Structural Operational Semantics for Continuous State Stochastic Transition Systems

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Abstract

In this paper we show how to model syntax and semantics of stochastic processes with continuous states, respectively as algebras and coalgebras of suitable endofunctors over the category of measurable spaces **Meas**. Moreover, we present an SOS-like rule format, called *MGSOS*, representing abstract GSOS over **Meas**, and yielding *fully abstract universal semantics*, for which behavioral equivalence is a congruence. An MGSOS specification defines how semantics of processes are composed by means of *measure terms*, which are expressions specifically designed for describing finite measures. The syntax of these measure terms, and their interpretation as measures, are part of the MGSOS specification. We give two example applications, with a simple and neat MGSOS specification: a "quantitative CCS", and a calculus of processes living in the plane $\mathbb{R}^2$ whose communication rate depends on their distance. The approach we follow in these cases can be readily adapted to deal with other quantitative aspects.

Keywords: Structural Operational Semantics, Rule formats, Stochastic Semantics, Markov Processes, Algebras, Coalgebras, Bialgebras, Continuous state systems, Quantitative Aspects.

1. Introduction

Process algebras are widely used for compositional modeling of nondeterministic, communicating, mobile systems. States of these systems are represented by syntactic-defined terms $P, Q, \ldots$; the semantics is represented by means of a (labelled) transition relation of the form $P \xrightarrow{\alpha} Q$ between these terms, where labels represent what can be observed by the environment. According to the Structural Operational Semantics (SOS) paradigm [38], this relation is specified by a set of inference rules whose application is driven by the syntactic structure of processes. In order to guarantee important properties about the resulting semantics, several formats for these SOS specifications have been studied. A

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well-known format is the so-called GSOS [13], which guarantees the bisimilarity to be a congruence. This framework is particularly appealing because it makes languages easier to understand, compare, and extend. In particular, a process algebra can be easily extended with new operators, without the need of time-consuming and error-prone proofs of congruence results.

In recent years this successful approach has been applied also to *stochastic* and *probabilistic* systems, which have received increasing attention due to their important applications to performance evaluation, systems biology, etc. [27, 14, 26, 21]. In order to deal with these quantitative aspects, the transition relation has to be modified by adding some real-valued parameter; often it has the form $P \xrightarrow{\alpha,r} Q$, meaning that "$P$ can do α and continue as Q, with probability (or rate) r". Bartels [10] and Klin and Sassone [31] have investigated rule formats for discrete probabilistic and stochastic systems, respectively, which guarantee probabilistic/stochastic bisimilarity to be a congruence. This approach has been further generalized in [34], with a rule format for general discrete, nondeterministic *weighted* labelled transition systems.

However, these formats do not cover the case of systems whose behavior is influenced by some continuous data, which may change as the system evolves. Typical examples are systems with spatial/geometric informations (e.g., in wireless networks, distance may affect data access and rates; in biological models, diffusion alters the signaling pathways, etc.), or intensional parameters like temperature, pressure, concentrations, etc.. A proper representation of the states of these systems must include these continuous data, hence the state space is a *continuous* set, and a term language representing these states cannot be countable any longer; see e.g. [15, 30]. Formally, this means that we cannot use transitions of the form $P \xrightarrow{\alpha,r} Q$, because the rate of reaching a precise Q from P may be zero, yet the rate of reaching any neighborhood of Q may be nonzero.

This leads us to consider transitions of the form $P \xrightarrow{\alpha} \mu$, where μ is a (finite) *measure* of the possible outcomes of P, as in [12]. This measure specifies a quantitative (probabilistic, stochastic,...) information about the different transitions. Stochastic calculi with a similar transition format have been considered also in [16, 6] for dealing with specific equational stochastic systems, and with behavioral equivalences which are proved to be congruences. However, differently from the case of discrete processes, these SOS specifications and results are rather *ad hoc*, not based on any general framework for operational descriptions.

In order to cover this gap, in [7] we introduced a new rule format for *probabilistic* systems with continuous states, which guarantees that the resulting probabilistic behavioral equivalence is a congruence. In this paper we continue this line of research, by focusing on *stochastic* systems. More precisely, we present a GSOS rule format for stochastic systems with continuous states, which ensures that the resulting stochastic behavioral equivalence is a congruence.

In order to achieve this goal, we follow the *bialgebraic* approach [44]. This approach relies on the fact that the syntax of processes can be represented as the initial algebra of a suitable signature functor, and their semantics as coalgebras of a suitable "behavioral" functor. The key idea of the bialgebraic framework is that rule specification systems can be formulated in terms of certain natural transformations called *distributive laws* between the signature and the behavioral functors. These distributive laws allow us to define a *fully abstract* denotational semantics with respect to the behavioral equivalence: two processes have the same denotation if and only if their behavior is the same. Moreover, this equivalence is a congruence and (if the behavioral functor preserves weak pullbacks) coincides with bisimilarity.

Usually, the bialgebraic construction has been applied in the **Set** or variations thereof like presheaf categories [44, 23, 24, 33]. This construction has been applied also to *discrete* probabilistic and stochastic systems: in [10], labelled probabilistic transition systems are shown to be coalgebras of the functor $(\mathcal{D}_\omega(\_) + 1)^L : \mathbf{Set} \to \mathbf{Set}$; in [31] labelled stochastic transition systems are shown to be coalgebras of the functor $\mathcal{R}_\omega(\_)^L : \mathbf{Set} \to \mathbf{Set}$ where L is the set of labels, and $\mathcal{R}_\omega$ is the functor over **Set** such that $\mathcal{R}_\omega(X)$ is the set of finitely supported measures over X. A more uniform and general treatment in the case of *weighted* labelled transition systems has been developed in [34].

However, in order to apply this approach to our setting we have to solve several issues. The crucial point is that our semantics have to deal with the probability of *sets* of possible outcomes, rather than of single states. This corresponds to move to the category **Meas** of *measurable spaces* and *measurable functions*, as advocated in [20, 22, 37]; for instance, the behavior of a stochastic system with continuous state can be modeled as a coalgebra of the functor $\Delta(\_)^L : \mathbf{Meas} \to \mathbf{Meas}$, where Δ associates with any measurable space (X, Σ) the set of finite measures over it.

Porting the bialgebraic approach from **Set** to **Meas** is not straightforward. First, the behavioral functor Δ does not preserve weak pullbacks [36, 48]; in fact, bisimilarity is strictly included in behavioral equivalence [43]. Hence, we focus on behavioral equivalence instead of bisimilarity.

Secondly, **Meas** is not known to be Cartesian closed; as a consequence, most constructions which can be carried out on **Set** and other topoi cannot be ported easily to **Meas**. In particular, we cannot follow Bartels's approach for deriving a rule format from a distributive law [10], because the extra structure given by σ-algebras does not allow one to decompose natural transformations of complex type as collections of natural transformations of simpler type.

Moreover, a SOS rule format should be as easy to apply, and as syntactic, as possible. In particular, it has to define a system's behavior in terms of those of its subsystems. In traditional GSOS format, this is reflected by the fact that the target of a transition is a process built from the components of the source process, and their corresponding semantics. In our settings, the target of a transition $P \xrightarrow{\alpha} \mu$ is not a process term, but a real-valued function over some measurable space, without any syntactic structure to leverage. In order to circumvent this problem, we propose to use *measure terms*, i.e., syntactic expressions purposely introduced to denote measures. The syntax of measure terms, and their interpretation as measures, is part of the specification. This new degree of freedom allows us to capture many examples from the literature.

Summarizing, we present a new operational semantics specification format, which we call *Measure GSOS format*; a MGSOS specification is given by a set of rules for deriving transitions of the form $P \xrightarrow{\alpha} \mu$ where both P and μ are syntactic objects (of possibly different languages), together with an interpretation of these measure terms into measures. We will show that any LTS specification in this format leads to a distributive law of type $S(Id \times \Delta^L) \Rightarrow (\Delta T^S)^L$, where S is the syntactic functor and T^S the corresponding free monad. As a consequence, the induced behavioral equivalence is always a congruence.

This paper is the extended and revised version of the conference paper [7]. Some original contributions of this version are: the generalization of the format to the stochastic case, several proofs and new examples and discussions (see Section 7); moreover, in Section 2.3 we provide new tools to prove the existence of initial algebras and final coalgebras in categories different from **Set**, but with some well-behaved factorization system. We apply these tools in Section 3.3 to prove that the functor Δ has final coalgebras, and in Section 4.2 to characterize freely generated monads of syntactic endofunctors over **Meas**.

Synopsis. In Section 2 we recall the basic results about algebras and coalgebras and some standard constructions [9] for the construction of initial and final objects in the categories of algebras and coalgebras for some endofunctor. Then, we provide new tools to prove the existence of such initial and final objects in the presence of some suitable factorizations systems on the underlying category.

In Section 3 we recall stochastic labelled Markov processes and show how they correspond to coalgebras of the (labelled) Giry endofunctor Δ^L on the category of measurable spaces. Moreover, using the results from the previous section, we show that Δ^L-coalgebras admit a final object.

In Section 4 we introduce the notions of measurable signatures and show how their interpretations correspond to algebras for syntactic measurable endofunctors on **Meas**. The main contributions are the proof of the existence of initial algebras for syntactic functors on **Meas** and the characterization of their freely generated monads, that are used to define the measurable space of terms.

Then, in Section 5 we introduce the MGSOS specification format for stochastic systems with continuous space. We show that every MGSOS specification corresponds to a distributive law of type $S(Id \times \Delta^L) \Rightarrow (\Delta T^S)^L$, for generic syntactic functors S on **Meas**, and that the behavioral equivalence between the induced Δ^L-coalgebras is always an S-congruence.

A key feature of this format is that the outcomes of a process are described by means of a specific language, whose terms have to be interpreted as measures. In Section 6 we show how to construct these interpretation functions using a generalized induction proof principle, dualizing a result in [10].

In Section 7 we demonstrate the usage of this new format. First, we consider a "quantitative CCS", i.e., a CCS with a real-valued prefix representing the "amount" of the given process. Then, we consider a calculus of processes living in the plane $\mathbb{R}^2$, where the communication rate depends on agents' distance. We show that both cases admit a simple and neat MGSOS specification.

Final remarks and direction for further work are in Section 8, and long proofs omitted in the paper are in Appendix A.

2. Algebras and Coalgebras

We recall the basic notions and facts about algebras and coalgebras for an endofunctor $F\colon \mathbf{C} \to \mathbf{C}$ in a generic category $\mathbf{C}$. For an overview of this theory, with many references and examples, see for example [28, 40, 39].

2.1. Algebras, Congruences and Induction

Let $\mathbf{C}$ be a category and $F\colon \mathbf{C} \to \mathbf{C}$ be a functor. An *F-algebra* is a pair (X, α) consisting of an object X, called *carrier*, and an arrow $\alpha\colon FX \to X$ in $\mathbf{C}$, called *algebra structure*. An *homomorphism* between two F-algebras (X, α) and (Y, β) is an arrow $f\colon X \to Y$ in $\mathbf{C}$ such that $f \circ \alpha = \beta \circ Ff$. F-algebras and homomorphisms between them form a category, denoted by F-**alg**.

An *F-congruence* (R, f, g) between two F-algebras (X, α) and (Y, β) is a monic span[1] between X and Y such that there exists a (necessarily unique) algebra structure $\gamma\colon FR \to R$ making f and g homomorphisms of F-algebras.

An *initial F-algebra* is an initial object in F-**alg**. The existence of an initial F-algebra (A, α) means that, for any F-algebra (Y, β) there exists a *unique* F-homomorphism of algebras f from (A, α) to (Y, β).

$$\begin{tikzcd} FA \arrow[r, "\alpha"] \arrow[d, "Ff"'] & A \arrow[d, dashed, "f"] \\ FY \arrow[r, "\beta"'] & Y \end{tikzcd} \qquad (F\text{-iteration})$$

The principle of defining arrows from the carrier of an initial F-algebra by providing another F-algebra is called *induction* (or *F-iteration*) *proof principle.*

Initial algebras may not exist, but if $\mathbf{C}$ has an initial object $\mathbf{0}$, and the forgetful functor $U^F\colon F\text{-}\mathbf{alg} \to \mathbf{C}$ mapping F-algebras to their carriers has a left adjoint $L^F\colon \mathbf{C} \to F\text{-}\mathbf{alg}$, mapping objects to the free F-algebras over them, then $L^F\mathbf{0}$ is the initial F-algebra. More in general, the adjunction $L^F \dashv U^F$ induces a monad $T^F = U^F L^F$, called the *monad freely generated by* F, and the induction proof principle is extended as:

Definition 2.1 (Structural induction). *Let (T^F, η^F, μ^F) be the monad freely generated by F. Then, for any object X, arrow $v\colon X \to Y$, and F-algebra (Y, β), there exists a unique arrow $f\colon T^F X \to Y$, called the* (free) inductive extension

[1] A *span* is given by a pair of arrows $f\colon R \to X$, $g\colon R \to Y$ sharing the same source object. A span is monic when, for all pair of morphisms $h, k\colon Z \to R$, it holds that $f \circ h = f \circ k$ and $g \circ h = g \circ k$ imply $h = k$.

of β along (the valuation) v*, making the following diagrams commute*

$$\begin{tikzcd}
X \arrow[r, "\eta^F_X"] \arrow[dr, bend right, "v"'] & T^F X \arrow[d, dashed, "f"'] & FT^F X \arrow[l, "\psi^F_X"'] \arrow[d, "Ff"] \\
& Y & FY \arrow[l, "\beta"']
\end{tikzcd}$$

where $\psi^F_X \colon FT^F X \to T^F X$ *is the free* F*-algebra structure over* X*.*

2.2. Coalgebras, Bisimulations, Cocongruences, and Coinduction

Let $\mathbf{C}$ be a category and $F \colon \mathbf{C} \to \mathbf{C}$ be a functor. An F*-coalgebra* is a pair (X, α) consisting of an object X, called *carrier*, and an arrow $\alpha \colon X \to FX$ in $\mathbf{C}$, called *coalgebra structure*. An *homomorphism* between two F-coalgebras (X, α) and (Y, β) is an arrow $f \colon X \to Y$ in $\mathbf{C}$ such that $Ff \circ \alpha = \beta \circ f$. F-coalgebras and homomorphisms between them form a category, denoted by F-**coalg**.

An F*-bisimulation* (R, f, g) between two F-coalgebras (X, α) and (Y, β) is a monic span between X and Y such that there exists a coalgebra structure $\gamma \colon R \to FR$ making f and g homomorphisms of F-coalgebras. Two F-coalgebras are *bisimilar* if there is a bisimulation between them. For coalgebras on a concrete category, two elements $x \in X$ and $y \in Y$ are bisimilar if there exists $R \subseteq X \times Y$ such that the (R, π_X, π_Y) is a bisimulation. Functors F that preserve weak pullbacks are particularly well behaved. For such functors, unions and relational compositions of F-bisimulations yield F-bisimulations again. As a consequence, bisimilarity is an equivalence relation.

A notion alternative to bisimilarity that has been proven very useful in reasoning about probabilistic systems [17], is *behavioral equivalence*, and it is based on the notion of cocongruence. An F*-cocongruence* (K, f, g) between F-coalgebras (X, α) and (Y, β) is an epic cospan[2] between X and Y such that there exists a (necessarily unique) coalgebra structure $\kappa \colon K \to FK$ on K making the following diagram commute

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\alpha"'] & K \arrow[d, dashed, "\kappa"] & Y \arrow[l, "g"'] \arrow[d, "\beta"] \\
FX \arrow[r, "Ff"'] & FK & FY \arrow[l, "Fg"]
\end{tikzcd}$$

Two F-coalgebras are *behaviorally equivalent* if they are related by a cocongruence. For coalgebras on a concrete category, we say that $x \in X$ and $y \in Y$ are

[2] A *cospan* is given by a pair of arrows $f \colon X \to K$, $g \colon Y \to K$ sharing the same target object. A cospan is epics when, for all pair of morphisms $h, k \colon K \to Z$, it holds that $h \circ f = k \circ f$ and $h \circ g = k \circ g$ imply $h = k$.

behaviorally equivalent, written $x \approx y$, if they are identified by some cocongruence (K, f, g) between them, i.e., if $f(x) = g(y)$. Bisimilarity always implies behavioral equivalence in categories with pushouts, and the two notions coincide if the behavior functor F preserves weak pullbacks. The use of cospans in cocongruences has advantages over spans, since behavioral equivalence is an equivalence even if the functor F does not preserve weak pullbacks (differently from bisimilarity).

A *final F-coalgebra* is a terminal object in F-**coalg**, i.e., an F-coalgebra such that for any F-coalgebra (X, α) there exists a unique F-homomorphism from (X, α) to it. Final F-coalgebras may or may not exist, but when they exist, they induce a *coinduction* (or *F-coiteration*) *proof principle*, a concept dual to that of induction in the case of algebras.

2.3. Initial Algebras and Final Coalgebras

We recall the definition of *initial* and *final sequences* for an endofunctor. These structures, which are dual to each other, were first given by Barr [8] and then used by other authors to provide sufficient conditions for a functor to admit an initial algebra or a final coalgebra (see Barr [9], Adámek [5, 4, 3, 2], Smyth and Plotkin [41], and Worrell [50, 49]).

We assume the reader to have elementary knowledge about ordinal numbers and transfinite induction. Ordinal numbers will be ranged over by $\alpha, \beta, \gamma, \kappa, \ldots$ and the class of ordinal numbers will be denoted by **Ord**. By a little abuse of notation, **Ord** will also denote the category of ordinal numbers with arrows $\alpha \to \beta$ iff $\alpha \leq \beta$, and by α we will also denote the full sub-category of **Ord** of all ordinal numbers less or equal than α. For a functor $F \colon \mathbf{Ord} \to \mathbf{C}$, we define the functor $F{\restriction}\alpha \colon \alpha \to \mathbf{C}$ as the composite $\iota \circ F$, where $\iota \colon \alpha \hookrightarrow \mathbf{Ord}$ is the inclusion functor.

Definition 2.2. *Let $F \colon \mathbf{C} \to \mathbf{C}$ be a functor on a category with initial object $\mathbf{0}$ and colimits of ordinal-indexed diagrams. The* initial sequence *of F is a colimit-preserving functor $A \colon \mathbf{Ord} \to \mathbf{C}$ such that, for all ordinals $\gamma \leq \beta$,*

i. $A(0) = \mathbf{0}$;
ii. $A(\beta{+}1) = FA(\beta)$;
iii. $A(\gamma{+}1 \to \beta{+}1) = FA(\gamma \to \beta)$.

The initial sequence stabilizes *at α, if $A(\alpha \to \alpha{+}1)$ is an isomorphism*[3]*.*

Note that, for all limit ordinals β, $(\gamma \to \beta)_{\gamma<\beta}$ is a colimit in **Ord**, and since A preserves colimits, $(A(\gamma \to \beta))_{\gamma<\beta}$ is a colimit in $\mathbf{C}$ for the diagram $A{\restriction}\alpha$. The dual of Definition 2.2 yields to the notion of final sequence.

[3] By definition of initial sequence and since functors preserve isomorphisms, if for some α, $A(\alpha \to \alpha{+}1)$ is an isomorphism, then for all ordinals $\beta \geq \gamma \geq \alpha$, $A(\gamma \to \beta)$ is an isomorphism [9]. This clarifies why the existence of an isomorphism is referred to as stabilization.

Definition 2.3. *Let $F\colon \mathbf{C} \to \mathbf{C}$ be a functor on a category with final object* $\mathbf{1}$ *and limits of ordinal-indexed diagrams. The* final sequence of T *is a limit-preserving functor $Z\colon \mathbf{Ord}^{op} \to \mathbf{C}$ such that, for all ordinals $\gamma \leq \beta$,*

i. $Z(0) = \mathbf{1}$;
ii. $Z(\beta{+}1) = TF(\beta)$;
iii. $Z(\beta{+}1 \to \gamma{+}1) = TF(\beta \to \gamma)$.

The final sequence stabilizes *at α, if $Z(\alpha{+}1 \to \alpha)$ is an isomorphism.*

Initial and final sequences give sufficient conditions for the existence of initial algebras and final coalgebras.

Theorem 2.4 (see Barr [8]). *Let $F\colon \mathbf{C} \to \mathbf{C}$ be a functor and assume A and Z be the initial and final sequence of F, respectively. Then,*

i. if A stabilizes at α, then $(A(\alpha), A(\alpha \to \alpha{+}1)^{-1})$ is an initial F-algebra;
ii. if Z stabilizes at α, then $(Z(\alpha), Z(\alpha{+}1 \to \alpha)^{-1})$ is a final F-algebra.

By Theorem 2.4, initial F-algebras and final F-coalgebras may be obtained looking for an isomorphism in the initial and final sequence of F, respectively.

If the functor F is κ-(co)continuous, for some limit ordinal κ, i.e., F preserves (co)limits for any κ-indexed diagram, then the final (resp. initial) sequence stabilizes (see [4]). However, (co)continuity for the endofunctor is too strong and may not hold.

An alternative strategy consists in relaxing the hypotheses on the functor by exploiting the structure of the underlying category instead. For instance, in [50, 49], Worrell proved that final sequences for κ-accessible endofunctors on locally presentable categories always stabilize at $\kappa + \kappa$ steps, for some regular ordinal κ. Differently from Worrell, here we combine the use of initial/final sequences and the existence of certain well-behaved factorization systems on the category and give new stabilization theorems for a wider class of categories then just the locally presentable ones (see also Remark 2.8).

Definition 2.5. *An* (orthogonal) factorization system *on a category* $\mathbf{C}$ *is a pair $(\mathcal{L}, \mathcal{R})$ of classes of $\mathbf{C}$-morphisms such that*

– *$\mathcal{L}$ and $\mathcal{R}$ are closed under composition with isomorphisms;*
– *every morphism f in $\mathbf{C}$ factors as $f = m \circ e$, for some $m \in \mathcal{R}$ and $e \in \mathcal{L}$;*
– *each commutative square $m \circ f = g \circ e$, where $m \in \mathcal{R}$ and $e \in \mathcal{L}$, admits a unique* diagonal *d such that $d \circ e = f$ and $m \circ d = g$.*

$$\begin{tikzcd} A \arrow[r, "e \in \mathcal{L}"] \arrow[d, "f"'] & B \arrow[d, "g"] \arrow[dl, dashed, "d"'] \\ C \arrow[r, "m \in \mathcal{R}"'] & D \end{tikzcd}$$

Factorization systems enjoy several properties, such as, uniqueness of the factorization up to isomorphism, closure under composition for $\mathcal{L}$ and $\mathcal{R}$, and isomorphisms are precisely those arrows that are both in $\mathcal{L}$ and $\mathcal{R}$. The classes $\mathcal{L}$ and $\mathcal{R}$ enjoy also useful cancellation properties:

– (Left-cancellation) if $f \circ g \in \mathcal{L}$ and $g \in \mathcal{L}$, then $f \in \mathcal{L}$;

– (Right-cancellation) if $f \circ g \in \mathcal{R}$ and $f \in \mathcal{R}$, then $g \in \mathcal{R}$.

Moreover, the classes $\mathcal{L}$ and $\mathcal{R}$ are respectively closed under limits and colimits over ordinal-indexed diagrams, i.e., given $F\colon \mathbf{Ord} \to \mathbf{C}$ and $G\colon \mathbf{Ord}^{op} \to \mathbf{C}$, respectively, colimit and limit preserving, and ordinal α, the following hold:

– if for all ordinals $\lambda \leq \beta < \alpha$, $F(\gamma \to \beta) \in \mathcal{L}$, then $F(\gamma \to \alpha) \in \mathcal{L}$;
– if for all ordinals $\lambda \leq \beta < \alpha$, $G(\beta \to \gamma) \in \mathcal{R}$, then $G(\alpha \to \gamma) \in \mathcal{R}$.

Notably, these properties generalize in a purely axiomatic way the concepts of subobject and quotient which are replaced by $\mathcal{R}$-morphism and $\mathcal{L}$-morphism, respectively. It is easy to see that the collection of $\mathcal{R}$-morphisms with codomain X is a partial order under the relation $f \leq g$ iff f factors through g (i.e., there exists an arrow h such that $f = g \circ h$). In general, the collection of $\mathcal{R}$-morphism of an object X may in fact be a proper class, but when for every object X the collection of its $\mathcal{R}$-morphisms is a proper set, we say that the category is *$\mathcal{R}$-well-powered*. These concepts dualize to collections of $\mathcal{L}$-morphisms with a common shared domain.

The following lemma uses the above properties of a factorization systems to ensure stabilization for the final sequence.

Lemma 2.6. *Let $Z\colon \mathbf{Ord}^{op} \to \mathbf{C}$ be the final sequence of an endofunctor F on a category $\mathbf{C}$ with factorization system $(\mathcal{L}, \mathcal{R})$, such that $\mathbf{C}$ is $\mathcal{R}$-well-powered.If F preserves $\mathcal{R}$-morphisms and, for some κ, $Z(\kappa{+}1 \to \kappa) \in \mathcal{R}$, then Z stabilizes.*

Proof. By [8, Proposition 1.1], if for some ordinals $\beta > \gamma$, $Z(\beta \to \gamma)$ is an isomorphism, then it is so $Z(\beta{+}1 \to \beta)$. Since $\mathbf{C}$ is $\mathcal{R}$-well-powered, to prove that $\beta > \gamma$ such as above exist, it suffices to show that for all ordinals $\alpha \geq \kappa$, $Z(\alpha \to \kappa) \in \mathcal{R}$. We proceed by transfinite induction on $\alpha \geq \kappa$. Base case: if $\alpha = \kappa$ then $Z(\alpha \to \kappa) = Z(\kappa \to \kappa) = id_{Z(\kappa)} \in \mathcal{R}$. Inductive step: assume, by inductive hypothesis that $Z(\alpha \to \kappa) \in \mathcal{R}$. Then the following holds

$$\begin{aligned} Z(\alpha{+}1 \to \kappa) &= Z(\kappa{+}1 \to \kappa) \circ Z(\alpha{+}1 \to \kappa{+}1) && \text{(by func. } Z\text{)} \\ &= Z(\kappa{+}1 \to \kappa) \circ FZ(\alpha \to \kappa) && \text{(by def. } Z\text{)} \end{aligned}$$

Since $Z(\kappa{+}1 \to \kappa) \in \mathcal{R}$ and F preserves $\mathcal{R}$-morphisms, $Z(\alpha{+}1 \to \kappa) \in \mathcal{R}$. Limit step: assume α is a limit ordinal and, by inductive hypothesis, that for all β such that $\kappa \leq \beta < \alpha$, the arrows $Z(\beta \to \kappa)$ are in $\mathcal{R}$. Since, the class $\mathcal{R}$ is closed under colimits over ordinal-indexed diagrams, to prove $Z(\alpha \to \kappa) \in \mathcal{R}$ it suffices to show that, for all ordinals δ such that $\kappa \leq \delta \leq \beta$, $Z(\beta \to \delta) \in \mathcal{R}$. We do this by transfinite induction on δ. The base case $\delta = \kappa$ follows by the inductive hypothesis on α. For the inductive step, assume that $Z(\beta \to \delta) \in \mathcal{R}$. We have

$$\begin{aligned} Z(\delta{+}1 \to \kappa) \circ Z(\beta \to \delta{+}1) &= Z(\beta \to \kappa) \circ Z(\beta{+}1 \to \delta{+}1) && \text{(by func. } Z\text{)} \\ &= Z(\beta \to \kappa) \circ FZ(\beta \to \delta) && \text{(by def. } Z\text{)} \end{aligned}$$

By inductive hypothesis on δ, $Z(\beta \to \delta) \in \mathcal{R}$, and since F preserves $\mathcal{R}$-morphisms, $FZ(\beta \to \delta) \in \mathcal{R}$. By inductive hypothesis on α, both $Z(\delta{+}1 \to \kappa)$

and $Z(\beta \to \kappa)$ are $\mathcal{R}$-morphisms, hence $Z(\beta \to \delta{+}1) \in \mathcal{R}$. Assume δ be a limit ordinal and, by inductive hypothesis, that for all γ such that $\kappa \leq \gamma < \delta$, the arrows $Z(\beta \to \gamma)$ are in $\mathcal{R}$. By functoriality of Z we have

$$Z(\delta \to \kappa) = Z(\gamma \to \kappa) \circ Z(\delta \to \gamma)$$

By inductive hypothesis on α, $Z(\delta \to \kappa)$ and $Z(\gamma \to \kappa)$ are in $\mathcal{R}$, hence, by the left-cancellation property, $Z(\delta \to \gamma) \in \mathcal{R}$. From this and by left-cancellation, also $Z(\beta \to \delta) \in \mathcal{R}$. Indeed, by functoriality, $Z(\beta \to \gamma) = Z(\delta \to \gamma) \circ Z(\beta \to \delta)$, and $Z(\beta \to \gamma) \in \mathcal{R}$ by inductive hypothesis on β. $\square$

Clearly, also the dual statement holds.

Lemma 2.7. *Let $A\colon \mathbf{Ord} \to \mathbf{C}$ be the initial sequence of an endofunctor F on a category $\mathbf{C}$ with factorization system $(\mathcal{L}, \mathcal{R})$, such that $\mathbf{C}$ is $\mathcal{L}$-cowell-powered. If F preserves $\mathcal{L}$-morphisms and, for some κ, $A(\kappa \to \kappa{+}1) \in \mathcal{R}$, then A stabilizes.*

Remark 2.8. Although the combination of Lemma 2.6 (resp. Lemma 2.7) and Theorem 2.4 ensures the existence of a final coalgebra (resp. initial algebra), we have no bounds on the ordinal at which the final (resp. initial) sequence stabilizes, differently from e.g. [50, 49]. However, those previous approaches require the category to be locally presentable and the functor to be accessible. Our constructions do not assume these properties, and hence they can be applied in categories such as **Top** (the category of topological spaces and continuous maps) and **Meas** which are not locally presentable.

3. Stochastic Labelled Markov Processes, Coalgebraically

We recall stochastic labelled Markov processes as introduced in [16], that is, dynamical systems with continuous state space, interacting with the environment by means of input labels and producing measurable events by means of transitions to a measurable set of successor states. The term "stochastic" is used to stress the fact that transition events can be measured by finite measures on the state space without assuming *a priori* that these are of (sub)probability.

We begin this section with some reminders about measure theory.

3.1. Measure Theoretic Preliminaries

A σ-algebra over a set X is a non-empty family Σ_X of subsets of X closed under complements and countable unions. The pair (X, Σ_X) is called *measurable space* and the members of Σ_X are its *measurable sets*. A *generator* $\mathcal{F}$ for Σ_X is a family of subsets of X such that the smallest σ-algebra containing $\mathcal{F}$ is Σ_X, denoted by $\sigma(\mathcal{F}) = \Sigma_X$.

Let (X, Σ_X), (Y, Σ_Y) be measurable spaces, a function $f\colon X \to Y$ is called *measurable* if $f^{-1}(E) = \{x \mid f(x) \in E\} \in \Sigma_X$, for all $E \in \Sigma_Y$ (notably, if Σ_Y is generated by $\mathcal{F}$, it suffices to show that $f^{-1}(F) \in \Sigma_X$, for all $F \in \mathcal{F}$).

Given a measurable space (X, Σ_X) and a set function $f\colon X \to Y$, then the family $\{E \subseteq Y \mid f^{-1}(E) \in \Sigma_X\}$, called the *final σ-algebra w.r.t. f*, is the largest σ-algebra over Y that renders f measurable. Dually, given $g\colon Y \to X$, the family $\{g^{-1}(E) \mid E \in \Sigma_X\}$ is the *initial σ-algebra w.r.t g*, and it is the smallest σ-algebra over Y that making g measurable. Initial and final σ-algebras generalize to families of maps in the obvious way.

A *measure* on (X, Σ_X) is a σ-additive function $\mu\colon \Sigma_X \to [0,\infty]$, that is, $\mu(\bigcup_{i\in I} E_i) = \sum_{i\in I} \mu(E_i)$ for all countable collections $\{E_i\}_{i\in I}$ of pairwise disjoint measurable sets. A measure $\mu\colon \Sigma_X \to [0,\infty]$ is of *(sub)-probability* if $\mu(X) = 1$ (resp. ≤ 1), is *finite* if $\mu(X) < \infty$, and is *σ-finite* if there exists a countable cover $\{E_i\}_{i\in I}$ of X, i.e., $\bigcup_{i\in I} E_i = X$, of measurable sets such that $\mu(E_i) \leq \infty$, for all $i \in I$.

Let (X, Σ_X) and (Y, Σ_Y) be measurable spaces, and μ and ν be σ-finite measures on these spaces. Denote by $\Sigma_X \otimes \Sigma_Y$ the σ-algebra on the Cartesian product $X \times Y$ generated by subsets of the form $E \times F$, said rectangles, where $E \in \Sigma_X$ and $F \in \Sigma_Y$. The *product measure* $\mu \times \nu$ is defined to be the unique measure on the *product measurable space* $(X \times Y, \Sigma_X \otimes \Sigma_Y)$ such that

$$(\mu \otimes \nu)(E \times F) = \mu(E) \cdot \nu(F)\,, \qquad \text{for all } E \in \Sigma_X \text{ and } F \in \Sigma_Y.$$

The existence of this measure is guaranteed by the Hahn-Kolmogorov theorem; uniqueness is ensured only when both μ and ν are σ-finite.

3.2. Coalgebraic Treatment of Continuous Stochastic Systems

Let (X, Σ) be a measurable space and $\Delta(X, \Sigma)$ be the set of all finite measures $\mu\colon \Sigma \to [0,\infty)$ on (X, Σ). For each measurable set $E \in \Sigma$, there is a canonical evaluation function $ev_E\colon \Delta(X,\Sigma) \to [0,\infty)$, defined by $ev_E(\mu) = \mu(E)$, for each measure $\mu \in \Delta(X, \Sigma)$, and called *evaluation at E*. By means of these evaluation maps, $\Delta(X, \Sigma)$ can be organized into a measurable space $(\Delta(X,\Sigma), \Sigma_{\Delta(X,\Sigma)})$, where $\Sigma_{\Delta(X,\Sigma)}$ the initial σ-algebra w.r.t. $\{ev_E \mid E \in \Sigma\}$, i.e., the smallest σ-algebra making ev_E measurable with respect to the Borel σ-algebra on $[0,\infty)$, for all $E \in \Sigma$. Note that the σ-algebra $\Sigma_{\Delta(X,\Sigma)}$ can be also generated by the collection $\{L_r(E) \mid r \in \mathbb{Q} \cap [0,\infty), E \in \Sigma\}$ where $L_r(E) = \{\mu \in \Delta(X,\Sigma) \mid \mu(E) \geq r\} = ev_E^{-1}([r,\infty))$ (see Lemma A.3).

Definition 3.1. *Let (X, Σ) be a measurable space and L be a set of action labels. A stochastic L-labelled* Markov kernel *is a tuple $\mathcal{M} = (X, \Sigma, \{\theta_a\}_{a\in L})$ where, for all $a \in L$*

$$\theta_a\colon X \to \Delta(X, \Sigma)$$

is a measurable function, called Markov a-transition function*. A stochastic L-labelled Markov kernel $\mathcal{M}$ with a distinguished initial state $x \in X$, is called a* Markov process*, and it is denoted by $(\mathcal{M}, x)$.*

The adjective "Markovian" is usually employed in the probabilistic setting; here it just indicates that the transitions depend entirely on the present state and not on the past history of the system. The labels in L represent all possible interactions of processes with the environment: for any $a \in L$, $x \in X$, and $E \in \Sigma$, the value $\theta_a(x)(E)$ represents the *rate* of an exponentially distributed random variable characterizing the duration of an a-transition from the current state x to arbitrary elements in E.

Let **Meas** be the category of measurable spaces and measurable functions. It is easy to see that the definition of the space of finite measures provided above is functorial. Formally, we define the functor $\Delta\colon \mathbf{Meas} \to \mathbf{Meas}$ acting on objects (X, Σ_X) and arrows $f\colon (X, \Sigma_X) \to (Y, \Sigma_Y)$ as

$$\Delta(X, \Sigma_X) = (\Delta(X, \Sigma_X), \Sigma_{\Delta(X,\Sigma_X)})\,, \qquad \Delta(f)(\mu) = \mu \circ f^{-1}\,.$$

This functor is commonly known as the *Giry functor*, in relation to the fact that is takes part to the so-called Giry (probabilistic) monad [25].

It is standard that L-Markov kernels can be modeled as Δ^L-coalgebras, where $(\cdot)^L\colon \mathbf{Meas} \to \mathbf{Meas}$ is the exponential functor, associating with each measurable space (X, Σ_X) the space of functions from L to X and endowed with the least σ-algebra making all evaluation maps at $a \in L$, $ev_a\colon X^L \to X$, measurable.

Proposition 3.2. *Stochastic L-Markov kernels are exactly the Δ^L-coalgebras.*

Proof. The correspondence is trivial. Given a stochastic L-labelled Markov kernel $\mathcal{M} = (X, \Sigma_X, \{\theta_a\}_{a\in L})$ we define a Δ^L-coalgebra on (X, Σ_X) with a coalgebra structure $\alpha\colon X \to \Delta^L X$ given by $\alpha(a) = \theta_a$, for all $a \in L$. A map $\alpha\colon X \to \Delta^L X$ is measurable iff $ev_E \circ ev_a \circ \alpha$ is measurable, for all $a \in L$ and $E \in \Sigma_X$. But $ev_a \circ \alpha = \theta_a$ is measurable by definition of Markov kernel, hence α is so. Conversely, given an Δ^L-coalgebra (X, α), we define a Markov kernel $\mathcal{M} = (X, \Sigma_X, \{\theta_a\}_{a\in L})$ as follows, where for all $a \in L$, $\theta_a = ev_a \circ \alpha = \alpha(\cdot)(a)$. Measurability of θ_a follows since it is the composite of measurable functions. It is immediate to see that the two translations are inverses of each other. □

For different reasons in most of the works about (probabilistic) Markov processes different categories were considered. In [19], de Vink and Rutten used ultrametric spaces arguing that the main reason for doing so was reusing a theorem that guarantees existence of a final coalgebra for locally contractive functors. Another reason for avoiding **Meas** is that the Giry functor does not preserve weak-pullbacks [48], which would be desirable for bisimilarity to be well-behaved. For instance, Desharnais, Edalat et al. [20] moved to the categories of *analytic spaces* in order to construct semi-pullback, providing a way to show that bisimilarity is transitive.

In this work we remain in **Meas** and argue that for the coalgebraic treatment of stochastic Markov processes it suffices to work with general measurable spaces unless one prefers bisimilarity to behavioral equivalence (i.e., the relation given by pullbacks on final coalgebra homomorphims). This choice was already discussed by several authors, who have observed that in the generalized probabilistic setting, behavioral equivalence is more sensible than bisimilarity [17, 11] and have suggested the use of cocongruence (also called event bisimulation) instead of the standard bisimulation. Similar arguments have been discussed with more generality by Kurz in his thesis [32], and by Staton [42] where four different notions of bisimulation were investigated.

3.3. Final Coalgebra for $\Delta\colon \mathbf{Meas} \to \mathbf{Meas}$

In this section, we prove the existence of final coalgebras for the functor $\Delta\colon \mathbf{Meas} \to \mathbf{Meas}$. To this end, we apply Lemma 2.6 considering the final sequence of Δ and noticing that $(\mathbf{Epic}, \mathbf{Emb})$ is a factorization system in **Meas**, where **Epic** denotes the class of epic morphisms (i.e., surjective measurable functions) and **Emb** is the class of measurable embeddings, that is injective measurable maps $f\colon (X, \Sigma_X) \to (Y, \Sigma_Y)$ such that Σ_X is initial w.r.t. f.

Since **Meas** has terminal object and is complete, the final sequence of Δ is well-defined. The following proposition shows that the final sequence of Δ reaches an embedding after ω steps, where ω denotes the first limit ordinal.

Proposition 3.3. *Let* $Z\colon \mathbf{Ord}^{op} \to \mathbf{Meas}$ *be the final sequence of* Δ*. Then* $Z(\omega{+}1 \to \omega)$ *is an embedding.*

The proof requires some technical lemmata, and can be found in Appendix.

As noticed in [29], Δ preserves embeddings, so that, by Lemma 2.6 and Proposition 3.3, it follows that Δ has a final coalgebra.

Theorem 3.4. *The functor* $\Delta\colon \mathbf{Meas} \to \mathbf{Meas}$ *has final coalgebra.*

Proof. **Meas** has terminal object and limits for ordinal indexed diagrams, thus the final sequence $Z\colon \mathbf{Ord}^{op} \to \mathbf{Meas}$ of Δ is well-defined. $(\mathbf{Epic}, \mathbf{Emb})$ is a factorization system in **Meas**, moreover, by $\mathbf{Emb} \subseteq \mathbf{Monic}$ and the fact that **Meas** is well-powered, it follows that it is also **Emb**-well-powered. By Proposition 3.3, $Z(\omega{+}1 \to \omega) \in \mathbf{Emb}$ so that, by Lemma 2.6, the final sequence stabilizes. The existence of Δ-coalgebra follows by Theorem 2.4. □

Remark 3.5. Proposition 3.3 extends also to the composite functor Δ^L, for any set L, therefore there exits a final stochastic L-labelled Markov kernel.

In [47], van Breugel et al. considered the functor $\Delta_{\leq 1}^L$ of subprobabilities over a measurable space. The construction of the final coalgebra is done using the connection between the final sequences of $\Delta_{\leq 1}^L$ and the one for the labelled probabilistic powerdomain functor in the category of ω-coherent domains. Although their approach gives better insights on the carrier of the final coalgebra, the entire construction is rather complicated and needs one to jump from a category to another. Viglizzo and Moss [36] used modal logics for proving the existence of final coalgebras for the Giry functor. Viglizzo [48] also provided another construction avoiding the logic in favor of final sequences.

4. Measurable Spaces of Terms

The development of a structural operational semantics over stochastic Markov processes demands for an algebraic description of their measurable space states. This can be formalized by means of *measurable signatures.*

Definition 4.1. *A* measurable signature *is a triple* $(\mathcal{S}, ar, \{(X_s, \Sigma_s)\}_{s\in\mathcal{S}})$*, where* $\mathcal{S}$ *is a set of* operator symbols*,* $ar\colon \mathcal{S} \to \mathbb{N}$ *is an arity function, and, for* $s \in \mathcal{S}$*,*

(X_s, Σ_s) *is a measurable space. An* interpretation *of* $(\mathcal{S}, ar, \{(X_s, \Sigma_s)\}_{s\in\mathcal{S}})$ *on a measurable set* (X, Σ) *is an* $\mathcal{S}$*-indexed collection of measurable functions* $(\llbracket s \rrbracket \colon X_s \times X^{ar(s)} \to X)_{s\in\mathcal{S}}$.

Differently from standard set-signatures, each operator symbol $s \in \mathcal{S}$ is associated with a measurable space (X_s, Σ_s). This, for example, allows for the definition of actual measurable spaces of constant symbols (i.e., 0-ary operators) and for the specification of nontrivial measurable spaces of terms.

Definition 4.2 (Measurable terms). *Let* $(\mathcal{S}, ar, \{(X_s, \Sigma_s)\}_{s\in\mathcal{S}})$ *be a measurable signature and* (X, Σ) *be a measurable space of variables. The measurable space of terms (freely) generated over* (X, Σ) *and* $(\mathcal{S}, ar, \{(X_s, \Sigma_s)\}_{s\in\mathcal{S}})$ *is defined as* (TX, Σ_{TX}) *where,* TX *and* Σ_{TX} *are, respectively, the smallest set and the smallest* σ*-algebra satisfying the following rules, for all* $s \in \mathcal{S}$

$$\frac{x \in X}{x \in TX} \qquad\qquad \frac{t_1, \dots, t_{ar(s)} \in TX \qquad k \in X_s}{s(k, t_1, \dots, t_{ar(s)}) \in TX}$$

$$\frac{E \in \Sigma}{E \in \Sigma_{TX}} \qquad\qquad \frac{T_1, \dots, T_{ar(s)} \in \Sigma_{TX} \qquad K \in \Sigma_s}{s(K, T_1, \dots, T_{ar(s)}) \in \Sigma_{TX}}$$

where $s(K, T_1, \dots, T_{ar(s)}) = \{s(k, t_1, \dots, t_{ar(s)}) \mid k \in K, t_i \in T_i\}$.

In the definition above the number of arguments for an operator symbol $s \in \mathcal{S}$ is augmented by one in the term $s(k, t_1, \dots, t_{ar(s)})$. This notation is particularly convenient for discriminating between measurable terms which differ only on the σ-algebra structure (a situation that does not happen in standard set-signatures). For example, assume the measurable spaces $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and $(\mathbb{R}, \{\emptyset, \mathbb{R}\})$ are associated with constant operators c and c', respectively, where $\mathcal{B}(\mathbb{R})$ denotes the Borel σ-algebra on $\mathbb{R}$. If we had considered $r \in \mathbb{R}$ as a term, it would have been impossible to discriminate between the elements in c or c'. By adopting the notation above these problems are overcome, since constants are denoted either by $c(r)$ or $c'(r)$.

4.1. Syntactic Endofunctors and Signature Interpretations as Algebras

Algebras are typically used to give an abstract categorical formalization to denotational semantics. Now we show how measurable signatures and their interpretations can be modeled as algebras for certain functors in **Meas**. Although this technique is standard in **Set**, the σ-algebra structures endowed with the objects in **Meas** makes the construction less easy to be treated.

For a measurable signature $(\mathcal{S}, ar, \{(X_s, \Sigma_s)\}_{s\in\mathcal{S}})$, the *syntactic endofunctor* in **Meas** associated with it is given by $S = \coprod_{s\in\mathcal{S}} (X_s, \Sigma_s) \times Id^{ar(s)}$. Explicitly, S acts on objects (X, Σ) and arrows $f \colon (X, \Sigma_X) \to (Y, \Sigma_Y)$ as follows

$$S(X, \Sigma) = (\{\langle s, (k, x_1, \dots, x_{ar(\sigma)})\rangle \mid s \in \mathcal{S}, k \in X_s,\, x_1, \dots, x_{ar(\sigma)} \in X\}, \Sigma_{S(X,\Sigma)}),$$
$$Sf = \left[\langle s, (k, x_1, \dots, x_{ar(\sigma)})\rangle \mapsto \langle s, (k, f(x_1), \dots, f(x_{ar(\sigma)}))\rangle\right],$$

where $\langle s,(k,x_1,\ldots,x_{ar(\sigma)})\rangle$ denotes $in_s^X(k,x_1,\ldots,x_{ar(\sigma)})$, and $\Sigma_{S(X,\Sigma)}$ is the final σ-algebra w.r.t. the injections $in_s^X\colon X_s \times X^{ar(s)} \to SX$, for $s \in \mathcal{S}$:

$$\Sigma_{S(X,\Sigma)} = \bigcap_{s\in\mathcal{S}} \{A \subseteq SX \mid (in_s^X)^{-1}(A) \in \Sigma_{X_s \times X^{ar(s)}}\}\,. \tag{1}$$

The σ-algebra $\Sigma_{S(X,\Sigma)}$ can be characterized in a more convenient way by means of a generating family of sets having a structure that is simpler to handle than the subsets $A \subseteq SX$ occurring in Eq. (1).

Proposition 4.3. *Let $(\mathcal{S}, ar, \{(X_s,\Sigma_s)\}_{s\in\mathcal{S}})$ be a measurable signature and S be the syntactic* **Meas***-endofunctor associated with it. Then, for any measurable space (X,Σ), the σ-algebra of $S(X,\Sigma)$ is generated by*

$$\{\textstyle\bigcup_{s\in\mathcal{S}} \langle s,(K_s,E_1,\ldots,E_{ar(s)})\rangle \mid \forall s\in\mathcal{S}.\, K_s \in \Sigma_s \text{ and } E_i \in \Sigma\}\,.$$

where $\langle s,(K_s,E_1,\ldots,E_{ar(s)})\rangle = \{\langle s,(k,x_1,\ldots,x_{ar(s)})\rangle \mid k \in K_s$ and $x_i \in E_i\}$.

Proof. Let first notice that, for each $s \in \mathcal{S}$, the σ-algebra $\Sigma_{X_s\times X^{ar(s)}}$ is generated by $\mathcal{R}_s = \{K \times E_1 \times \cdots \times E_{ar(s)} \mid K \in \Sigma_s, E_i \in \Sigma\}$, that is, the family of measurable rectangles. So, we have that

$$\begin{aligned}
\Sigma_{S(X,\Sigma)} &= \textstyle\bigcap_{s\in\mathcal{S}} \{A \subseteq SX \mid (in_s^X)^{-1}(A) \in \Sigma_{X_s\times X^{ar(s)}}\} \\
&= \textstyle\bigcap_{s\in\mathcal{S}} \{A \subseteq SX \mid (in_s^X)^{-1}(A) \in \sigma(\mathcal{R}_s)\} \\
&= \textstyle\bigcap_{s\in\mathcal{S}} \sigma\big(\{A \subseteq SX \mid (in_s^X)^{-1}(A) \in \mathcal{R}_s\}\big) && \text{(by Prop. A.2)} \\
&= \sigma\big(\{\textstyle\bigcup_{s\in\mathcal{S}} \langle s,(K_s,E_1,\ldots,E_{ar(s)})\rangle \mid \forall s\in\mathcal{S}.\, K_s\in\Sigma_s \text{ and } E_i \in \Sigma\}\big)\,,
\end{aligned}$$

where the last equality holds since each $A \subseteq SX$ such that $(in_s^X)^{-1}(A) \in \mathcal{R}_s$ can always be given as a disjoint union of the form $\bigcup_{s\in\mathcal{S}} \langle s,(K_s,E_1,\ldots,E_{ar(s)})\rangle$ for some $K_s \in \Sigma_s$ and $E_i \in \Sigma$, for $1 \leq i \leq ar(s)$. $\square$

Remark 4.4. In case the measurable signature $(\mathcal{S}, ar, \{(X_s,\Sigma_s)\}_{s\in\mathcal{S}})$ is assumed to have (only) a countable set $\mathcal{S}$ of operator symbols, the generator given in Proposition 4.3 can be simplified as follows

$$\begin{aligned}
\Sigma_{S(X,\Sigma)} &= \sigma\big(\{\textstyle\bigcup_{s\in\mathcal{S}} \langle s,(K_s,E_1,\ldots,E_{ar(s)})\rangle \mid \forall s\in\mathcal{S}.\, K_s\in\Sigma_s \text{ and } E_i \in \Sigma\}\big) \\
&= \sigma\big(\{\langle s,(K_s,E_1,\ldots,E_{ar(s)})\rangle \mid s\in\mathcal{S},\, K_s\in\Sigma_s \text{ and } E_i \in \Sigma\}\big)\,,
\end{aligned}$$

since all σ-algebras are already closed by countable unions.

The following correspondence is standard.

Proposition 4.5. *Let $(\mathcal{S}, ar, \{(X_s,\Sigma_s)\}_{s\in\mathcal{S}})$ be a measurable signature and S be the syntactic* **Meas***-endofunctor associated with it. Then, interpretations for the signature are in one-to-one correspondence with S-algebras.*

Proof. For any interpretation $(\llbracket s\rrbracket\colon X^{ar(s)} \to X)_{s\in\mathcal{S}}$ and S-algebra (X,α), the correspondence is given by

$$(X,\alpha) \mapsto (\alpha \circ in_s^X)_{s\in\mathcal{S}} \qquad (\llbracket s\rrbracket\colon X^{ar(s)} \to X)_{s\in\mathcal{S}} \mapsto (X, \textstyle\coprod_{s\in\mathcal{S}} \llbracket s\rrbracket)$$

which is clearly bijective by the universal property of coproducts. $\square$

4.2. Term Monad over Measurable Spaces

In this section we show how the measurable space of terms over a measurable signature can be elegantly modeled as the free monad over the syntactic endofunctor associated with the signature. As a consequence we obtain a principle of structural induction over measurable terms which extends the well-known principle of structural induction for terms over standard set-signatures.

To this end, for any measurable signature $(\mathcal{S}, ar, \{(X_s, \Sigma_s)\}_{s\in\mathcal{S}})$, we need to show that the forgetful functor from the category of algebras for the syntactic **Meas**-endofunctor S associated with the signature has a left adjoint. It is standard that, for endofunctors $F\colon \mathbf{C} \to \mathbf{C}$ in a category $\mathbf{C}$ with binary coproducts, the forgetful functor $U^F\colon F\text{-}\mathbf{alg} \to \mathbf{C}$ has a left adjoint $L^F\colon \mathbf{C} \to F\text{-}\mathbf{alg}$, if for any object X in $\mathbf{C}$ the functor $X + F$ has an initial algebra. Therefore, it suffices to show that for any measurable space (X, Σ) the functor $(X, \Sigma) + S$ has an initial algebra.

Theorem 4.6. *Let $(\mathcal{S}, ar, \{(X_s, \Sigma_s)\}_{s\in\mathcal{S}})$ be a measurable signature, S be the syntactic endofunctor associated with it, and (X, Σ) be a measurable set. Then $((TX, \Sigma_{TX}), [\eta_X, \psi_X])$ is the initial $((X, \Sigma) + S)$-algebra, where (TX, Σ_{TX}) is the measurable space of terms generated over (X, Σ) and $(\mathcal{S}, ar, \{(X_s, \Sigma_s)\}_{s\in\mathcal{S}})$ and $\eta_X\colon X \to TX$ and $\psi_X\colon STX \to TX$ are defined as follows*

$$\eta_X(x) = x \qquad \textit{and} \qquad \psi_X(\langle s, (k, t_1, \dots, t_{ar(s)})\rangle) = s(k, t_1 \dots t_{ar(\sigma)})\,,$$

for all $x \in X$, $s \in \mathcal{S}$, $k \in X_s$, and $t_1, \dots, t_{ar(s)} \in TX$.

Proof. By Proposition 4.3 it is immediate to see that both η_X and ψ_X are measurable, hence $[\eta_X, \psi_X]$ is a well-defined $((X, \Sigma) + S)$-algebra structure. We proceed first proving that $(X, \Sigma) + S$ has an initial algebra, then we prove that it is isomorphic to $((TX, \Sigma_{TX}), [\eta_X, \psi_X])$. Let $A\colon \mathbf{Ord} \to \mathbf{Meas}$ be the initial sequence of $(X, \Sigma) + S$, we prove that $A(\omega \to \omega{+}1)$ is an epimorphism. To this end, consider the **Set**-endofunctor $S' = \coprod_{s\in\mathcal{S}} X_s \times Id^{ar(s)}$. It is immediate to see that $(X + S')U = U((X, \Sigma) + S)$, where $U\colon \mathbf{Meas} \to \mathbf{Set}$ is the forgetful functor forgetting the σ-algebra structure of a measurable space.

We prove that $UA\colon \mathbf{Ord} \to \mathbf{Set}$ is the initial sequence for $X + S'$. Clearly, since both A and U preserves colimits, also UA preserves them. Moreover, for all ordinals $\gamma \leq \beta$ the following holds

$$UA(0) = U\mathbf{0} = \mathbf{0}$$
$$UA(\beta{+}1) = U((X, \Sigma) + S)A(\beta) = (X + S')UA(\beta)$$
$$UA(\gamma{+}1 \to \beta{+}1) = U((X, \Sigma) + S)A(\gamma \to \beta) = (X + S')UA(\gamma \to \beta)\,.$$

Therefore $UA\colon \mathbf{Ord} \to \mathbf{Set}$ is the initial sequence of $X + S'$. Recall that polynomial functors in **Set** are ω-cocontinuous, that is, preserves colimits of ω-sequences. Therefore the initial sequence UA of $X + S'$ stabilizes at ω, thus, $UA(\omega \to \omega{+}1)$ is an isomorphism and, in particular, also an epimorphism. Since U reflects epimorphisms, $A(\omega \to \omega{+}1)$ is an epic arrow in **Meas**.

Polynomial endofunctors in **Meas** preserves epimorphism, moreover **Meas** is cowell-powered and has $(\mathbf{Epic}, \mathbf{Emb})$ as a factorization system. Thus, by Lemma 2.7, A stabilizes at some ordinal $\kappa \geq \omega$ hence, $(A(\kappa), A(\kappa \to \kappa{+}1)^{-1})$ is an initial $((X, \Sigma) + S)$-algebra (Theorem 2.4).

To prove that $((TX, \Sigma_{TX}), [\eta_X, \psi_X])$ is isomorphic to $(A(\kappa), A(\kappa \to \kappa{+}1)^{-1})$, we exploit the connection between the the initial sequences A and UA. The key observation is that both $(TX, [\eta_X, \psi_X])$ and $(UA(\kappa), UA(\kappa \to \kappa{+}1)^{-1})$ are initial $(X + S')$-algebras, so that, denoting $A(\kappa) = (I, \Sigma_I)$ and $\iota = A(\kappa \to \kappa{+}1)^{-1}$ we have that the $((X, \Sigma) + S)$-homomorphism $\varphi \colon I \to TX$ given by initiality of $((I, \Sigma_I), \iota)$ is a bijection:

$$\begin{tikzcd}
(X,\Sigma) + S(I,\Sigma_I) \arrow[r, "\iota"] \arrow[d, "(X,\Sigma)+S'\varphi"'] & (I,\Sigma_I) \arrow[d, dashed, "\varphi"] \\
(X,\Sigma) + S(TX,\Sigma_{TX}) \arrow[r, "{[\eta_X,\psi_X]}"'] & (TX,\Sigma_{TX})
\end{tikzcd}
\quad \overset{U}{\longmapsto} \quad
\begin{tikzcd}
X + S'I \arrow[r, "\iota"] \arrow[d, "X+S'\varphi"'] & I \arrow[d, dashed, "\varphi"] \\
X + S'TX \arrow[r, "{[\eta_X,\psi_X]}"'] & TX
\end{tikzcd}$$

So $(I, \Sigma_I) \cong (\varphi(I), \varphi(\Sigma_I)) = (TX, \varphi(\Sigma_I))$, and $((TX, \varphi(\Sigma_I)), [\eta_X, \psi_X])$ is an isomorphic initial $(X, \Sigma) + S$-algebras. By initiality, there exists $\varphi' \colon TX \to TX$ such that the following diagrams commute

$$\begin{tikzcd}
(X,\Sigma) + S(TX,\varphi(\Sigma_I)) \arrow[r, "{[\eta_X,\psi_X]}"] \arrow[d, "(X,\Sigma)+S'\varphi'"'] & (TX,\varphi(\Sigma_I)) \arrow[d, dashed, "\varphi'"] \\
(X,\Sigma) + S(TX,\Sigma_{TX}) \arrow[r, "{[\eta_X,\psi_X]}"'] & (TX,\Sigma_{TX})
\end{tikzcd}
\quad \overset{U}{\longmapsto} \quad
\begin{tikzcd}
X + S'TX \arrow[r, "{[\eta_X,\psi_X]}"] \arrow[d, "X+S'\varphi'"'] & TX \arrow[d, dashed, "\varphi'"] \\
X + S'TX \arrow[r, "{[\eta_X,\psi_X]}"'] & TX
\end{tikzcd}$$

but, by unicity of the initial $(X + S')$-homomorphism $\varphi' = id_{TX}$. Since φ' is measurable, we have that for all $E \in \Sigma_{TX}$, $id^{-1}(E) = \varphi^{-1}(E) \in \varphi(\Sigma_I)$, hence $\Sigma_{TX} \subseteq \varphi(\Sigma_I)$. By Lambek's lemma, $[\eta_X, \psi_X]$ is an isomorphism between $(X, \Sigma) + S(TX, \varphi(\Sigma_I))$ and $(TX, \varphi(\Sigma_I))$ so it is also an embedding, and $\varphi(\Sigma_I)$ is the smallest σ-algebra rendering $[\eta_X, \psi_X]$ measurable. Thus, by the fact that $[\eta_X, \psi_X] \colon (X, \Sigma) + S(TX, \Sigma_{TX}) \to (TX, \Sigma_{TX})$ is measurable and Σ_{TX} is contained in $\varphi(\Sigma_I)$, we have that $\Sigma_{TX} = \varphi(\Sigma_I)$. Hence $((TX, \Sigma_{TX}), [\eta_X, \psi_X])$ is isomorphic to the initial $((X, \Sigma) + S)$-algebra, so it is initial. □

By Theorem 4.6, for any syntactic endofunctor S associated with a measurable signature $(\mathcal{S}, ar, \{(X_s, \Sigma_s)\}_{s \in \mathcal{S}})$, the forgetful functor $U^S \colon S\text{-}\mathbf{alg} \to \mathbf{Meas}$ has a left adjoint $L^S \colon \mathbf{Meas} \to S\text{-}\mathbf{alg}$ defined as follows, for any measurable space (X, Σ), and measurable function $f \colon (X, \Sigma_X) \to (Y, \Sigma_Y)$,

$$L^S(X, \Sigma) = ((TX, \Sigma_{TX}), \psi_X)$$
$$L^S f = (f^{\#} \colon TX \to TY)$$

where $f^{\#}$ is the unique $((X, \Sigma) + S)$-homomorphism from the initial algebra $((TX, \Sigma_{TX}), [\eta_X, \psi_X])$ to $((TY, \Sigma_{TY}), [f \circ \eta_Y, \psi_Y])$.

Next we give a more explicit characterization of the monad arising from the adjunction $U^S \dashv L^S$. Remarkably, due to Theorem 4.6 this monad is essentially defined as the term monad freely generated by syntactic **Set**-endofunctors arising from set-signatures.

Definition 4.7 (Term monad). *Let* $(\mathcal{S}, ar, \{(X_s, \Sigma_s)\}_{s\in\mathcal{S}})$ *be a measurable signature and* S *be the syntactic* **Meas***-endofunctor associated with it. The monad freely generated by* S*, called* measurable term monad over S*, is given by the triple* (T^S, η^S, μ^S)*, where* $T^S \colon \mathbf{Meas} \to \mathbf{Meas}$ *is defined as follows, for* (X, Σ) *a measurable space and* $f \colon (X, \Sigma_X) \to (Y, \Sigma_Y)$ *a measurable map*

$$T^S(X,\Sigma) = (TX, \Sigma_{TX})$$
$$T^S f(x) = f(x)$$
$$T^S f(s(k, t_1, \dots, t_{ar(s)})) = s(k, T^S f(t_1), \dots, T^S f(t_{ar(s)}))$$

where $x \in X$*,* $s \in \mathcal{S}$*, and* $k \in X_s$*,* $t_1, \dots, t_{ar(s)} \in TX$*; the unit given by* $\eta^S_X = \eta_X \colon (X, \Sigma) \to (TX, \Sigma_{TX})$ *(the insertion-of-variables function); and multiplication* $\mu^S_X \colon (TTX, \Sigma_{TTX}) \to (TX, \Sigma_{TX})$ *(the operation which allows one to plug measurable terms into contexts) inductively defined as follows*

$$\mu^S_X(t) = t$$
$$\mu^S_X(s(k, C_1, \dots, C_{ar(s)})) = s(k, \mu^S(C_1), \dots, \mu^S(C_{ar(s)}))$$

for all $t \in TX$*,* $s \in \mathcal{S}$*,* $k \in X_s$*, and* $C_1, \dots, C_{ar(s)} \in TTX$ *(i.e., contexts).*

5. Measure GSOS Specification Rule Format

We are now going to present a concrete well-behaved specification rule format for stochastic transition systems with continuous state space as a particular instance of abstract GSOS distributive laws of [44].

Briefly, our approach consists in instantiating the bialgebraic framework of Turi and Plotkin [44] to L-labelled Markov kernels, that is, to the coalgebras for the functor $\Delta^L \colon \mathbf{Meas} \to \mathbf{Meas}$ (Proposition 3.2). The key intuition behind the bialgebraic framework is that rule specification systems can be formulated in terms of certain natural transformations, called *distributive laws*. The models for these distributive laws are *bialgebras*, that is, pairs consisting of a T-algebra $\alpha \colon TX \to X$ and a D-coalgebra $\beta \colon X \to DX$ on the same carrier and such that they are related by a distributive law $\lambda \colon TD \Rightarrow DT$ of a monad T over a comonad D as follows:

$$\begin{tikzcd}
TX \arrow[r, "\alpha"] \arrow[d, "T\beta"'] & X \arrow[r, "\beta"] & DX \\
TDX \arrow[rr, "\lambda_X"] & & DTX \arrow[u, "D\alpha"']
\end{tikzcd}$$

Intuitively, the monad T represents the syntax of the programming language and the comonad D models the shape of computations. The algebra $\alpha\colon TX \to X$ and coalgebra $\beta\colon X \to DX$, respectively, denote the denotational and operational models of the system, and the distributive law $\lambda\colon TD \Rightarrow DT$ explains how the syntax distributes over the computations, that is, how the computations of a composed term depend on the executions of its components. Bialgebras form a category, where any unique morphism from the initial object represents a denotational semantics, and any unique morphism to the final one an operational semantics. Hence, one can always find a canonical fully-abstract semantics with respect to behavioral equivalence: the universal morphism from the initial to the final bialgebras.

The distributive laws we are interested in this paper are those of type $S(Id\times\Delta^L) \Rightarrow (\Delta T^S)^L$, that is, abstract GSOS distributive laws of a monad T^S over the copointed functor $(Id \times \Delta^L)$, where T^S is freely generated by a syntactic functor $S\colon \mathbf{Meas} \to \mathbf{Meas}$ associated with some measurable signature. Our aim is to describe these distributive laws by means of a set of derivation rules similar to the well-know GSOS format of Bloom et al. [13].

In the GSOS format, the target of a transition is a term built from the components of the source process, and their corresponding semantics. In our settings, the target of a transition is not a term, but a (finite) measure over a generic measurable space, hence the derivations of the stochastic transitions becomes more complicated. We cope with this problem proposing labelled transitions of the form $t \xrightarrow{a} \mu$, where μ is no more a measure but a syntactic expression intended to denote a measure, which we call *measure term.*

Definition 5.1. *Let* $(\mathcal{S}, ar_S, \{(X_s,\Sigma_s)\}_{s\in\mathcal{S}})$ *and* $(\mathcal{M}, ar_M, \{(X'_m,\Sigma'_m)\}_{m\in\mathcal{M}})$ *be the measurable signatures for processes and measure terms, respectively. An* MGSOS rule *over them, for a finite set L of labels, is an expression of the form*

$$\frac{\left\{x_i \xrightarrow{a_{ij}} \mu_{ij}\right\}^{1\le j,k\le m_i,\,(a_{ij}=a_{ik}\Rightarrow j=k)}_{1\le i\le ar_S(s),\,a_{ij}\in A_i} \qquad \left\{x_i \stackrel{b}{\not\longrightarrow}\right\}^{b\in B_i}_{1\le i\le ar_S(s)}}{s(k, x_1, \dots, x_{ar_S(s)}) \xrightarrow{c} \mu}$$

where

- $s \in S$, $k \in X_s$;
- $\{x_i \mid 1 \le i \le ar_S(s)\}$ *and* $\{\mu_{ij} \mid 1 \le i \le ar_S(s),\, 1 \le j \le m_i\}$ *are pairwise distinct* process and measure term variables*, respectively;*
- $A_i \cap B_i = \emptyset$ *are disjoint subsets of labels in L, for all* $1 \le i \le n$, *and* $c \in L$;
- μ *is a measurable term for the signature* $(\mathcal{M}, ar_M, \{(X'_m,\Sigma'_m)\}_{m\in\mathcal{M}})$ *with variables in* $\{x_i \mid 1 \le i \le ar_S(s)\}$ *and* $\{\mu_{ij} \mid 1 \le i \le ar_S(s),\, 1 \le j \le m_i\}$.

Note that, differently from the standard GSOS rule format of [13], in the premises one is not allowed to use the same label twice for the same variable x_i.

Measure terms have to be interpreted as actual measures (over process terms) by means of some suitable interpretation function. As one may expect, not all interpretations guarantee that the behavioral equivalence is a congruence. A condition which ensures that these are well-behaved is that they are natural transformations of a particular type:

Definition 5.2. *Let $(\mathcal{S}, ar_S, \{(X_s, \Sigma_s)\}_{s\in\mathcal{S}})$ and $(\mathcal{M}, ar_M, \{(X'_m, \Sigma'_m)\}_{m\in\mathcal{M}})$ be measurable signatures for processes and measure terms, respectively, and $S, M\colon \mathbf{Meas} \to \mathbf{Meas}$ be the syntactic functors associated with them.*

A measure term interpretation *over these signatures is a natural transformation of type $T^M\Delta \Rightarrow \Delta T^S$, where T^S and T^M are the free monads over S and M respectively.*

The operational specification is given by a set of rules along with a measure term interpretation that describes how measure terms should be interpreted as measure over measurable terms.

Definition 5.3. *Let $(\mathcal{S}, ar_S, \{(X_s, \Sigma_s)\}_{s\in\mathcal{S}})$ and $(\mathcal{M}, ar_M, \{(X'_m, \Sigma'_m)\}_{m\in\mathcal{M}})$ be measurable signatures for process and measure terms, respectively.*

An MGSOS specification system *is a pair $(\mathcal{R}, \llparenthesis \cdot \rrparenthesis)$, such that $\mathcal{R}$ is a set of image finite MGSOS rules and $\llparenthesis \cdot \rrparenthesis$ is a measure term interpretation over the process and measure term signatures.*

Similarly to GSOS transition systems specifications, also MGSOS specification systems define a (structural) operational semantics, but in this particular case in the form of a labelled Markov kernel over the measurable space of process terms. Intuitively, its definition can be summarized in two stages. First, an image finite labelled transition system $(T^S\mathbf{0}, \{\xrightarrow{\alpha} \subseteq T^S\mathbf{0} \times T^M(T^S\mathbf{0})\}_{\alpha\in L})$ is (inductively) defined from the set of MGSOS derivation rules, then the associated Markov kernel is obtained evaluating measure terms to measures. Formally, the associated Δ^L-coalgebra γ over $T^S\mathbf{0}$ is defined, for $\alpha \in L$ and $t \in T^S\mathbf{0}$, by

$$\gamma(t)(\alpha) = \oplus_{T^S\mathbf{0}}\big(\{\llparenthesis\mu\rrparenthesis_{T^S\mathbf{0}} \mid t \xrightarrow{\alpha} \mu\}\big)\,.$$

where, for arbitrary measurable spaces (X, Σ), we define $\oplus_X\colon \mathcal{P}_{fin}\Delta X \to \Delta X$ by $\oplus_X(\{\mu_1, \dots, \mu_n\}) = \sum_i \mu_i$. Note that $\oplus_X$ is easily seen to measurable and natural in (X, Σ).

The following is the main theorem of this section: it shows how MGSOS specification systems induce natural transformation of type $S(Id \times \Delta^L) \Rightarrow (\Delta T^S)^L$, i.e., abstract GSOS laws of [44].

Theorem 5.4. *An MGSOS specification system $(\mathcal{R}, \llparenthesis \cdot \rrparenthesis)$ over the measurable signatures $(\mathcal{S}, ar_S, \{(X_s, \Sigma_s)\}_{s\in\mathcal{S}})$ and $(\mathcal{M}, ar_M, \{(X'_m, \Sigma'_m)\}_{m\in\mathcal{M}})$, where S and M are the associated syntactic functors, and set of labels L, determines a natural transformation $[\![\mathcal{R}]\!]\colon S(Id \times \Delta^L) \Rightarrow (\Delta T^S)^L$.*

Proof. For any measurable space (X, Σ), define the (set) function $[\![\mathcal{R}]\!]_X$ as the composite:

$$S(X \times (\Delta X)^L) \xrightarrow{\nu_X} (\mathcal{P}_{\mathrm{fin}} T^M \Delta X)^L \xrightarrow{(\oplus T^S \circ \mathcal{P}_{\mathrm{fin}} \llparenthesis \cdot \rrparenthesis)^L_X} (\Delta T^S X)^L\,,$$

where, ν_X is defined as follows: for all $\mu' \in T^M \Delta X$, $c \in L$, $s \in S$, $k \in X_s$, $x_i \in X$, and $\beta_i \in (\Delta X)^L$, put

$$\mu' \in \nu_X(s(k, (x_1, \beta_1), \dots, (x_{ar_S(s)}, \beta_{ar_S(s)})))(c)$$

if and only if there exists a (possibly renamed) rule in $\mathcal{R}$ of the form

$$\frac{\{x_i \xrightarrow{a_{ij}} \mu_{ij}\}^{1\le j,k\le m_i,\,(a_{ij}=a_{ik}\Rightarrow j=k)}_{1\le i\le ar_S(s),\,a_{ij}\in A_i} \qquad \{x_i \stackrel{b}{\not\to}\}^{b\in B_i}_{1\le i\le ar_S(s)}}{s(k, x_1, \dots, x_{ar_S(s)}) \xrightarrow{c} \mu}$$

such that $\beta_i(b) = \mathbf{0}$ (the zero measure), for $b \in B_i$, and $\mu' = (T^M\sigma)(\mu)$ for a substitution map σ such that $\sigma(x_i) = \delta_{x_i}$ (the Dirac measure at x_i) and $\sigma(\mu_{ij}) = \beta_i(a_{ij})$.

Naturality of $[\![\mathcal{R}]\!]$ is proved separately for the two components. To prove the naturality of ν one proceeds as in [44, Th. 1.1]. The composite $(\oplus T^S \circ \mathcal{P}_{\text{fin}} \langle\!| \cdot |\!\rangle)^L$, is natural since $\langle\!| \cdot |\!\rangle$ and $\oplus$ are natural.

As for measurability of $[\![\mathcal{R}]\!]_X$, it suffices to check that $[\![\mathcal{R}]\!]_X^{-1}(U_\alpha[E])$ is measurable in $S(X \times (\Delta X)^L)$, for $U_\alpha^E = \{\beta' \in (\Delta T^S X)^L \mid \beta'(\alpha) \in E\}$, where $\alpha \in L$, and $E \in \Sigma_{\Delta T^S X}$.

Let $s(k, \overline{(x, \beta)})$ abbreviates $s(k, (x_1, \beta_1), \dots, (x_n, \beta_n)) \in S(X \times (\Delta X)^L)$, then

$$\begin{aligned}
[\![\mathcal{R}]\!]_X^{-1}(U_\alpha^E) &= \left\{ s(k, \overline{x, \beta}) \mid [\![\mathcal{R}]\!]_X(s(k, \overline{x, \beta})) \in U_\alpha^E \right\} \\
&= \left\{ s(k, \overline{x, \beta}) \mid ((\oplus T^S \circ \mathcal{P}_{\text{fin}} \langle\!| \cdot |\!\rangle)^L \circ \nu)_X(s(k, \overline{x, \beta})) \in U_\alpha^E \right\} \\
&= \left\{ s(k, \overline{x, \beta}) \mid ((\oplus T^S \circ \mathcal{P}_{\text{fin}} \langle\!| \cdot |\!\rangle)^L \circ \nu)_X(s(k, \overline{x, \beta}))(\alpha) \in E \right\} \\
&= \left\{ s(k, \overline{x, \beta}) \mid (\oplus T^S \circ \mathcal{P}_{\text{fin}} \langle\!| \cdot |\!\rangle \circ \nu^\alpha)_X(s(k, \overline{x, \beta})) \in E \right\}
\end{aligned}$$

where $\nu_X^\alpha \triangleq \nu_X(\cdot)(\alpha)$ is the specialization of ν_X on a fixed $\alpha \in L$,

$$\begin{aligned}
&= (\oplus T^S \circ \mathcal{P}_{\text{fin}} \langle\!| \cdot |\!\rangle \circ \nu^\alpha)_X^{-1}(E) \\
&= (\nu_X^\alpha)^{-1} \circ (\oplus T^S \circ \mathcal{P}_{\text{fin}} \langle\!| \cdot |\!\rangle)_X^{-1}(E)
\end{aligned}$$

Now is easy to prove measurability. Since $\langle\!| \cdot |\!\rangle$ is measurable and sums and products of measurable functions is measurable, $(\oplus T^S \circ \mathcal{P}_{\text{fin}} \langle\!| \cdot |\!\rangle)_X^{-1}(E) \in \Sigma_{\mathcal{P}_{\text{fin}} T^M \Delta X}$.

To prove measurability of ν_X^α we need only to check that $(\nu_X^\alpha)^{-1}(\bigcup_{j=0}^k \{E_j\})$ is a measurable set in $S(X \times (\Delta X)^L)$, for $E_j \in \Sigma_{T^M \Delta X}$, $0 \le j \le k$.

$$\begin{aligned}
(\nu_X^\alpha)^{-1}(\textstyle\bigcup_{j=0}^k \{E_j\}) &= \left\{ s(k, \overline{x, \beta}) \mid \nu_X^\alpha(s(k, \overline{x, \beta})) \in \textstyle\bigcup_{j=0}^k \{E_j\} \right\} \\
&= \textstyle\bigcup_{j=0}^k \left\{ s(k, \overline{x, \beta}) \mid \nu_X^\alpha(s(k, \overline{x, \beta})) = E_j \right\}
\end{aligned}$$

But $\nu_X^\alpha(s(k, \overline{x, \beta})) = E_j$ iff there exists some rule in $\mathcal{R}$ with conclusion of the form $f(x_1, \dots, x_n) \xrightarrow{\alpha} \mu$ and $(T^M\sigma)(\mu) \in E_j$ (obviously, all the other conditions given above have to be satisfied too). By construction, σ is defined by sums of Dirac measures δ_X and $ev_{a_{ij}}$, which are measurable. Therefore, $T^M\sigma$ is measurable, and as a consequence also ν_X^α is measurable. $\square$

Remark 5.5. In the proof of Theorem 5.4, measure term variables are interpreted as Dirac measures via the natural transformation $\delta\colon Id \Rightarrow \Delta$. This together with the assumption that $\langle\!|\cdot|\!\rangle\colon T^M\Delta \Rightarrow \Delta T^S$ is a natural transformation in **Meas** are crucial to prove measurability of $[\![\mathcal{R}]\!]$.

Remark 5.6 (Related work). To establish a correspondence between abstract GSOS distributive laws and concrete rule formats, in [10] Bartels proposed an elegant decomposition strategy to recover congruential specification systems from distributive laws. The method uses a collection of representation lemmas for distributive laws, which allows to explain natural transformations of complex type in terms of collections of natural transformations of simpler type. Using this technique Bartels has been able to give the first detailed proof of correspondence between natural transformations of type $S(Id \times \mathcal{P}_{fin}^L) \Rightarrow (\mathcal{P}_{fin}T^S)^L$ for labelled transition systems and GSOS specification systems. Moreover, he further extended the technique in order to derive from scratch a rule specification format for discrete probabilistic transition systems, the so called PGSOS. The very same technique has been applied also by Klin and Sassone [31] in the definition of a rule format for discrete state stochastic transition systems.

Unfortunately, this method applies only in the category **Set** and cannot be ported easily to **Meas**. This is due to the fact that many of the decomposition lemmas used in [10] require the objects of the category to be representable as indexed coproducts of simpler canonical subobjects. This clearly works in **Set**, since, for any set X, the isomorphism $X \cong \coprod_{x\in X}\{x\}$ holds, but it does not work in **Meas**, due to the presence of the σ-algebra structure in the objects.

In [44], it is shown that a distributive law $\rho\colon S(Id \times B) \Rightarrow BT^S$, where T^S is the free monad over S, for endofunctors S and B admitting initial S-algebra $(T^S\mathbf{0}, \alpha)$ and final B-coalgebra (F, ω), gives rise to a unique B-coalgebra structure $\beta_\rho\colon T^S\mathbf{0} \to BT^S\mathbf{0}$ such that $(T^S\mathbf{0}, \alpha, \beta_\rho)$ is the initial ρ-bialgebra. Dually, there is a unique S-algebra structure $\alpha_\rho\colon SF \to F$ such that (F, α_ρ, β) is the final ρ-bialgebra. The unique (by both initiality and finality) homomorphism from the initial to the final ρ-model is both the initial and final semantics for ρ, and it is called *universal semantics* for ρ. Note, that two B-coalgebras have the same universal semantics if and only if they are behaviorally equivalent, therefore B-behavioral equivalence is an S-congruence.

As for abstract GSOS laws of type $S(Id \times \Delta^L) \Rightarrow (\Delta T^S)^L$ we have the following result for behavioral equivalence on probabilistic processes on **Meas**.

Theorem 5.7. *Behavioral equivalence on the Δ^L-coalgebras over $T^S\mathbf{0}$ inductively induced by MGSOS specification systems over S and M, and set of labels L, is an S-congruence.*

Proof. By Theorem 4.6 every syntactic functor $S\colon \mathbf{Meas} \to \mathbf{Meas}$ have initial algebra. As for the endofunctor $\Delta^L\colon \mathbf{Meas} \to \mathbf{Meas}$, the existence of the final coalgebra is given by Theorem 3.4 and Remark 3.5. The thesis follows combining Theorems 5.4 and [44, Corollary 7.3]. □

Remark 5.8. In fact, the two stage construction described before Theorem 5.4 is not the standard abstract GSOS derivation procedure, as per [44]. However it can be easily proven that the two procedures give the same Δ^L-coalgebra structure by showing that both constructions actually fit the induction proof principle schema given by initiality.

6. Measure Term Interpretations by Structural Induction

So far we only focused on the format of MGSOS rules. However, the naturality condition required in the definition of measure terms interpretation is as crucial as the rule format in order for Theorem 5.4 to hold. In this section, given two syntactic (freely generated) monads T^S and T^M in **Meas**, we provide methods in order to define natural transformations of type $T^M\Delta \Rightarrow \Delta T^S$, i.e., measure term interpretations.

6.1. Simple Structural Induction

Given that the natural transformations we are interested in are of the form $\lambda\colon T^F G \Rightarrow H$, for (T^F, η^F, μ^F) a freely generated monad and G, H endofunctors, one may consider to use structural induction (recall Definition 2.1) in order to define λ. Indeed, the following (folklore) results hold.

Lemma 6.1. *Let $F, G, H\colon \mathbf{C} \to \mathbf{C}$ be endofunctors and (T^F, η^F, μ^F) be the monad freely generated by F. Then, for any pair of natural transformations $f\colon G \Rightarrow H$ and $\beta\colon FH \Rightarrow H$, there exists a unique natural transformation $f^\#\colon T^F G \Rightarrow H$ making the following diagrams commute*

$$\begin{tikzcd} G \arrow[r, "\eta^F G"] \arrow[dr, bend right, "f"'] & T^F G \arrow[d, dashed, "f^\#"'] & FT^F G \arrow[l, "\psi^F G"'] \arrow[d, "Ff^\#"] \\ & H & FH \arrow[l, "\beta"] \end{tikzcd}$$

where $\psi^F\colon FT^F \Rightarrow T^F$ is the canonical natural transformation induced by the free F-algebra structures over $\mathbf{C}$-objects.

Proof. Each X-component of $f^\#$ is uniquely determined as the (free) inductive extension of β_X along the valuation f_X, by usual structural induction (Definition 2.1). Naturality follows immediately by uniqueness aspect of the structural induction proof principle and naturality of f and β. □

An immediate corollary of Lemma 6.1 provides a simple method for defining measure term interpretations by induction on the structure measure terms.

Corollary 6.2. *Let $S, M\colon \mathbf{Meas} \to \mathbf{Meas}$ be the syntactic endofunctors, and $\varphi\colon M\Delta T^S \Rightarrow \Delta T^S$ be a natural transformation. Then, there exists a unique* *measure term interpretation* $\llparenthesis \cdot \rrparenthesis \colon T^M\Delta \Rightarrow \Delta T^S$ *over* S *and* M *such that the following diagrams commute.*

$$\begin{tikzcd}
\Delta \arrow[r, "\eta^M\Delta"] \arrow[dr, bend right, "\Delta\eta^S"'] & T^M\Delta \arrow[d, dashed, "\llparenthesis \cdot \rrparenthesis"'] & MT^M\Delta \arrow[l, "\psi^M\Delta"'] \arrow[d, "M\llparenthesis \cdot \rrparenthesis"] \\
& \Delta T^S & M\Delta T^S \arrow[l, "\varphi"']
\end{tikzcd}$$

Therefore, to define a measure term interpretation $\llparenthesis \cdot \rrparenthesis$ one only needs to provide two natural transformations ϕ and φ as in Corollary 6.2, which will be uniquely extended by structural induction on measure terms. The actual benefit in using the structural induction proof principle consists in easing the checks of naturality and measurability for $\llparenthesis \cdot \rrparenthesis$, which are usually simpler for ϕ and φ.

6.2. Structural λ-iteration Induction Proof Principle

It is well known that the basic induction proof principle induced by intiality in the category of algebras of some functor does not capture the schema of many useful definitions (for examples and discussions on this subject see [45, 46]). In fact, Corollary 6.2 can be used to define only a small subclass of measure term interpretations and it may not be enough to meet the expressivity demanded.

In this section, we propose a different recursion proof principle, called *iterative proof principle*, that will be used to extend Corollary 6.2 to provide more expressivity in the definition of measure term interpretations.

The following lemma is be the core our technique, and it is the dual of the "coiterative proof principle" introduced by Bartels in [10, Chapter 4].

Lemma 6.3. *Let* $F, G \colon \mathbf{C} \to \mathbf{C}$ *be functors on a category with countable products,* (A, α) *be the initial* F*-algebra, and* $\lambda \colon FG \Rightarrow FG$ *be a simple distributive law*[4] *of* F *over* G*. For any* FG*-algebra* (X, φ) *there exists a unique arrow* $f \colon A \to X$*, such that the following diagrams commute*

$$\begin{tikzcd}
FGX \arrow[d, "\varphi"'] & FGA \arrow[l, dashed, "FGf"'] & FA \arrow[l, "F\beta_\lambda"'] \arrow[d, "\alpha"] \\
X & & A \arrow[ll, dashed, "f"]
\end{tikzcd}
\qquad
\begin{tikzcd}
FA \arrow[r, "\alpha"] \arrow[d, "F\beta_\lambda"'] & A \arrow[r, "\beta_\lambda"] & GA \\
FGA \arrow[rr, "\lambda_A"'] & & GFA \arrow[u, "G\alpha"']
\end{tikzcd}$$

Proof. Dualize [10, Theorem 4.2.2]. □

We denote this induction proof principle as λ*-iteration proof principle*, and we call $f \colon A \to X$ as the λ*-iterative arrow induced by* φ. Note that, the diagram

[4] A simple distributive laws of F over G, for $F, G \colon \mathbf{C} \to \mathbf{C}$ endofunctors, is a natural transformation of type $FG \Rightarrow GF$. Differently from distributive laws between (co)monads, no other conditions are required.

on the right is the initial λ-bialgebra, and in particular β_λ is uniquely determined by (standard) induction on the F-algebra $(BA, B\alpha \circ \lambda_A)$.

The λ-iteration proof principle of Lemma 6.3 can be extended as a proof principle on the monad T^F freely generated by F as follows:

Proposition 6.4 (Structural λ-iteration). *Let $F, G \colon \mathbf{C} \to \mathbf{C}$ be endofunctors on a category with binary coproducts and countable products, (T^F, η^F, μ^F) be the free monad over F, $\lambda \colon FG \Rightarrow GF$ be a simple distributive law, and $\psi_X \colon FT^F X \to T^F X$ be the free F-algebra structure over X. Then, for any FG-algebra (Y, φ), G-coalgebra (X, k), and arrow $\phi \colon X \to Y$, there exists a unique arrow $f \colon T^F X \to Y$ such that the following diagrams commute*

$$
\begin{tikzcd}
FGY \arrow[d, "\varphi"'] & FGT^F X \arrow[l, dashed, "FGf"'] & FT^F X \arrow[l, "F\beta_\lambda"'] \arrow[d, "\psi^F_X"] \\
Y & & T^F X \arrow[ll, dashed, "f"'] \\
 & & X \arrow[u, "\eta^F_X"'] \arrow[ull, bend left, "\phi"]
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "\eta^F_X"] \arrow[d, "k"'] & T^F X \arrow[d, "\beta_\lambda"'] & FT^F X \arrow[l, "\psi^F_X"'] \arrow[d, "F\beta_\lambda"] \\
BX \arrow[r, "G\eta^F_X"'] & BT^F X & FBT^F X \arrow[l, "G\psi^F_X \circ \lambda_{T^F X}"]
\end{tikzcd}
$$

Proof. First, notice that β_λ is the inductive extension of the $(X + F)$-algebra structure on $GT^F X$ given by the copair $[G\eta^F_X \circ k, G\psi^F_X \circ \lambda_{T^F X}]$, along the initial $(X + F)$-algebra structure $[\eta^F_X, \psi^F_X]$ on $T^F X$. Now, define the distributive law $\lambda' \colon (X + F)G \Rightarrow G(X + F)$ as $\lambda'_Y = [Gin_l, Gin_r] \circ (k + \lambda_Y)$ (the proof of naturality is straightforward). By definition of λ' we have

$$[G\eta^F_X \circ k, G\psi^F_X \circ \lambda_{T^F X}] = G[\eta^F_X, \psi^F_X] \circ \lambda'_{T^F X}\,.$$

Therefore, by unicity of the inductive extension, $(T^F X, \psi^F_X, \beta_\lambda)$ turns out to be a λ'-model on $T^F X$. This allows one to apply Lemma 6.3 on the $(X + F)G$-algebra structure $(Y, [\phi, \varphi])$ obtaining a unique λ'-iterative arrow $f \colon T^F X \to Y$ making the required diagrams above commute. $\square$

We denote this proof principle by *structural λ-induction proof principle*, and we say that f is the *λ-iterative extension of φ along the (pair of) valuations ϕ and k*. Note that, the diagram on the right define β_λ as the structural inductive extension of $G\psi^F_X \circ \lambda_{T^F X}$ along $G\eta^F_X \circ k$.

Proposition 6.4 can be turned into an induction proof principle on natural transformations in the following way:

Corollary 6.5. *Let $F, G, H, K \colon \mathbf{C} \to \mathbf{C}$ be functors on a category with binary coproducts and countable products, (T^F, η^F, μ^F) be the free monad over F, $\lambda \colon FG \Rightarrow GF$ be a simple distributive law. For any $\varphi \colon FGH \Rightarrow H$, $k \colon Id \Rightarrow G$, and $\phi \colon K \Rightarrow H$, there exist unique natural transformations $\beta_\lambda \colon T^F \Rightarrow GT^F$ and*

$f\colon T^F \Rightarrow H$ *such that the following (natural) diagrams commute*

$$\begin{tikzcd}
FGH \arrow[d, "\varphi"'] & FGT^F K \arrow[l, dashed, "FGf"'] & FT^F K \arrow[l, "F\beta_\lambda K"'] \arrow[d, "\psi^F K"] \\
H & & T^F K \arrow[ll, dashed, "f"'] \\
& & K \arrow[u, "\eta^F K"'] \arrow[llu, bend left, "\phi"]
\end{tikzcd}
\qquad
\begin{tikzcd}
Id \arrow[r, "\eta^F"] \arrow[d, "k"'] & T^F \arrow[d, "\beta_\lambda"'] & FT^F \arrow[l, "\psi^F"'] \arrow[d, "F\beta_\lambda"] \\
G \arrow[r, "G\eta^F"'] & GT^F & FGT^F \arrow[l, "G\psi^F \circ \lambda T^F"]
\end{tikzcd}$$

Proof. We first prove naturality of β_λ, i.e., that for any morphism $g\colon X \to Y$, $(\beta_\lambda)_Y \circ T^F g = GT^F g \circ (\beta_\lambda)_X$. The commuting diagrams

$$\begin{tikzcd}
X \arrow[r, "\eta^F_X"] \arrow[d, "k_X"'] \arrow[dr, phantom, "\text{(by def.)}"] & T^F X \arrow[d, "(\beta_\lambda)_X"'] \arrow[dr, phantom, "\text{(by def.)}"] & FT^F X \arrow[l, "\psi^F_X"'] \arrow[d, "F(\beta_\lambda)_X"] \\
GX \arrow[r, "G\eta^F_X"'] \arrow[d, "Gg"'] \arrow[dr, phantom, "\text{(nat. } \eta^F\text{)}"] & GT^F X \arrow[d, "GT^F g"'] \arrow[dr, phantom, "\text{(nat. } \psi^F \text{ and } \lambda\text{)}"] & FGT^F X \arrow[l, "G\psi^F_X \circ \lambda_{T^F X}"] \arrow[d, "FGT^F g"] \\
GY \arrow[r, "G\eta^F_Y"'] & GT^F Y & FGT^F Y \arrow[l, "G\psi^F_Y \circ \lambda_{T^F Y}"]
\end{tikzcd}$$

and

$$\begin{tikzcd}
& X \arrow[r, "\eta^F_X"] \arrow[d, "g"'] \arrow[dl, "k_X"'] \arrow[dr, phantom, "\text{(nat. } \eta^F\text{)}"] & T^F X \arrow[d, "T^F g"'] \arrow[dr, phantom, "\text{(nat. } \psi^F\text{)}"] & FT^F X \arrow[l, "\psi^F_X"'] \arrow[d, "FT^F g"] \\
GX \arrow[dr, "Gg"'] \arrow[r, phantom, "\text{(nat. } k\text{)}"] & Y \arrow[r, "\eta^F_Y"] \arrow[d, "k_Y"'] \arrow[dr, phantom, "\text{(by def.)}"] & T^F Y \arrow[d, "(\beta_\lambda)_Y"'] \arrow[dr, phantom, "\text{(by def.)}"] & FT^F Y \arrow[l, "\psi^F_Y"'] \arrow[d, "F(\beta_\lambda)_Y"] \\
& GY \arrow[r, "G\eta^F_Y"'] & GT^F Y & FGT^F Y \arrow[l, "G\psi^F_Y \circ \lambda_{T^F Y}"]
\end{tikzcd}$$

assert that $(\beta_\lambda)_Y \circ T^F g$ and $GT^F g \circ (\beta_\lambda)_X$ are both inductive extensions of $G\psi^F_Y \circ \lambda_{T^F Y}$ along $G\eta^F_X \circ Gg \circ k_X$, hence they necessarily coincide. Naturality of f follows similarly by unicity of the λ-iterative extension.

$$
\begin{tikzcd}
 & & & FGT^F KX \arrow[dll, bend right, "FGT^F Kg"'] \\
FGHY \arrow[d, "\varphi_Y"'] & FGT^F KY \arrow[l, "FBf_Y"'] \arrow[d, phantom, "\text{(by def.)}"] & FT^F KY \arrow[l, "F(\beta_\lambda)_{KY}"] \arrow[d, "\psi^F_{KY}"'] \arrow[r, phantom, "\text{(nat. } \psi^F\text{)}"'] & FT^F KX \arrow[l, "FT^F Kg"] \arrow[u, "F(\beta_\lambda)_{KX}"'] \arrow[d, "\psi^F_{KX}"] \arrow[ul, phantom, "\text{(nat. } \beta_\lambda\text{)}"] \\
HY & & T^F KY \arrow[ll, "f_Y"] \arrow[dl, phantom, "\text{(by def.)}"] & T^F KX \arrow[l, "T^F Kg"'] \\
 & & KY \arrow[u, "\eta^F_{KY}"'] \arrow[ull, bend left, "\phi_Y"] \arrow[r, phantom, "\text{(nat. } \eta^F\text{)}"] \arrow[dl, phantom, "\text{(nat. } \phi\text{)}"] & \\
HX \arrow[uu, "Hg"] & & & KX \arrow[lll, "\phi_X"] \arrow[ul, "Kg"] \arrow[uu, "\eta^F_{KX}"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
FGHY \arrow[d, "\varphi_Y"'] \arrow[dr, phantom, "\text{(nat. } \varphi\text{)}"] & FGHX \arrow[l, "FGHg"'] \arrow[d, "\varphi_X"] & FGT^F KX \arrow[l, "FGf_X"'] \arrow[d, phantom, "\text{(by def.)}"] & FT^F KX \arrow[l, "F(\beta_\lambda)_{KX}"'] \arrow[d, "\psi^F_{KX}"] \\
HY & HX \arrow[l, "Hg"] & & T^F KX \arrow[ll, "f_X"] \arrow[dl, phantom, "\text{(by def.)}"] \\
 & & & KX \arrow[u, "\eta^F_{KX}"'] \arrow[ull, bend left, "\phi_X"]
\end{tikzcd}
$$

Indeed, both $f_Y \circ T^F Kg$ and $Hg \circ f_X$ are λ-iterative extensions of φ_Y along the pair of valuations $Hg \circ \phi_X$ and k_{KX}, as proved by the commutative diagrams above. $\square$

An immediate consequence of Corollary 6.5 is the following method for defining measure term interpretations by structural λ-iteration.

Corollary 6.6. *Let $S, M\colon$ **Meas** $\to$ **Meas** be the syntactic measurable endofunctors, $G\colon$ **Meas** $\to$ **Meas** be a functor, $\lambda\colon MG \Rightarrow GM$ be a simple distributive law, and $\varphi\colon MG\Delta T^S \Rightarrow \Delta T^S$, $k\colon Id \Rightarrow G$ be natural transformations. Then, there exists a unique measure term interpretation $\llparenthesis \cdot \rrparenthesis\colon T^M\Delta \Rightarrow \Delta T^S$ over S and M such that the following diagrams commute*

$$
\begin{tikzcd}
MG\Delta T^S \arrow[d, "\varphi"'] & MGT^M\Delta \arrow[l, dashed, "MG\llparenthesis \cdot \rrparenthesis"'] & MT^M\Delta \arrow[l, "M\beta_\lambda\Delta"'] \arrow[d, "\psi^M\Delta"] \\
\Delta T^S & & T^M\Delta \arrow[ll, dashed, "\llparenthesis \cdot \rrparenthesis"'] \\
 & & \Delta \arrow[u, "\eta^M\Delta"'] \arrow[ull, bend left, "\Delta\eta^S"]
\end{tikzcd}
\qquad
\begin{tikzcd}
Id \arrow[r, "\eta^M"] \arrow[d, "k"'] & T^M \arrow[d, "\beta_\lambda"'] & MT^M \arrow[l, "\psi^M"'] \arrow[d, "F\beta_\lambda"] \\
G \arrow[r, "G\eta^M"'] & GT^M & MGT^M \arrow[l, "G\psi^M \circ \lambda T^M"]
\end{tikzcd}
$$

The actual benefit in using the λ-iteration induction proof principle consists in extending the class of definable measure terms interpretations. It is trivial to notice that Corollary 6.6 subsumes Corollary 6.2 in the case when G, λ, and k are identities. Naturality and measurability follows immediately by checking it on λ, φ, and k, which are usually simpler.

7. Examples of MGSOS Specifications

To illustrate the expressiveness of the MGSOS format, we present the specifications for two simple process calculi which extend Milner's CCS [35] (without restriction) with continuous data information. Remarkably, the MGSOS format allows for a very simple presentation of continuous state stochastic semantics, ensuring important properties (e.g., congruence) that are notoriously difficult to obtain even in the discrete case.

7.1. Quantitative CCS

In this section we present a CCS-like process calculus capable to model continuous occurrences of agents. The syntax is defined as follows:

$$\begin{aligned} P, Q &::= \mathsf{nil} \;\mid\; \alpha.P \;\mid\; P+Q \;\mid\; P \parallel Q \;\mid\; c \textbf{ of } P \\ \alpha &::= a \;\mid\; \overline{a} \;\mid\; \tau \end{aligned}$$

where $c \in \mathbb{R}_{\geq 0}$ is a non-negative real number, and $a \in A$ is an action label taken from a finite set A. The nil operator denotes the null process, $\alpha.P$ the action prefix, $P+Q$ the stochastic choice operator, $P \parallel Q$ the parallel composition. The *concentration* operator c **of** P models a process with "concentration" (or quantity) c of agent P.

As for the semantics, we aim to give to processes a stochastic behavior which is faithful with the standard nondeterministic CCS, but such that the execution rate of each action depends on the availability of the involved agents. The introduction of the concentration operator c **of** P opens several problems that cannot be solved by a discrete state semantics. Indeed, a discrete semantics for c **of** P forces to decide a priori the quantity of P to be consumed, with a rule of the following form

$$\frac{x \xrightarrow{\alpha[r]} x'}{c \textbf{ of } x \xrightarrow{\alpha[c' \cdot r]} c' \textbf{ of } x' \parallel (c - c') \textbf{ of } x} \tag{2}$$

where r denotes the execution rate of the stochastic a-transition in the premise, and c' denotes the concentration of the agent consumed by the transition. Having to deal with continuous concentrations, any fixed choice of $c' \leq c$ is unreasonable since the uniform probability of choosing the exact value of c' in the interval $[0, c]$ would be always zero. The only satisfactory choice is to give an actual continuous state operational semantics to the calculus. We will achieve this by means of an MGSOS specification system.

$$
\frac{}{\alpha.x \xrightarrow{\alpha} x} \qquad \frac{x \xrightarrow{\alpha} \mu}{x + x' \xrightarrow{\alpha} \mu} \qquad \frac{x' \xrightarrow{\alpha} \mu}{x + x' \xrightarrow{\alpha} \mu}
$$

$$
\frac{x \xrightarrow{\alpha} \mu}{c \textbf{ of } x \xrightarrow{\alpha} U_c(\mu)} \qquad \frac{x \xrightarrow{\alpha} \mu}{x \parallel x' \xrightarrow{\alpha} \mu \mid x'} \qquad \frac{x' \xrightarrow{\alpha} \mu}{x \parallel x' \xrightarrow{\alpha} x \mid \mu}
$$

$$
\frac{x \xrightarrow{a} \mu \qquad x' \xrightarrow{\overline{a}} \mu'}{x \parallel x' \xrightarrow{\tau} \mu \mid \mu'} \qquad \frac{x \xrightarrow{\overline{a}} \mu \qquad x' \xrightarrow{a} \mu'}{x \parallel x' \xrightarrow{\tau} \mu \mid \mu'}
$$

$$
\begin{aligned}
\llparenthesis \beta \rrparenthesis^{\mathsf{q}}_X &= \beta \\
\llparenthesis \mu \mid \mu' \rrparenthesis^{\mathsf{q}}_X &= (\llparenthesis \mu \rrparenthesis^{\mathsf{q}}_X \times \llparenthesis \mu' \rrparenthesis^{\mathsf{q}}_X) \circ (\lambda(x, x').\, x \parallel x')^{-1} \\
\llparenthesis U_c(\mu) \rrparenthesis^{\mathsf{q}}_X &= c \cdot (U[0, c] \times \llparenthesis \mu \rrparenthesis^{\mathsf{q}}_X \times \delta_X) \circ (\lambda(c', x', x).\, c' \textbf{ of } x' \parallel (c - c') \textbf{ of } x)^{-1}
\end{aligned}
$$

Figure 1: MGSOS specification system $(\mathcal{R}^{\mathsf{q}}, \llparenthesis \cdot \rrparenthesis^{\mathsf{q}})$ for Quantitative CCS.

We start by defining the measurable signatures for processes and measure terms in order to formally determine which are the σ-algebras associated with each operator symbols and, consequently, to the measurable spaces of terms freely generated over them. We do this directly by giving the syntactic functors $S, M \colon \mathbf{Meas} \to \mathbf{Meas}$ associated with the measurable signatures for processes and measure terms, respectively:

$$
\begin{aligned}
SX &= \overbrace{\mathbf{1}}^{\mathsf{nil}} + \overbrace{A \times X}^{a.x} + \overbrace{A \times X}^{\overline{a}.x} + \overbrace{X}^{\tau.x} + \overbrace{X \times X}^{x+x} + \overbrace{X \times X}^{x \parallel x} + \overbrace{\mathbb{R}_{\geq 0} \times X}^{c \textbf{ of } x}, \\
MX &= \overbrace{X \times X}^{x|x} + \overbrace{\mathbb{R}_{\geq 0} \times X}^{U_c(x)},
\end{aligned}
$$

where A is the set of action labels endowed with the discrete σ-algebra, and $\mathbb{R}_{\geq 0}$ is the set of positive real numbers with Borel σ-algebra.

The stochastic semantics for Quantitative CCS is described by means of the MGSOS specification system $(\mathcal{R}^{\mathsf{q}}, \llparenthesis \cdot \rrparenthesis^{\mathsf{q}})$ given in Figure 1. The measure term interpretation $\llparenthesis \cdot \rrparenthesis^{\mathsf{q}} \colon T^M \Delta \Rightarrow \Delta T^S$ is obtained by using Corollary 6.2 on $\varphi \colon M \Delta T^S \Rightarrow \Delta T^S$ defined as follows, for $\beta, \beta' \in \Delta T^S$,

$$
\begin{aligned}
\varphi(\beta)_X &= \beta \\
\varphi(\beta \mid \beta')_X &= (\beta \times \beta') \circ (\lambda(x, x').\, x \parallel x')^{-1} \\
\varphi(U_c(\beta))_X &= c \cdot (U[0, c] \times \beta \times \delta_X) \circ (\lambda(c', x', x).\, c' \textbf{ of } x' \parallel (c - c') \textbf{ of } x)^{-1}
\end{aligned}
$$

where we have denoted by $U[0, c](E) = \int_{[0,c] \cap E} \frac{1}{c}\, dx$ the uniform probability measure over the interval $[0, c]$, for any Borel set $E \subseteq \mathbb{R}_{\geq 0}$; δ_X is the Dirac measure; $\beta \times \beta'$ denotes the product measure, and $(r\beta)(E) = r \cdot \beta(E)$, for $0 \leq r \leq 1$. Naturality of φ can be easily proven noticing that product measure is uniquely defined on measurable rectangles, so that one can restrict the naturality check only on those measurable sets. By Corollary 6.2, then also $\llparenthesis \cdot \rrparenthesis^{\mathsf{q}}$ is a natural transformation.

The rules in $\mathcal{R}^{\mathsf{q}}$ regarding the choice operator and the parallel operator are very simple. The former choses the behavior of one their two arguments, the latter is defied following the standard CSS semantics for channel synchronization. As for the quantitative operator, c **of** x, the actual semantics is hidden in the definition of the measure term interpretation $\llparenthesis \cdot \rrparenthesis^{\mathsf{q}}$. Intuitively, for a measurable set E, the pre-image $(\lambda(c', x', x).\, c' \textbf{ of } x' \parallel (c-c') \textbf{ of } x)^{-1}(E)$ gives probabilities different from zero only to those triples (c', x', x) for which a term of the form $c' \textbf{ of } x' \parallel (c-c') \textbf{ of } x$ belongs to E. Since the function is measurable, the set of all such triples belongs to the σ-algebra of the product space $\mathbb{R}_{\geq 0} \times T^S X \times T^S X$, so that it can be measured by $(U[0,c] \times \llparenthesis \mu \rrparenthesis^{\mathsf{q}}_X \times \delta_X)$. In this way, the concentration c acts as a formal parameter for the uniform distribution $U[0,c]$ (note the analogy with rule (2)). This solves the problem posed at the beginning of the section, when a discrete semantics was shown to be inadequate.

7.2. FlatCCS

In this section we introduce a simple yet paradigmatic calculus of agents living in the Euclidean plane $\mathbb{R}^2$, which we call *FlatCCS* (of course other variants can be considered, e.g. LineCCS, SpaceCCS, etc. [1]). The idea we aim to model is that the rate of communications between two agents depends on their distance (like, e.g., in wireless networks). To this end, FlatCCS extends CCS with a syntactic *frame operator* representing a process' displacement:

$$\begin{aligned} P, Q &::= \mathsf{nil} \mid \alpha.P \mid P + Q \mid P \parallel Q \mid [\,P\,]_z \\ \alpha &::= a \mid \overline{a} \mid \tau \end{aligned}$$

where $a \in A$ ranges over a finite set of action labels, and z over the plane $\mathbb{R}^2$. Intuitively, if the P is in position $z' \in \mathbb{R}^2$, the process $[\,P\,]_z$ is in $z' + z$. If no frame operator occurs, processes are assumed to be in the origin $(0,0)$. Thus, in $[\,P \parallel [\,Q\,]_{(0,1)}\,]_{(1,0)}$, P is (externally) seen to be in $(1,0)$ and Q in $(1,1)$.

As for the semantics, we assume that the communication probability decreases exponentially with the distance. Thus, we expect the FlatCCS process $a.\mathsf{nil} \parallel [\overline{a}.\mathsf{nil}]_{(r,0)}$ (with $r \in \mathbb{R}$) to perform an internal communication evolving into $\mathsf{nil} \parallel [\mathsf{nil}]_{(r,0)}$ at rate $|r|$.

We start by defining the measurable signatures for processes and measure terms in order to formally determine which are the σ-algebras associated with each operator symbols and, consequently, to the measurable spaces of terms freely generated over them. The syntactic functors $S, M \colon \mathbf{Meas} \to \mathbf{Meas}$ for FlatCCS processes and measure terms are given as follows

$$\begin{aligned} SX &= \overbrace{1}^{\mathsf{nil}} + \overbrace{A \times X}^{a.x} + \overbrace{A \times X}^{\overline{a}.x} + \overbrace{X}^{\tau.x} + \overbrace{X \times X}^{x+x} + \overbrace{X \times X}^{x\parallel x} + \overbrace{\mathbb{R}^2 \times X}^{[\,x\,]_z}, \\ MX &= \overbrace{X \times X}^{x \blacktriangleleft x} + \overbrace{X \times X}^{x \blacktriangleright x} + \overbrace{X \times X}^{x \blacktriangledown x} + \overbrace{\mathbb{R}^2 \times X}^{\langle\, x\,\rangle_z}\ . \end{aligned}$$

$$\frac{}{\alpha.x \xrightarrow{\alpha} x} \qquad \frac{x \xrightarrow{\alpha} \mu}{x + x' \xrightarrow{\alpha} \mu} \qquad \frac{x' \xrightarrow{\alpha} \mu}{x + x' \xrightarrow{\alpha} \mu}$$

$$\frac{x \xrightarrow{\alpha} \mu}{[\,x\,]_z \xrightarrow{\alpha} \langle\, \mu \,\rangle_z} \qquad \frac{x \xrightarrow{\alpha} \mu}{x \parallel x' \xrightarrow{\alpha} \mu \blacktriangleleft x'} \qquad \frac{x' \xrightarrow{\alpha} \mu}{x \parallel x' \xrightarrow{\alpha} x \blacktriangleright \mu}$$

$$\frac{x \xrightarrow{a} \mu \qquad x' \xrightarrow{\overline{a}} \mu'}{x \parallel x' \xrightarrow{\tau} \mu \blacktriangledown \mu'} \qquad \frac{x \xrightarrow{\overline{a}} \mu \qquad x' \xrightarrow{a} \mu'}{x \parallel x' \xrightarrow{\tau} \mu \blacktriangledown \mu'}$$

$$\begin{aligned}
\llparenthesis \beta \rrparenthesis^{\mathsf{fl}}_X &= \beta \\
\llparenthesis \mu \blacktriangleleft \mu' \rrparenthesis^{\mathsf{fl}}_X &= (\llparenthesis \mu \rrparenthesis^{\mathsf{fl}}_X \times \llparenthesis \mu' \rrparenthesis^{\mathsf{fl}}_X) \circ (\lambda(x,x').\, x \parallel x')^{-1} \\
\llparenthesis \mu \blacktriangleright \mu' \rrparenthesis^{\mathsf{fl}}_X &= (\llparenthesis \mu \rrparenthesis^{\mathsf{fl}}_X \times \llparenthesis \mu' \rrparenthesis^{\mathsf{fl}}_X) \circ (\lambda(x,x').\, x \parallel x')^{-1} \\
\llparenthesis \mu \blacktriangledown \mu' \rrparenthesis^{\mathsf{fl}}_X &= \Big(\| pos(\mu) - pos(\mu') \| \cdot (\llparenthesis \mu \rrparenthesis^{\mathsf{fl}}_X \times \llparenthesis \mu' \rrparenthesis^{\mathsf{fl}}_X) \Big) \circ (\lambda(x,x').\, x \parallel x')^{-1} \\
\llparenthesis \langle\, \mu \,\rangle_z \rrparenthesis^{\mathsf{fl}}_X &= \llparenthesis \mu \rrparenthesis^{\mathsf{fl}}_X \circ (\lambda x.\, [\,x\,]_z)^{-1}
\end{aligned}$$

Figure 2: MGSOS specification system $(\mathcal{R}^{\mathsf{fl}}, \llparenthesis \cdot \rrparenthesis^{\mathsf{fl}})$ for FlatCCS.

where the set A is endowed with the discrete σ-algebra, and $\mathbb{R}^2$ is the real plane endowed with its Borel σ-algebra.

The stochastic semantics is defined by means of the specification system $(\mathcal{R}^{\mathsf{fl}}, \llparenthesis \cdot \rrparenthesis^{\mathsf{fl}})$ given in Figure 2. According to these rules, the term $\mu \blacktriangleleft \mu'$ indicates that an action has been performed on the left hand side (dually in $\mu \blacktriangleright \mu'$), $\mu \blacktriangledown \mu'$ denotes that the process succeeded in a synchronization, and $\langle\, \mu \,\rangle_z$ encodes the absolute position. In the measure term interpretation $\llparenthesis \cdot \rrparenthesis^{\mathsf{fl}} \colon T^M \Delta \Rightarrow \Delta T^S$, the function $pos \colon T^M \Delta X \to \mathbb{R}^2$ determines the position of an action by inspecting the syntactic structure of μ; this is inductively defined as follows:

$$pos(\beta) = (0,0) \qquad pos(\mu \blacktriangleleft \mu') = pos(\mu) \qquad pos(\mu \blacktriangleright \mu') = pos(\mu')$$
$$pos(\mu \blacktriangledown \mu') = \tfrac{1}{2}(pos(\mu) + pos(\mu')) \qquad pos(\langle\, \mu \,\rangle_z) = z + pos(\mu)\,.$$

Note that the use of a syntax for representing the measures occurring in the target of the transitions is essential in order to define this position function.

The measure term interpretation $\llparenthesis \cdot \rrparenthesis^{\mathsf{fl}}$ as defined above, arises as an instance of the structural λ-iteration of Corollary 6.5, for a suitable (simple) distributive law. Let $\lambda \colon M(\mathbb{R}^2 \times Id) \Rightarrow (\mathbb{R}^2 \times Id)M$ be, for $x, x' \in X$ and $z, z' \in \mathbb{R}^2$,

$$\begin{aligned}
\lambda_X((z,x) \blacktriangleleft (z',x')) &= (z, x \blacktriangleleft x') \\
\lambda_X((z,x) \blacktriangleright (z',x')) &= (z', x \blacktriangleright x') \\
\lambda_X((z,x) \blacktriangledown (z',x')) &= (\tfrac{1}{2}(z+z'), x \blacktriangledown x') \\
\lambda_X(\langle\, (z,x) \,\rangle_{z'}) &= (z+z', \langle\, x \,\rangle_{z'})\,.
\end{aligned}$$

Then, define $k\colon Id \Rightarrow (\mathbb{R}^2 \times Id)$, and a $\varphi\colon M(\mathbb{R}^2 \times \Delta T^S) \Rightarrow \Delta T^S$ as follows, for $x \in X$, $z, z' \in \mathbb{R}^2$, and $\beta, \beta' \in \Delta T^S X$,

$$\begin{aligned}
k_X(x) &= ((0,0), x)\,, \\
\varphi_X((z,\beta) \blacktriangleleft (z',\beta')) &= (\beta \times \beta') \circ (\lambda(x,x').\, x \parallel x')^{-1} \\
\varphi_X((z,\beta) \blacktriangleright (z',\beta')) &= (\beta \times \beta') \circ (\lambda(x,x').\, x \parallel x')^{-1} \\
\varphi_X((z,\beta) \blacktriangledown (z',\beta')) &= \left(e^{-\|z-z'\|} \cdot (\beta \times \beta')\right) \circ (\lambda(x,x').\, x \parallel x')^{-1} \\
\varphi_X(\langle\, (z,\beta)\,\rangle_{z'}) &= \beta \circ (\lambda x.\, [\, x\,]_{z'})^{-1}\,.
\end{aligned}$$

These are easily seen to be natural in X and measurable. Now, applying Corollary 6.6 we obtain

$$\begin{tikzcd}
M(\mathbb{R}^2 \times \Delta T^S X) \arrow[d, "\varphi_X"'] & M(\mathbb{R}^2 \times T^M \Delta X) \arrow[l, dashed, "M(\mathbb{R}^2 \times \llparenthesis \cdot \rrparenthesis_X)"'] & MT^M \Delta X \arrow[l, "M\langle pos, id\rangle_{\Delta X}"'] \arrow[d, "\psi^M_{\Delta X}"] \\
\Delta T^S X & & T^M \Delta X \arrow[ll, dashed, "\llparenthesis \cdot \rrparenthesis_X"'] \\
& & \Delta X \arrow[u, "\eta^M_{\Delta X}"'] \arrow[llu, bend left, "\Delta \eta^S_X"]
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[r, "\eta^M_X"] \arrow[d, "k_X"'] & T^M X \arrow[d, "\langle pos, id\rangle_X"'] & MT^M X \arrow[l, "\psi^M_X"'] \arrow[d, "M\langle pos, id\rangle_X"] \\
\mathbb{R}^2 \times X \arrow[r, "(\mathbb{R}^2 \times \eta^M_X)"'] & \mathbb{R}^2 \times T^M X & M(\mathbb{R}^2 \times T^M X) \arrow[l, "(\mathbb{R}^2 \times \psi^M_X) \circ \lambda_{(T^M X)}"]
\end{tikzcd}$$

It is easy to check that the diagrams above commute also when $\llparenthesis \cdot \rrparenthesis^{\mathsf{fl}}_X$ is used in place of $\llparenthesis \cdot \rrparenthesis_X$, hence $\llparenthesis \cdot \rrparenthesis^{\mathsf{fl}}_X = \llparenthesis \cdot \rrparenthesis_X$, by unicity of $\llparenthesis \cdot \rrparenthesis_X$.

We conclude the description of the MGSOS rule format with an example of how the construction in Theorem 5.4 applies to the specification $(\mathcal{R}^{\mathsf{fl}}, \llparenthesis \cdot \rrparenthesis^{\mathsf{fl}})$ for FlatCCS. The abstract GSOS distributive law $[\![\cdot]\!]\colon S(Id \times \Delta^L) \Rightarrow (\Delta T^S)^L$ for $x, x' \in X$, $L = A \cup \overline{A} \cup \{\tau\}$, and $\beta, \beta' \in (\Delta X)^L$, is given by

$$\begin{aligned}
[\![\mathsf{nil}]\!]_X &= \lambda\alpha.\, \mathbf{0} \\
[\![\alpha.\,]\!]_X(x,\beta) &= \lambda\alpha'.\begin{cases} \llparenthesis \delta_x \rrparenthesis^{\mathsf{fl}}_X & \text{if } \alpha' = \alpha \\ \mathbf{0} & \text{otherwise} \end{cases} \\
(x,\beta)[\![\, + \,]\!]_X(x',\beta') &= \lambda\alpha.\, \llparenthesis \beta(\alpha) \rrparenthesis^{\mathsf{fl}}_X \oplus \llparenthesis \beta'(\alpha) \rrparenthesis^{\mathsf{fl}}_X \\
(x,\beta)[\![\, \parallel \,]\!]_X(x',\beta') &= \lambda\alpha.\begin{cases} \llparenthesis \beta(\alpha) \blacktriangleleft \delta_{x'} \rrparenthesis^{\mathsf{fl}}_X \oplus \llparenthesis \delta_x \blacktriangleright \beta'(\alpha) \rrparenthesis^{\mathsf{fl}}_X & \text{if } \alpha \neq \tau \\ \bigoplus_{a \in A} \left\{ \begin{array}{c} \llparenthesis \beta(\alpha) \blacktriangleleft \delta_{x'} \rrparenthesis^{\mathsf{fl}}_X, \llparenthesis \delta_x \blacktriangleright \beta'(\alpha) \rrparenthesis^{\mathsf{fl}}_X, \\ \llparenthesis \beta(a) \blacktriangledown \beta'(\overline{a}) \rrparenthesis^{\mathsf{fl}}_X, \llparenthesis \beta(\overline{a}) \blacktriangledown \beta'(a) \rrparenthesis^{\mathsf{fl}}_X \end{array} \right\} & \text{if } \alpha = \tau \end{cases} \\
[\![[\ \ ]_z]\!]_X(x,\beta) &= \lambda\alpha'.\llparenthesis \beta(\alpha) \rrparenthesis^{\mathsf{fl}}_X
\end{aligned}$$

where $\mathbf{0}$ is the null measure (assigning 0 to each measurable set), and δ_x is the Dirac measure at $x \in X$. When the measurable space of variables X is taken to be the space $T^S\mathbf{0}$ of ground measurable terms, the above definition gives rise to the "canonical" universal (initial and final) semantics for FlatCCS.

8. Conclusions and Future Work

In this paper, we have introduced an SOS specification format for continuous state stochastic calculi, called *MGSOS*. In this format, transitions have the form $P \xrightarrow{\alpha} \mu$, where μ is a *measure term*, i.e., an expression denoting a finite measure over sets of processes. An MGSOS specification $(\mathcal{R}, \langle\!| \cdot |\!\rangle)$ is composed by a set $\mathcal{R}$ of GSOS-like rules, and a measure terms interpretation $\langle\!| \cdot |\!\rangle \colon T^M\Delta \Rightarrow \Delta T^S$. The rule set yields a LTS corresponding to the collective semantics of all the $\mathcal{R}$-derivable measure terms for a given process. Then, each measure term is given an "interpretation" via $\langle\!| \cdot |\!\rangle$, and the overall stochastic semantics is given by summing up the set of partial behaviors. MGSOS specification systems yield an abstract GSOS distributive law of type $S(Id \times \Delta^L) \Rightarrow (\Delta T^S)^L$. We have then proved that the behavioral equivalence induced by a rule specification in MGSOS format is always a congruence.

The MGSOS format is general enough to cover several examples, including those in [16, 6], and can be applied to different quantitative aspects, e.g. Quality of Service, physical and chemical parameters in biological systems, etc. We have provided two case studies: a "quantitative CCS" (i.e., a calculus with a real-valued prefix denoting the "amount" of a process), and a calculus of processes whose communication rate depends on their distance. We have shown that both cases admit an MGSOS specification.

It is interesting to compare this format with the usual GSOS. In particular, in a transition $P \xrightarrow{\alpha} \mu$, the source term P and the target term μ are from different languages, defined by two different syntactic monads (T^S and T^M respectively). The connection between these two languages is provided by $\langle\!| \cdot |\!\rangle$, which is a kind of "distributive law" across two languages. The usual GSOS format can be seen as a special case, when $T^S = T^M$ and Δ is not present; in this case, $\langle\!| \cdot |\!\rangle = Id$. After [7], the idea of using different languages for sources and targets has been used in other works; e.g. [34] introduces a general GSOS-like format for discrete, non-deterministic systems with quantitative aspects, and [18] discusses bisimulation equivalences from probabilistic SOS rules. A possible future work is to investigate a more general GSOS-like format, encompassing both MGSOS and GSOS specifications.

As a side contribution, we have provided a new construction of initial algebras and final coalgebras for functors which do not necessarily preserve colimits of ω-sequences; we just require the underlying category to have a well-behaved factorization system. We have used it for constructing initial algebras of polynomial functors and the final coalgebra of the measure functor Δ on **Meas**, but clearly this result is quite general and can be used in many other situations.

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Appendix A. Technical proofs

A key fact that will be used in the following proofs is that the pre-image operation commutes with respect to all σ-algebra operations: if $f\colon X \to Y$ and $A_i, A, B \subseteq Y$, for all $i \in I$, then

(i) $f^{-1}(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} f^{-1}(A_i)$;

(ii) $f^{-1}(A \setminus B) = f^{-1}(A) \setminus f^{-1}(B)$; in particular $f^{-1}(Y \setminus A) = X \setminus f^{-1}(A)$.

Proposition A.1. *Let X,Y be sets, $\mathcal{F} \subseteq \mathcal{P}(Y)$, and $f\colon X \to Y$ be a function. Then $f^{-1}(\sigma(\mathcal{F})) = \sigma(f^{-1}(\mathcal{F}))$.*

Proof. We prove the two inclusions separately.

($\supseteq$) Since by definition $\sigma(f^{-1}(\mathcal{F}))$ is the intersection of all σ-algebras over X containing $f^{-1}(\mathcal{F})$, the inclusion can be proved just showing that $f^{-1}(\sigma(\mathcal{F}))$ is a σ-algebra and that it contains $f^{-1}(\mathcal{F})$. By $\mathcal{F} \subseteq \sigma(\mathcal{F})$, we have that the inclusion $f^{-1}(\mathcal{F}) \subseteq f^{-1}(\sigma(\mathcal{F}))$. Let us prove that $f^{-1}(\sigma(\mathcal{F}))$ is a σ-algebra. By $\emptyset = f^{-1}(\emptyset)$ and $\emptyset \in \sigma(\mathcal{F})$, we have $\emptyset \in f^{-1}(\sigma(\mathcal{F}))$. Let $F_0, F_1, \ldots \in f^{-1}(\sigma(\mathcal{F}))$, then, for each $n \in \mathbb{N}$, there exists $A_n \in \sigma(\mathcal{F})$, s.t. $F_n = f^{-1}(A_n)$. Then $\bigcup_{n\in\mathbb{N}} A_n \in \sigma(\mathcal{F})$ and in particular $f^{-1}(\bigcup_{n\in\mathbb{N}} A_n) \in f^{-1}(\sigma(\mathcal{F}))$. Since $f^{-1}(\bigcup_{n\in\mathbb{N}} A_n) = \bigcup_{n\in\mathbb{N}} f^{-1}(A_n)$, thus $\bigcup_{n\in\mathbb{N}} F_n \in f^{-1}(\sigma(\mathcal{F}))$. Closure under complements is proved similarly.

($\subseteq$) It suffices to show that $D = \{A \subseteq Y \mid f^{-1}(A) \in \sigma(f^{-1}(\mathcal{F}))\}$ is a σ-algebra containing $\mathcal{F}$. Indeed, if it is so, $\sigma(\mathcal{F}) \subseteq D$, hence, by definition of D, we have $f^{-1}(\sigma(\mathcal{F})) \subseteq \sigma(f^{-1}(\mathcal{F}))$. The inclusion $\mathcal{F} \subseteq D$, follows since $\mathcal{F} \in \sigma(\mathcal{F})$. It remains to prove that D is a σ-algebra. $\emptyset \in D$, since $\emptyset = f^{-1}(\emptyset)$, and $\emptyset \in \sigma(f^{-1}(\mathcal{F}))$. Let $A_1, A_2, \ldots \in D$, then, for all $n \in \mathbb{N}$, $f^{-1}(A_n) \in \sigma(f^{-1}(\mathcal{F}))$. Since $\sigma(f^{-1}(\mathcal{F}))$ is closed by countable unions, we have $f^{-1}(\bigcup_{n\in\mathbb{N}} A_n) = \bigcup_{n\in\mathbb{N}} f^{-1}(A_n) \in \sigma(f^{-1}(\mathcal{F}))$, therefore $\bigcup_{n\in\mathbb{N}} A_n \in D$. Assume $A \in D$, then $f^{-1}(A) \in \sigma(f^{-1}(\mathcal{F}))$. Since $\sigma(f^{-1}(\mathcal{F}))$ is closed under complements, $f^{-1}(Y \setminus A) = X \setminus f^{-1}(A) \in \sigma(f^{-1}(\mathcal{F}))$, therefore $Y \setminus A \in D$. □

Proposition A.2. *Let X,Y be sets, $\mathcal{F} \subseteq \mathcal{P}(X)$, and $f \colon X \to Y$ be a function. Then $\sigma(\{A \subseteq Y \mid f^{-1}(A) \in \mathcal{F}\}) = \{A \subseteq Y \mid f^{-1}(A) \in \sigma(\mathcal{F})\}$.*

Proof. Let $\mathcal{K} = \{A \subseteq Y \mid f^{-1}(A) \in \mathcal{F}\}$ and $\mathcal{J} = \{A \subseteq Y \mid f^{-1}(A) \in \sigma(\mathcal{F})\}$. We show that $\sigma(\mathcal{K}) = \mathcal{J}$ proving the two inclusions simultaneously. Since $\mathcal{F} = f^{-1}(\mathcal{K})$, we have that $\sigma(\mathcal{F}) = \sigma(f^{-1}(\mathcal{K}))$. By Proposition A.1, $\sigma(f^{-1}(\mathcal{K})) = f^{-1}(\sigma(\mathcal{K}))$, hence $\sigma(\mathcal{F}) = f^{-1}(\sigma(\mathcal{K}))$. The following sequence of equivalences

$$A \in \mathcal{J} \iff f^{-1}(A) \in \sigma(\mathcal{F}) \iff f^{-1}(A) \in f^{-1}(\sigma(\mathcal{K})) \iff A \in \sigma(\mathcal{K}),$$

proves the equality. □

Lemma A.3. *Let $\Delta(X, \Sigma)$ be the set of measures on (X, Σ). Then the following families of sets generate the same σ-algebra*

(i) $\mathcal{F}_0 = \{ev_E^{-1}(O) \mid E \in \Sigma,\ O$ *Borel-open in* $[0, \infty)\}$

(ii) $\mathcal{F}_1 = \{L_r(E) \mid r \in [0, \infty) \cap \mathbb{Q},\ E \in \Sigma\}$.

where $L_r(E) = \{\mu \in \Delta(X, \Sigma) \mid \mu(E) \geq r\}$. *In particular,* $\Sigma_{\Delta(X,\Sigma)} = \sigma(\mathcal{F}_1)$.

Proof. The Borel σ-algebra on $[0, \infty)$ can be generated by the open intervals $[r, \infty)$, for $r \in [0, \infty)$. The thesis follows since $ev_E^{-1}(\sigma(\mathcal{F})) = \sigma(ev_E^{-1}(\mathcal{F}))$ for any family $\mathcal{F}$ of subsets in $[0, \infty)$ (Proposition A.1), and $L_r(E) = ev_E^{-1}([r, \infty))$. □

Lemma A.4. *Let $\mathcal{A}$ be a boolean algebra and (X, Σ) a measurable space with σ-algebra Σ generated by $\mathcal{A}$. Then $\sigma(\mathcal{F}) = \sigma(\mathcal{G})$ where*

$$\mathcal{F} = \{L_r(E) \mid E \in \Sigma \text{ and } r \in \mathbb{Q} \cap [0, \infty)\},$$
$$\mathcal{G} = \{L_r(A) \mid A \in \mathcal{A} \text{ and } r \in \mathbb{Q} \cap [0, \infty)\}.$$

Proof. We will prove the two inclusions separately.

$\sigma(\mathcal{G}) \subseteq \sigma(\mathcal{F})$**:** Since $\sigma(\mathcal{G})$ is the smallest σ-algebra that contains $\mathcal{G}$, to prove the inclusion it suffices to show that $\mathcal{G} \subseteq \sigma(\mathcal{F})$. By definition of generated σ-algebra $\mathcal{F} \subseteq \sigma(\mathcal{F})$, and $\mathcal{A} \subseteq \sigma(\mathcal{A}) = \Sigma$. From this it is clear that $\mathcal{G} \subseteq \mathcal{F}$, and therefore that $\mathcal{G} \subseteq \sigma(\mathcal{F})$.

$\sigma(\mathcal{F}) \subseteq \sigma(\mathcal{G})$**:** Let $D = \{E \in \Sigma \mid L_r(E) \in \sigma(\mathcal{G})\}$. Notice that $\mathcal{A} \subseteq D$, indeed, for every $r \in \mathbb{Q} \cap [0, \infty)$ and $A \in \mathcal{A}$, $L_r(A) \in \mathcal{G} \subseteq \sigma(\mathcal{G})$. So that, if we were able to show that D is a σ-algebra, $\Sigma = \sigma(\mathcal{A}) = D$ and by definition of D this will imply that $\sigma(\mathcal{F}) \subseteq \sigma(\mathcal{G})$. So, let us prove that D is a σ-algebra. Since $\mathcal{A} \subseteq D$ and $\mathcal{A}$ is a boolean algebra, by the monotone class theorem it is enough to show that D is a monotone class. Assume $E_0 \supseteq E_1 \supseteq E_2 \supseteq \ldots$ be a decreasing countable collection of elements in D, i.e., $L_r(E) \in \sigma(\mathcal{G})$. We show $L_r(\bigcap_{n\in\mathbb{N}} E_n) = \bigcap_{n\in N} L_r(E)$, thus that $\bigcap_{n\in\mathbb{N}} E_n \in D$.

($\subseteq$) Since, for all $k \in \mathbb{N}$, $\bigcap_{n\in\mathbb{N}} E_n \subset E_k$, we have that $L_r(\bigcap_{n\in\mathbb{N}} E_n) \subseteq L_r(E_k)$, therefore $L_r(\bigcap_{n\in\mathbb{N}} E_n) \subseteq \bigcap_{n\in N} L_r(E)$.

($\supseteq$) Let $\mu \in \bigcap_{n\in N} L_r(E)$, hence, for all $n \in \mathbb{N}$, $\mu(E_n) \geq r$. This means that r is a lower bound for $\{\mu(E_n) \mid n \in \mathbb{N}\}$. Since all measures are assumed to be finite, $\mu(\bigcap_{n\in\mathbb{N}} E_n) = \inf_{n\in\mathbb{N}} \mu(E_n)$, so that $\mu(\bigcap_{n\in\mathbb{N}} E_n) \geq r$, thus $\mu \in L_r(\bigcap_{n\in\mathbb{N}} E_n)$.

Assume $E_0 \subseteq E_1 \subseteq E_2 \subseteq \dots$ be an increasing countable collection of elements in D, that is, $L_r(E) \in \sigma(\mathcal{G})$. We show $L_r(\bigcup_{n\in\mathbb{N}} E_n) = \bigcap_{k>0} \bigcup_{n\in N} L_{r-\frac{1}{k}}(E)$, which implies $\bigcap_{n\in\mathbb{N}} E_n \in D$. This follows from the following sequence of equivalent statements:

$$
\begin{aligned}
\mu \in L_r(\textstyle\bigcup_{n\in\mathbb{N}} E_n) &\iff \mu(\textstyle\bigcup_{n\in\mathbb{N}} E_n) \geq r && \text{(by def. } L_r(E))\\
&\iff \lim_{n\in\mathbb{N}} \mu(E_n) && \text{(by } \mu \text{ monotone)}\\
&\iff \forall k > 0 \,.\exists n \in \mathbb{N}\,.\mu(E_n) \geq r - \tfrac{1}{k} && \text{(by convergence)}\\
&\iff \forall k > 0 \,.\exists n \in \mathbb{N}\,.\mu \in L_{r-\frac{1}{k}}(E_n) && \text{(by def. } L_r(E))\\
&\iff \forall k > 0 \,.\mu \in \textstyle\bigcup_{n\in\mathbb{N}} L_{r-\frac{1}{k}}(E_n) && \text{(by union)}\\
&\iff \mu \in \textstyle\bigcap_{k>0} \bigcup_{n\in\mathbb{N}} L_{r-\frac{1}{k}}(E_n) && \text{(by intersection)}
\end{aligned}
$$

□

Lemma A.5. *Let $Z\colon \mathbf{Ord}^{op} \to \mathbf{Meas}$ be the final sequence of Δ. Then, the σ-algebra $\Sigma_{Z(\omega+1)}$ on $Z(\omega+1)$ is generated by the following collection of subsets*

$$
\mathcal{F} = \{ev^{-1}_{Z(\omega\to n)^{-1}(E)}([r,\infty)) \mid r \in [0,\infty) \cap \mathbb{Q} \text{ and } E \in \Sigma_{Z(n)}\}\,.
$$

Proof. $(Z(\omega \to n))_{n<\omega}$ is a limit cone over $Z{\restriction}\omega$, hence $Z(\omega)$ has initial σ-algebra w.r.t. its cone, i.e., $\Sigma_{Z(\omega)} = \sigma(\{Z(\omega \to n)^{-1}(E) \mid E \in \Sigma_{Z(n)}, n < \omega\})$. We show that the family $\mathcal{A} = \{Z(\omega \to n)^{-1}(E) \mid E \in \Sigma_{Z(n)}, n < \omega\}$ is a boolean algebra. The empty set $\emptyset$ is contained in $\mathcal{A}$, since $Z(\omega \to n)^{-1}(\emptyset) = \emptyset$, for all $n < \omega$. Assume $A \in \mathcal{A}$, then there exists $n < \omega$ and $E \in \Sigma_{Z(n)}$ such that $A = Z(\omega \to n)^{-1}(E)$. The following hold by definition of pre-image

$$
Z(\omega) \setminus A = Z(\omega) \setminus Z(\omega \to n)^{-1}(E) = Z(\omega \to n)^{-1}(Z(n) \setminus E)\,.
$$

Since $Z(n) \setminus E \in \Sigma_{Z(n)}$, then $Z(\omega) \setminus A \in \mathcal{A}$. Assume $A, B \in \mathcal{A}$, then there exits $m, n < \omega$, $E_m \in \Sigma_{Z(m)}$, and $E_n \in \Sigma_{Z(n)}$ such that $A = Z(\omega \to m)^{-1}(E_m)$ and $B = Z(\omega \to n)^{-1}(E_n)$. Without loss of generality assume that $m \leq n$, then

$$
\begin{aligned}
A = Z(\omega \to m)^{-1}(E_m) &= (Z(n \to m) \circ Z(\omega \to n))^{-1}(E_m) && \text{(by func. } Z)\\
&= Z(\omega \to n)^{-1} \circ Z(n \to m)^{-1}(E_m) && \text{(by inv.)}\\
&= Z(\omega \to n)^{-1}\big(Z(n \to m)^{-1}(E_m)\big)\,. && \text{(by comp.)}
\end{aligned}
$$

From this we derive the following equality

$$
\begin{aligned}
A \cup B &= Z(\omega \to n)^{-1}\big(Z(n \to m)^{-1}(E_m)\big) \cup Z(\omega \to n)^{-1}(E_n)\\
&= Z(\omega \to n)^{-1}\big(Z(n \to m)^{-1}(E_m) \cup E_n\big)\,.
\end{aligned}
$$

Since $Z(n \to m)$ is measurable, $Z(n \to m)^{-1}(E_m) \cup E_n \in \Sigma_{Z(n)}$. Therefore $A \cup B \in \mathcal{A}$. From this we conclude that $\mathcal{A}$ is a boolean algebra. Now, the thesis follows by Lemma A.4, noticing that, by definition, $Z(\omega{+}1) = \Delta Z(\omega)$, and $L_r(E) = ev_E^{-1}([r, \infty))$. □

Proposition A.6. *Let $Z \colon \mathbf{Ord}^{op} \to \mathbf{Meas}$ be the final sequence of Δ. Then, the σ-algebra on $Z(\omega{+}1)$ is $\Sigma_{Z(\omega+1)} = \{Z(\omega{+}1 \to \omega)^{-1}(E) \mid E \in \Sigma_{Z(\omega)}\}$.*

Proof. Let $\mathcal{E} = \{Z(\omega{+}1 \to \omega)^{-1}(E) \mid E \in \Sigma_{Z(\omega)}\}$. The inclusion $\mathcal{E} \subseteq \Sigma_{Z(\omega+1)}$ follows since $Z(\omega{+}1 \to \omega)$ is measurable. As for the reverse inclusion, we know, by Lemma A.5, that $\Sigma_{Z(\omega+1)}$ is generated by the following collection of subsets

$$\mathcal{F} = \{ev^{-1}_{Z(\omega \to n)^{-1}(E)}([r, \infty)) \mid r \in [0, \infty) \cap \mathbb{Q} \text{ and } E \in \Sigma_{Z(n)}\}\,.$$

Since $\mathcal{E}$ is already a σ-algebra, to prove $\sigma(\mathcal{F}) \subseteq \mathcal{E}$, we only need to show that $\mathcal{F} \subseteq \mathcal{E}$. This is done noticing that, for all $n < \omega$, $r \in [0, \infty)$ and $E \in \Sigma_{Z(n)}$

$$\begin{aligned}
& ev^{-1}_{Z(\omega \to n)^{-1}(E)}([r, \infty)) \\
&= \{\mu \in Z(\omega{+}1) \mid \mu(Z(\omega \to n)^{-1}(E)) \geq r\} && \text{(by inv.)} \\
&= \{\mu \in Z(\omega{+}1) \mid \big(\Delta Z(\omega \to n)(\mu)\big)(E) \geq r\} && \text{(by def. } \Delta\text{)} \\
&= \{\mu \in Z(\omega{+}1) \mid \big(Z(\omega{+}1 \to n{+}1)(\mu)\big)(E) \geq r\} && \text{(by def. } Z\text{)} \\
&= \big(ev_E \circ Z(\omega{+}1 \to n{+}1)\big)^{-1}([r, \infty)) && \text{(by def. } ev_E \text{ and inv.)} \\
&= \big(ev_E \circ Z(\omega \to n{+}1) \circ Z(\omega{+}1 \to \omega)\big)^{-1}([r, \infty)) && \text{(by func. } Z\text{)} \\
&= Z(\omega{+}1 \to \omega)^{-1}\big(Z(\omega \to n{+}1)^{-1} \circ ev_E^{-1}([r, \infty))\big)\,. && \text{(by inv.)}
\end{aligned}$$

Clearly, $ev_E^{-1}([r, \infty)) \in \Sigma_{Z(n+1)}$, so that $Z(\omega \to n{+}1)^{-1}(ev_E^{-1}([r, \infty))) \in \Sigma_{Z(\omega)}$ follows by measurability of $Z(\omega \to n{+}1)$. This proves that for all $r \in [0, \infty)$ and $E \in \Sigma_{Z(n)}$, $ev^{-1}_{Z(\omega \to n)^{-1}(E)}([r, \infty)) \in \mathcal{E}$, therefore $\mathcal{F} \subseteq \mathcal{E}$. □

Proposition A.7. *Let $Z \colon \mathbf{Ord}^{op} \to \mathbf{Meas}$ be the final sequence of Δ. Then $Z(\omega{+}1 \to \omega)$ is injective.*

Proof. By definition of final sequence, $(Z(\omega \to n))_{n<\omega}$ is a limit cone over $Z{\restriction}\omega$, therefore $Z(\omega)$ is equipped with the initial σ-algebra w.r.t. its cone, that is $\Sigma_{Z(\omega)}$ is generated by

$$\mathcal{A} = \sigma(\{Z(\omega \to n)^{-1}(E) \mid E \in \Sigma_{Z(n)}, n < \omega\})\,.$$

By definition of final sequence $Z(\omega{+}1) = \Delta Z(\omega)$. Let $\mu, \nu \in \Delta Z(\omega)$ be to measure on $Z(\omega)$, and assume $Z(\omega{+}1 \to \omega)(\mu) = Z(\omega{+}1 \to \omega)(\nu)$. We have to show that $\mu = \nu$. Since μ and ν are clearly σ-finite, and they are also pre-measures on the boolean algebra $\mathcal{A}$ (see the proof of Lemma A.5), hence it suffices to show that μ and ν agree on all subsets in $\mathcal{A}$. This is shown below, for all $n < \omega$ and $E \in \Sigma_{Z(n)}$

$$
\begin{aligned}
\mu(Z(\omega \to n)^{-1}(E)) &= \Delta Z(\omega \to n)(\mu)(E) && \text{(by def. } \Delta) \\
&= Z(\omega{+}1 \to n{+}1)(\mu)(E) && \text{(by def. } Z) \\
&= Z(\omega \to n{+}1) \circ Z(\omega{+}1 \to \omega)(\mu)(E) && \text{(by func. } Z) \\
&= Z(\omega \to n{+}1) \circ Z(\omega{+}1 \to \omega)(\nu)(E) && \text{(by hp.)} \\
&= Z(\omega{+}1 \to n{+}1)(\nu)(E) && \text{(by func. } Z) \\
&= \Delta Z(\omega \to n)(\nu)(E) && \text{(by def. } Z) \\
&= \nu(Z(\omega \to n)^{-1}(E))\,. && \text{(by def. } \Delta)
\end{aligned}
$$

Therefore, $Z(\omega{+}1 \to \omega)$ is injective. □

Proof (of Proposition 3.3). A measurable function $f \colon X \to Y$ between measurable spaces (X, Σ) and (Y, Σ_Y) is an embedding if it is injective and Σ_X is the initial σ-algebra with respect to f, i.e., $\Sigma_X = \{f^{-1}(E) \mid E \in \Sigma_Y\}$. Therefore, the thesis follows by Propositions A.6 and A.7. □