon.P \to [{}^{\varepsilon}_{P}]}$$

$$(Sum).\qquad \frac{P \to \mu' \qquad Q \to \mu''}{P + Q \to \mu' \oplus \mu''}$$

$$(Par).\qquad \frac{P \to \mu' \qquad Q \to \mu''}{P|Q \to \mu' \; {}_{P}\!\otimes_{Q} \; \mu''}$$

Table I
STRUCTURAL OPERATIONAL SEMANTICS

Earlier attempts. . .

[Cardelli-Mardare'10]

The Measurable Space of Stochastic Processes

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$(Sum).\quad \dfrac{P \to \mu' \qquad Q \to \mu''}{P + Q \to \mu' \oplus \mu''}$

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Table I
STRUCTURAL OPERATIONAL SEMANTICS

Earlier attempts. . .

[Cardelli-Mardare'10]

The Measurable Space of Stochastic Processes

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rather ad

(no general

$(Null).\quad \dfrac{}{0 \to \overline{\omega}}$

$(Guard).\quad \dfrac{}{\varepsilon.P \to [^{\varepsilon}_{P}]}$

$(Sum).\quad \dfrac{P \to \mu' \qquad Q \to \mu''}{P + Q \to \mu' \oplus \mu''}$

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Table I
STRUCTURAL OPERATIONAL SEMANTICS

Measure terms

We adopt a new syntax to handle measures syntactically

$$\Sigma : \mathbf{Meas} \to \mathbf{Meas} \qquad \text{(process syntax)}$$

$$M : \mathbf{Meas} \to \mathbf{Meas} \qquad \text{(measure syntax)}$$

$$t \xrightarrow{a} \mu$$

it's a M-term!

Measure GSOS rule format

$$\frac{\left\{x_i \xrightarrow{a_{ij}} \mu_{ij}\right\}_{1 \leq i \leq n,\, a_{ij} \in A_i}^{1 \leq j \leq m_i} \quad \left\{x_i \stackrel{b}{\not\longrightarrow}\right\}_{1 \leq i \leq n}^{b \in B_i}}{f(x_1, \ldots, x_n) \xrightarrow{c} \mu} \qquad \text{(MGSOS)}$$

where

- $f \in \Sigma$ with $ar(f) = n$;
- $\{x_1, \ldots, x_n\}$ and $\{\mu_{ij} \mid 1 \leq i \leq n,\ 1 \leq j \leq m_i\}$ are pairwise distinct process and measure *variables*;
- $A_i \cap B_i = \emptyset$ are disjoint subsets of labels in L, and $c \in L$;
- μ is a M-term with variables in $\{x_1, \ldots, x_n\}$ and $\{\mu_{ij} \mid 1 \leq i \leq n, 1 \leq j \leq m_i\}$.

An MGSOS specification system consists of

Set of MGSOS rules:

$$\mathcal{R} = \left\{ \frac{\left\{x_i \xrightarrow{a_{ij}} \mu_{ij}\right\}^{1\leq j\leq m_i}_{1\leq i\leq n,\, a_{ij}\in A_i} \quad \left\{x_i \stackrel{b}{\not\rightarrow}\right\}^{b\in B_i}_{1\leq i\leq n}}{f(x_1,\ldots,x_n) \xrightarrow{c} \mu} \right\}_{\text{image finte}}$$

Measure terms interpretation:

$$\langle\!| \cdot |\!\rangle : T_M\Delta \Rightarrow \Delta T_\Sigma$$

From MGSOS to labelled Markov processes

We can obtain a Δ^L-coalgebra on the set of closed Σ-terms

$$\gamma\colon T_\Sigma 0 \to \Delta^L T_\Sigma 0$$

as

$$\gamma(t)(\alpha) = \bigoplus_{T_\Sigma 0}\Big(\{\,\llparenthesis \mu \rrparenthesis_{T_\Sigma 0} \mid t \xrightarrow{\alpha} \mu\}\Big)$$

where, for a finite set of $U = \{\mu_1, \dots, \mu_n\}$ of sub-probability measures over X,

$$\oplus_X(\{\mu_1, \dots, \mu_n\})(E) = \frac{\mu_1(E) + \cdots + \mu_n(E)}{\mu_1(X) + \cdots + \mu_n(X)}$$

(weighted sum of sub-probability measures)

Example: Quantitative CCS

Measure terms syntax:

$$\mu ::= U^{\alpha}_{c}[P] \;\mid\; D[P] \;\mid\; \mu|\mu \;\mid\; \mu \blacktriangledown_{c,c'} \mu \qquad (c, c' \in \mathbb{R}_{\geq 0})$$

Measure GSOS Rules*:

$$\frac{}{(c \text{ of } \alpha).x \xrightarrow{\alpha,c} U^{\alpha}_{c}[x]} \qquad \frac{}{(0 \text{ of } \alpha).x \xrightarrow{\tau} D[x]}$$

$$\frac{x \xrightarrow{\alpha,c} \mu}{x + x' \xrightarrow{\alpha,c} \mu} \qquad \frac{x \xrightarrow{\alpha,c} \mu}{x \parallel x' \xrightarrow{\alpha,c} \mu|D[x']}$$

$$\frac{x \xrightarrow{\alpha,c} \mu \qquad x' \xrightarrow{\alpha,c'} \mu'}{x \parallel x' \xrightarrow{\alpha,c+c'} \mu|\mu'} \qquad \frac{x \xrightarrow{a,c} \mu \qquad x' \xrightarrow{\bar{a},c'} \mu'}{x \parallel x' \xrightarrow{\tau} \mu \blacktriangledown_{c,c'} \mu'}$$

(*) dual rules for $+$ and $\parallel$ are omitted

Example: Quantitative CCS

Measure term interpretation:

$$\langle\!|\cdot|\!\rangle_X \colon T_M \Delta X \Rightarrow \Delta T_\Sigma X$$

$$\langle\!| U_c^\alpha[x] |\!\rangle_X(E) = \int_{E'} \frac{1}{c}\, dy \qquad \text{where } E' = [0,c] \cap (\lambda\epsilon.\,(\epsilon \text{ of } \alpha).x)^{-1}(E)$$

$$\langle\!| D[x] |\!\rangle_X(E) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{otherwise} \end{cases}$$

$$\langle\!| \mu | \mu' |\!\rangle_X(E) = (\langle\!|\mu|\!\rangle_X \otimes \langle\!|\mu'|\!\rangle_X) \circ (\lambda(x,x').\, x \parallel x')^{-1}(E)$$

$$\langle\!| \mu \blacktriangledown_{c,c'} \mu' |\!\rangle_X(E) = \begin{cases} 1 & \text{if } c \cdot \langle\!|\mu|\!\rangle_X(A_1) = c' \cdot \langle\!|\mu'|\!\rangle_X(A_2), \\ & \quad \text{for } A_i = \pi_i((\lambda(x,x').\, x \parallel x')^{-1}(E)) \\ 0 & \text{otherwise} \end{cases}$$

From MGSOS to distributive laws

$$\Sigma(Id \times \Delta^L) \overset{\lambda}{\Longrightarrow} (\Delta T_\Sigma)^L$$

how do we get the distributive law λ
out of an MGSOS specification systems?

From MGSOS to distributive laws

$$\Sigma(Id \times \Delta^L)$$

$$\Big\Downarrow \llbracket\mathcal{R}\rrbracket$$

$$(\mathcal{P}_{\text{fin}} T_M \Delta)^L$$

1. define the natural transformation $\llbracket\mathcal{R}\rrbracket$ from the image finite set MGSOS rules

$$(\Delta T_\Sigma)^L$$

From MGSOS to distributive laws

$$\Sigma(Id \times \Delta^L) \overset{[\![\mathcal{R}]\!]}{\Longrightarrow} (\mathcal{P}_{\text{fin}} T_M \Delta)^L \overset{(\mathcal{P}_{\text{fin}} \langle\!| \cdot |\!\rangle)^L}{\Longrightarrow} (\mathcal{P}_{\text{fin}} \Delta T_\Sigma)^L$$

$$(\Delta T_\Sigma)^L$$

1. define the natural transformation $[\![\mathcal{R}]\!]$ from the image finite set MGSOS rules
2. apply the measure terms interpretation

$$\langle\!| \cdot |\!\rangle \colon T_M \Delta \Rightarrow \Delta T_\Sigma$$

From MGSOS to distributive laws

$$\Sigma(Id \times \Delta^L)$$
$$\Downarrow [\![\mathcal{R}]\!]$$
$$(\mathcal{P}_{\mathrm{fin}} T_M \Delta)^L$$
$$\Downarrow (\mathcal{P}_{\mathrm{fin}} \langle\!| \cdot |\!\rangle)^L$$
$$(\mathcal{P}_{\mathrm{fin}} \Delta T_\Sigma)^L$$
$$\Downarrow (\oplus T_\Sigma)^L$$
$$(\Delta T_\Sigma)^L$$

1. define the natural transformation $[\![\mathcal{R}]\!]$ from the image finite set MGSOS rules
2. apply the measure terms interpretation

   $$\langle\!| \cdot |\!\rangle \colon T_M \Delta \Rightarrow \Delta T_\Sigma$$

3. obtain the actual measure by averaging

   $$\oplus_X(\{\mu_1, \ldots, \mu_n\})(E) = \frac{\mu_1(E) + \cdots + \mu_n(E)}{\mu_1(X) + \cdots + \mu_n(X)}$$

Benefits from the bialgebraic framework

For continuous state probabilistic processes described by means of MGSOS specification systems we have:

- + denotational model on the final Δ^L-coalgebra
- + operational model on the initial Σ-algebra

- + universal semantics (full-abstaction)

  initial algebra semantics = final coalgebra semantics
- + Δ^L-behavioural equivalence is a Σ-congruence
- + is Δ^L-bisimilarity a Σ-congruence?

  (Δ^L does not preserves weak pullbacks! [Viglizzo'05])

From MGSOS to distributive laws

$$\Sigma(Id \times \Delta^L)$$
$$\Downarrow [\![\mathcal{R}]\!]$$
$$(\mathcal{P}_{\text{fin}} T_M \Delta)^L$$
$$\Downarrow (\mathcal{P}_{\text{fin}} \langle\!| \cdot |\!\rangle)^L$$
$$(\mathcal{P}_{\text{fin}} \Delta T_\Sigma)^L$$
$$\Downarrow (\oplus T_\Sigma)^L$$
$$(\Delta T_\Sigma)^L$$

Naturality of the distributive laws depends on naturality of $\langle\!| \cdot |\!\rangle \colon T_M\Delta \Rightarrow \Delta T_\Sigma$

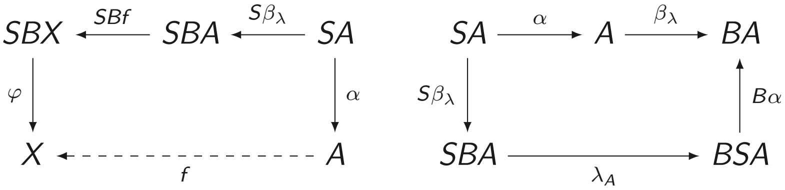
we need (general) techniques in order to derive natural transformations of type

$$T_M\Delta \Rightarrow \Delta T_\Sigma$$

λ-iterative recursion

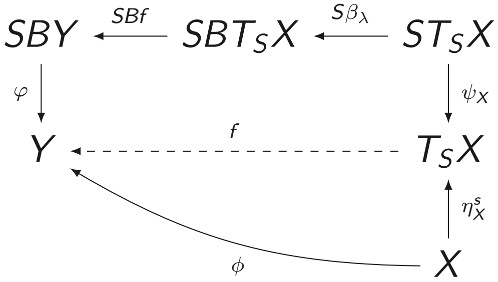
We adopt a generalized induction proof principle...

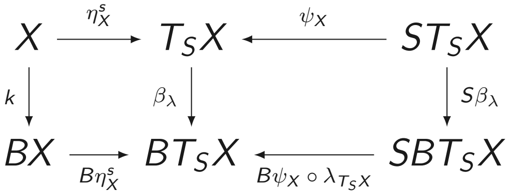
For any distributive law $\lambda\colon SB \Rightarrow BS$ and SB-algebra (X, φ) there exists a unique $f\colon A \to X$ making the following commute



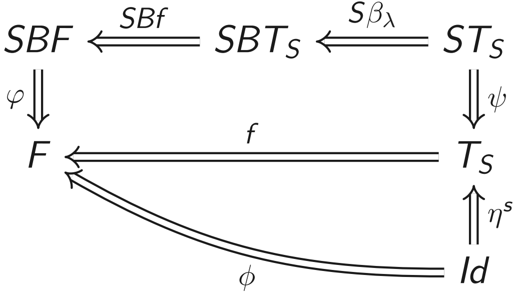
Structural λ-iterative recursion

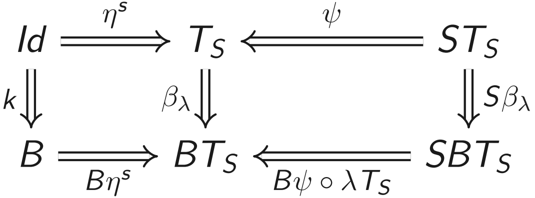
...can be extended on the free monad (T_S, η^s, μ^s)



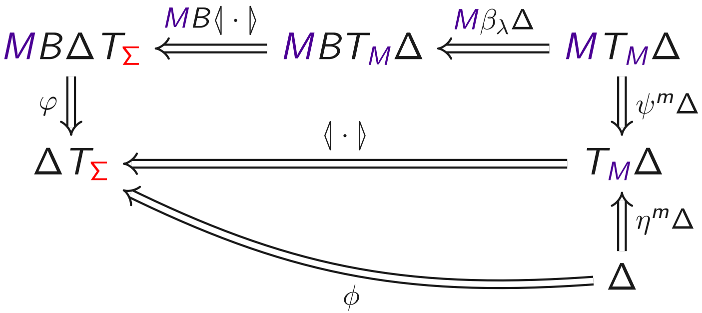


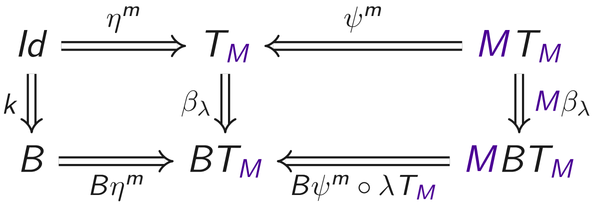
. . . and can be turned to a proof principle on natural transformations





. . . to be used to derive measure terms interpretations





Conclusions and future work

Done:

- rule format for continuous state probabilistic processes
- syntactical treatment of measures via M-terms
- general techniques for defining interpretations
- initial algebra for polynomial functors in **Meas** (not in this talk)

To do:

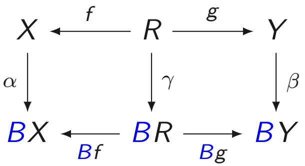
- move from probabilistic to general measures (bounded?)
- find a rule format that coincides with the distributive law
- formal expressivity analysis of the intermediate syntax + interpretation method

Thanks

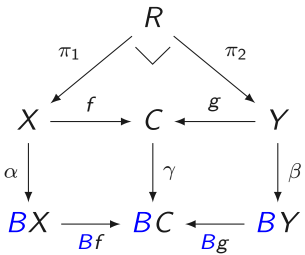
Appendix

Bisimulation vs Kernel-bisimulation

Bisimulation
(a span)



Kernel-bisimulation
(pullback of a cospan)



if B preserves weak-pullbacks, bisimulation and kernel-bisimulation coincide (provided that **C** has pullbacks and pushouts) [Staton'11]