Comments on Proposition 13 in [BBLM14]

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9th September 2016

Abstract

In this note we consider Proposition 13 of [BBLM14] that yields a sufficient criterion for the existence of a joint distribution of two marginals with given support. We argue that the proof presented in Appendix A is flawed and give an alternative argumentation for an instance of the claim. However, we have no counterexample for (a slightly modified) Proposition 13.

1 Overview

Before we recall Proposition 13 in [BBLM14] let us first introduce the following notations: given a measurable space X, we denote the set of all probability measures on X by $\mathrm{Prob}[X]$. Let X and Y be measurable spaces and $f\colon X \to Y$ be a measurable function. The *pushforward of* f is given by the function $f_\sharp\colon \mathrm{Prob}[X] \to \mathrm{Prob}[Y]$ defined as follows: for all $\mu \in \mathrm{Prob}[X]$ and measurable $F \subseteq Y$ let $f_\sharp(\mu)(F) = \mu(f^{-1}(F))$. Using these notations, we are ready to state the aforementioned proposition as follows:

Claim 1. *Let Z, X and Y be measurable spaces, $\mu \in \mathrm{Prob}[X]$ and $\nu \in \mathrm{Prob}[Y]$, and $r_1\colon Z \to X$ and $r_2\colon Z \to Y$ be surjective and measurable functions. Assume the sigma-algebra on Z coincides with the initial sigma-algebra on Z for $\{r_1, r_2\}$. If for all measurable sets $E \subseteq X$ and $F \subseteq Y$,*

$$r_1^{-1}(E) = r_2^{-1}(F) \quad \textit{implies} \quad \mu(E) = \nu(F),$$

then there exists $\mu \wedge \nu \in \mathrm{Prob}[Z]$ such that

$$(r_1)_\sharp(\mu \wedge \nu) = \mu \quad \textit{and} \quad (r_2)_\sharp(\mu \wedge \nu) = \nu.$$

Simple counterexample. It is easy to produce counterexamples for Claim 1 when imposing no restrictions on the involved measurable spaces Z, X, and Y. Indeed, consider the discrete spaces Z, X and Y defined by $X = Y = \{0, 1\}$ and $Z = (X \times Y) \setminus \{\langle 0, 1\rangle\}$. Set $\mu = \mathrm{Dirac}[0]$ and $\nu = \mathrm{Dirac}[1]$ and let r_1

and r_2 be natural projections, i.e., $r_1\colon Z \to X$, $r_1(x,y) = x$ and $r_2\colon Z \to Y$, $r_2(x,y) = y$. Given $E \subseteq X$ and $F \subseteq Y$ with $r_1^{-1}(E) = r_2^{-1}(F)$, one has $E = F = \varnothing$ or $E = F = \{0,1\}$ and therefore, $\mu(E) = \nu(F)$. However, there is no $\mu \wedge \nu \in \text{Prob}[Z]$ where $(r_1)_\sharp(\mu \wedge \nu) = \mu$ and $(r_2)_\sharp(\mu \wedge \nu) = \nu$ relying on the fact that $\langle 0,1\rangle \notin Z$.

Claim for quasi-equivalence relations. Let us consider an instance of Claim 1 where $Z = R$ for some quasi-equivalence relation R on $X \times Y$. Here, $R \subseteq X \times Y$ is called a *quasi-equivalence relation (on $X \times Y$)* if the following two statements hold:

- R is *lr-total (on $X \times Y$)*: for all $x \in X$ there is $y \in Y$ where $\langle x,y\rangle \in R$ and vice versa, i.e., for all $y \in Y$ there is $x \in X$ where $\langle x,y\rangle \in R$.
- R is *z-transitive*: for all $\langle x,y\rangle, \langle x',y\rangle, \langle x',y'\rangle \in R$ one has $\langle x,y'\rangle \in R$.

Quasi-equivalence relations generalize the idea of equivalence relations: In case $X = Y$ every equivalence relation on X constitutes a quasi-equivalence relation on $X \times Y$.

First of all, the conditions for a quasi-equivalence relations are rather natural in the context of bisimulations for probabilistic systems (cf. Lemma 11 in [BBLM14]). Besides this, in the presented setting using quasi-equivalences, one cannot expect a simple counterexample for Claim 1 as presented above. We illustrate the crux and suppose an quasi-equivalence relation R on $X \times Y$. Define $r_1\colon R \to X$, $r_1(x,y) = x$ and $r_2\colon R \to Y$, $r_2(x,y) = y$. For every $x \in X$ and $y \in Y$ introduce $F_x = \{y' \in Y \,;\, \langle x,y'\rangle \in R\}$ and $E_y = \{x' \in X \,;\, \langle x',y\rangle \in R\}$, respectively. Given $x \in X$ and $y \in Y$, we have

$$\langle x,y\rangle \in R \quad \text{implies} \quad r_1^{-1}(E_y) = r_2^{-1}(F_x).$$

Therefore, apart from trivial cases, the sets $\{\langle\varnothing,\varnothing\rangle, \langle X,Y\rangle\}$ and $\{\langle E,F\rangle \in 2^X \times 2^Y \,;\, r_1^{-1}(E) = r_2^{-1}(F)\}$ do not coincide for the quasi-equivalence relation R. However, this is the key point for our simple counterexample presented before since $\mu(\varnothing) = \nu(\varnothing)$ and $\mu(X) = \nu(Y)$ for all probability measures $\mu \in \text{Prob}[X]$ and $\nu \in \text{Prob}[Y]$.

Outline. In Section 2 we investigate the proof of Proposition 13 of [BBLM14] given in Appendix A. Section 3 presents an instance of Claim 1 with quasi-equivalences between Polish spaces, which we prove to be correct using results from descriptive set theory.

2 Review of a proof given in [BBLM14]

We illustrate a flaw in the proof of the proposition under consideration presented in Appendix A. Unfortunately, the function $\mu \wedge \nu$ defined as in the

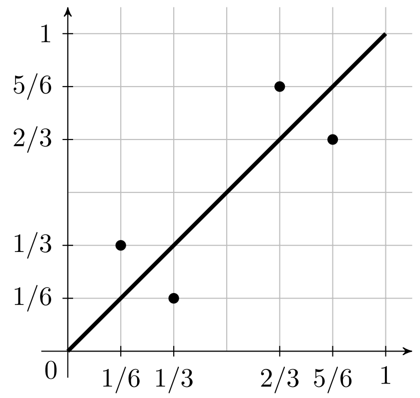

Figure 1: Equivalence relation R on $[0,1]$

given proof is generally not well-defined on G-sets. Let us give a concrete example where we assume the notations as in [BBLM14].

We consider the case where $X = Y = [0,1]$ and $\mu = \nu = \text{Leb}$ with Leb being the Lebesgue measure on $[0,1]$. Let R be the equivalence relation on $[0,1]$ depicted in Figure 1, i.e.,

$$R = \{\langle x, y\rangle \in [0,1] \times [0,1]\ ;\ x = y\} \\ \cup \{\langle 1/6, 1/3\rangle, \langle 1/3, 1/6\rangle, \langle 2/3, 5/6\rangle, \langle 5/6, 2/3\rangle\}.$$

Let $r_1 \colon R \to [0,1]$ and $r_2 \colon R \to [0,1]$ be the natural projections, hence, $r_1(x,y) = x$ and $r_2(x,y) = y$ for all $x \in X$ and $y \in Y$. Obviously, r_1 and r_2 are surjective. Using that R is an equivalence relation on $[0,1]$ we obtain the following statement: for every $E, F \subseteq [0,1]$ where $r_1^{-1}(E) = r_2^{-1}(F)$ one has $E = F$ and thus, provided E and F are Borel in $[0,1]$, it holds $\mu(E) = \nu(E)$. Putting things together the assumptions of Proposition 13 are fulfilled.

In what follows we illustrate why the function $\mu \wedge \nu$ as introduced in Appendix A is not well-defined in general. To that end let

$$M_1 = [0, 2/3] \quad \text{and} \quad M_2 = [1/3, 1]$$

and define $M = r_1^{-1}(M_1)$, $M' = r_1^{-1}(M_1) \cap r_2^{-1}(M_2)$, and $M'' = r_1^{-1}(M_1) \cap r_2^{-1}([0,1] \setminus M_2)$. The introduced sets M, M', and M'' are G-sets in the sense of [BBLM14]. Based on equation (8) in [BBLM14] we justify

$$\mu \wedge \nu(M) = 2/3, \quad \mu \wedge \nu(M') = 0, \quad \text{and} \quad \mu \wedge \nu(M'') = 1/3.$$

Let us consider equation (8) carefully. Relying on the fact that for all Borel sets $E, F \subseteq [0,1]$ one has $\mu(E) = \nu(F)$ if $r_1^{-1}(E) = r_2^{-1}(F)$, equation (8) can be simplified as follows: given Borel sets $E, F \subseteq [0,1]$, we abbreviate

$G = r_1^{-1}(E) \cap r_2^{-1}(F)$ and obtain

$$\mu \wedge \nu(G) = \begin{cases} \mu(\tilde{E}), & \text{if there is } \tilde{E} \subseteq [0,1] \text{ Borel where } G = r_1^{-1}(\tilde{E}) \\ \nu(\tilde{F}), & \text{if there is } \tilde{F} \subseteq [0,1] \text{ Borel where } G = r_2^{-1}(\tilde{F}) \\ 0, & \text{otherwise} \end{cases} .$$

It directly follows $\mu \wedge \nu(M) = \mu \wedge \nu(r_1^{-1}([0,2/3])) = \mu([0,2/3]) = 2/3$. Considering the set M'', we have $M'' = r_2^{-1}([0,1/3))$ and also, $\mu \wedge \nu(M'') = \mu \wedge \nu(r_2^{-1}([0,1/3))) = \nu([0,1/3)) = 1/3$. It remains to argue $\mu \wedge \nu(M') = 0$. Note that

$$M' = \{\langle x,y\rangle \in [0,1] \times [0,1] \; ; \; x = y \text{ and } x \in [1/3,2/3]\} \\ \cup \{\langle 1/6,1/3\rangle, \langle 2/3,5/6\rangle\}.$$

Assume that there is a set $\tilde{E} \subseteq [0,1]$ where $M' = r_1^{-1}(\tilde{E})$. Since $\langle 1/3,1/3\rangle \in M'$, it follows $1/3 \in \tilde{E}$. As $\langle 1/3,1/6\rangle \in r_1^{-1}(\tilde{E})$, we have $\langle 1/3,1/6\rangle \in M'$. Contradiction. Suppose there exists $\tilde{F} \subseteq [0,1]$ where $M' = r_2^{-1}(\tilde{F})$. Using $\langle 2/3,5/6\rangle \in M'$, we obtain $5/6 \in \tilde{F}$. Since $\langle 5/6,5/6\rangle \in r_2^{-1}(\tilde{F})$, it hence holds $\langle 5/6,5/6\rangle \in M'$. Contradiction. We conclude $\mu \wedge \nu(M') = 0$.

We derive a contradiction using equation (9) in [BBLM14]: since M is the union of the disjoint sets M' and M'', the mentioned equation yields

$$\mu \wedge \nu(M) = \mu \wedge \nu(M') + \mu \wedge \nu(M'') = 0 + 1/3 = 1/3.$$

Putting things together we thus have $\mu \wedge \nu(M) = 2/3$ (using equation (8)) and $\mu \wedge \nu(M) = 1/3$ (using equation (9)). Contradiction. We conclude that $\mu \wedge \nu$ as introduced in Appendix A is not well-defined.

However, our example does not contradict Proposition 13 since there indeed exists a probability measure φ on R such that $(r_1)_\sharp(\varphi) = \mu$ and $(r_2)_\sharp(\varphi) = \nu$. To see this define $f\colon [0,1] \to R$, $f(x) = \langle x,x\rangle$. Remind, R is equipped with the initial sigma-algebra for $\{r_1, r_2\}$. This sigma-algebra coincides with the trace sigma-algebra from $[0,1] \times [0,1]$ using that R is Borel in $[0,1] \times [0,1]$. We conclude that f is measurable and can thus safely define $\varphi = f_\sharp(\mathrm{Leb})$. It is easy to see that φ satisfies the required properties, i.e., $\varphi(R) = 1$, $\varphi(E \times [0,1]) = \mathrm{Leb}(E)$ for all Borel sets $E \subseteq [0,1]$, and $\varphi([0,1] \times F) = \mathrm{Leb}(F)$ for all Borel sets $F \subseteq [0,1]$.

3 Thoughts towards a proof

We give an alternative formulation of Claim 1 with quasi-equivalences over Polish spaces first. After that we present a proof for an instance of the statement where, among others, the involved relation is supposed to be countably separated. The section ends with some concluding remarks.

Restatement of the claim. As before let X and Y be sets, $\mu \in \mathrm{Prob}[X]$ and $\nu \in \mathrm{Prob}[Y]$, and suppose a measurable set $R \subseteq X \times Y$. Define $r_1 \colon R \to X$, $r_1(x, y) = x$ and $r_2 \colon R \to Y$, $r_2(x, y) = y$. For all $E \subseteq X$ and $Y \subseteq F$ we have

$$r_1^{-1}(E) = r_2^{-1}(F) \quad \text{iff} \quad (E, F) \text{ is } R\text{-stable.}$$

Here, the pair (E, F) is *R-stable* if $R \cap (E \times Y) = R \cap (X \times F)$. We denote $\mu\, R^{\mathrm{stb}}\, \nu$ if $\mu(E) = \nu(F)$ for all measurable sets $E \subseteq X$ and $F \subseteq Y$ where (E, F) is R-stable.

A *weight function for* (μ, R, ν) is a probability measure W on $X \times Y$ such that $W(R) = 1$ and W is a coupling of (μ, ν), i.e., for all measurable sets $E \subseteq X$ and$F \subseteq Y$,

$$W(E \times Y) = \mu(E) \quad \text{and} \quad W(X \times F) = \nu(F).$$

We write $\mu\, R^{\mathrm{wgt}}\, \nu$ if there exists a weight function for (μ, R, ν). Putting things together we can rewrite an instance of Claim 1 as follows:

Claim 2. *Let X and Y be measurable spaces and R be a quasi-equivalence in $X \times Y$. For all $\mu \in \mathrm{Prob}[X]$ and $\nu \in \mathrm{Prob}[Y]$ we have*

$$\mu\, R^{\mathrm{stb}}\, \nu \quad \textit{implies} \quad \mu\, R^{\mathrm{wgt}}\, \nu.$$

Given the notations of Claim 2 it is easy to see that the reverse implication holds, i.e., $\mu\, R^{\mathrm{wgt}}\, \nu$ implies $\mu\, R^{\mathrm{stb}}\, \nu$. Indeed, given a weight function W for (μ, R, ν), then for all measurable sets $E \subseteq X$ and $F \subseteq Y$ one has the following statement using that $W(R) = 1$ and W is a coupling of (μ, ν): If (E, F) is R-stable, then $R \cap (E \times Y) = R \cap (X \times F)$ and therefore

$$\mu(E) = W(E \times Y) = W(X \times F) = \nu(F).$$

Countably-separated relations. We consider the instance of Claim 2 where X and Y are supposed to be Polish spaces and $R \subseteq X \times Y$ is countably separated, i.e., there exists a Polish space $\underline{Z}$ and Borel functions $g_1 \colon X \to \underline{Z}$ as well as $g_2 \colon Y \to \underline{Z}$ such that

$$R = \{\langle x, y\rangle \in X \times Y \; ; \; g_1(x) = g_2(y)\}.$$

Given a countably-separated relation $R \subseteq X \times Y$, then R is Borel in $X \times Y$. From the literature the following statement is known:

Theorem 3 (Proposition A.7 in [Lov12a, Lov12b])**.** *Let X and Y be Polish spaces and R be a countably-separated relation in $X \times Y$. For all $\mu \in \mathrm{Prob}[X]$ and $\nu \in \mathrm{Prob}[Y]$ one has the following equivalence,*

$$\mu\, R^{\mathrm{stb}}\, \nu \quad \textit{iff} \quad \mu\, R^{\mathrm{wgt}}\, \nu.$$

In the paper [GBK16a] the authors present a sufficient criterion for a relation between Polish spaces to be countably separated. For the sake of completeness let us recall the theorem and its proof. After that we present two important corollaries.

Theorem 4 (Theorem 32 in [GBK16b]). *Let X and Y be Polish spaces and R be a quasi-equivalence relation on $X \times Y$. Assume that R is closed in $X \times Y$ and there are Borel functions $f_1 \colon X \to Y$ and $f_2 \colon Y \to X$ where $x \, R \, f_1(x)$ and $f_2(y) \, R \, y$ for all $x \in X$ and $y \in Y$. Then, R is countably separated.*

Proof. Define the relation $\hat{R} \subseteq R \times R$ by

$$\hat{R} = \{\langle\langle x_1, y_1\rangle, \langle x_2, y_2\rangle\rangle \in R \times R \, ; \, x_1 \, R \, y_2\}.$$

We argue that $\hat{R}$ is an equivalence relation on R. Reflexivity is obvious. Symmetry can be seen as follows: if $\langle x_1, y_1\rangle \, \hat{R} \, \langle x_2, y_2\rangle$, then $x_1 \, R \, y_1$, $x_1 \, R \, y_2$, and $x_2 \, R \, y_2$ and as R is z-transitive we therefore obtain $x_2 \, R \, y_1$ and also $\langle x_2, y_2\rangle \, \hat{R} \, \langle x_1, y_1\rangle$. We justify transitivity of $\hat{R}$ and suppose $\langle x_1, y_1\rangle \, \hat{R} \, \langle x_2, y_2\rangle$ and $\langle x_2, y_2\rangle \, \hat{R} \, \langle x_3, y_3\rangle$. Since $\hat{R}$ is symmetric we obtain $\langle x_3, y_3\rangle \, \hat{R} \, \langle x_2, y_2\rangle$ and thus $x_1 \, R \, y_2$, $x_3 \, R \, y_2$, and $x_3 \, R \, y_3$. The z-transitivity of R yields $x_1 \, R \, y_3$ and hence $\langle x_1, y_1\rangle \, \hat{R} \, \langle x_3, y_3\rangle$. Finally, $\hat{R}$ is an equivalence relation in R.

Moreover, $\hat{R}$ is closed in $R \times R$ that can be seen as follows: define the continuous function $h \colon (X \times Y) \times (X \times Y) \to X \times Y$, $h(x_1, y_1, x_2, y_2) = \langle x_1, y_2\rangle$. Using that R is closed in $X \times Y$, the set $h^{-1}(R)$ is closed in $(X \times Y) \times (X \times Y)$ and thus $\hat{R} = h^{-1}(R) \cap (R \times R)$ is closed in $R \times R$.

We are in the situation of Proposition 5.1.11 in [Sri08] (in the given reference the notion of countably-separated relations slightly differs from the one given here). There hence exist a Polish space $\underline{Z}$ and a Borel function $\hat{g} \colon R \to \underline{Z}$ such that

$$\hat{R} = \{\langle r_1, r_2\rangle \in R \times R \, ; \, \hat{g}(r_1) = \hat{g}(r_2)\}.$$

Let $\mu \in \mathrm{Prob}[X]$ and $\nu \in \mathrm{Prob}[Y]$. Let $f_1 \colon X \to Y$ and $f_2 \colon Y \to X$ be as in the formulation of the theorem, i.e., $x \, R \, f_1(x)$ and $f_2(y) \, R \, y$ for all $x \in X$ and $y \in Y$. Define the Borel functions $g_1 \colon X \to [0,1]$, $g_1(x) = \hat{g}(x, f_1(x))$ and $g_2 \colon Y \to [0,1]$, $g_2(y) = \hat{g}(f_2(y), y)$. Define the relation $\tilde{R} \subseteq X \times Y$ by

$$\tilde{R} = \{\langle x, y\rangle \in X \times Y \, ; \, g_1(x) = g_2(y)\}.$$

Obviously, the relation $\tilde{R}$ is countably separated. It remains to argue $R = \tilde{R}$. For that purpose let us observe the following statement first. For all $x_1 \, R \, y_1$ and $x_2 \, R \, y_2$,

$$x_1 = x_2 \text{ or } y_1 = y_2 \quad \text{implies} \quad \hat{g}(x_1, y_1) = \hat{g}(x_2, y_2).$$

Indeed, given $x_1 \, R \, y_1$ and $x_2 \, R \, y_2$ such that $x_1 = x_2$ or $y_1 = y_2$, then $x_1 \, R \, y_2$ and also $\langle x_1, y_1\rangle \, \hat{R} \, \langle x_2, y_2\rangle$, which yields $\hat{g}(x_1, y_1) = \hat{g}(x_2, y_2)$.

In what follows we finally conclude $R = \tilde{R}$. To this end let $x \in X$ and $y \in Y$. In case $x\,R\,y$ we obtain

$$g_1(x) = \hat{g}(x, f_1(x)) = \hat{g}(x, y) = \hat{g}(f_2(y), y) = g_2(y)$$

and hence $x\,\tilde{R}\,y$. Conversely, if $g_1(x) = g_2(y)$, then

$$\hat{g}(x, f_1(x)) = g_1(x) = g_2(y) = \hat{g}(f_2(y), y),$$

also $\langle x, f_1(x)\rangle\,\hat{R}\,\langle f_2(y), y\rangle$, and thus $x\,R\,y$. □

Based on a measurable selection theorem from the literature (cf. Theorem 6.9.6 in [Bog07]) we obtain the following corollary.

Corollary 5. *Let X and Y be sigma-compact Polish spaces and R be an quasi-equivalence relation on $X \times Y$. Assume R is closed in $X \times Y$. Then for all $\mu \in \mathrm{Prob}[X]$ and $\nu \in \mathrm{Prob}[Y]$ one has the following statement,*

$$\mu\,R^{\mathrm{stb}}\,\nu \quad \textit{iff} \quad \mu\,R^{\mathrm{wgt}}\,\nu.$$

Proof. Relying on Theorems 3 and 4 we have to show that there are Borel functions $f_1 \colon X \to Y$ and $f_2 \colon Y \to X$ such that $x\,R\,f_1(x)$ and $f_2(y)\,R\,y$ for all $x \in X$ and $y \in Y$. However, this is a consequence of Theorem 6.9.6 in [Bog07]. Indeed, we can apply this theorem for the following reasons: since R is closed in $X \times Y$, the set $\{y \in Y\,;\, x\,R\,y\}$ is closed in Y and thus, using that Y is sigma-compact, $\{y \in Y\,;\, x\,R\,y\}$ is sigma-compact in Y for all $x \in X$. Moreover, $\{y \in Y\,;\, x\,R\,y\}$ is not empty for all $x \in X$ as every quasi-equivalence relation is lr-total. The same arguments apply for the set $\{x \in X\,;\, x\,R\,y\}$ where $y \in Y$. □

Every locally-compact Polish space is trivially sigma-compact because Polish spaces have a countable and dense subset. Thus, Corollary 5 covers an important class of topological spaces in the context of probabilistic models as, e.g., $\mathbb{R}^k$ is a locally-compact Polish space for every $k \in \mathbb{N}_{>0}$.

Essentially, the next corollary corresponds to Corollary 5 in the context of equivalence relations instead of quasi-equivalence relations (for the following, reflexivity is crucial).

Corollary 6. *Let X be a Polish space and R be an equivalence relation on X. If R is closed in $X \times X$, then for all $\mu \in \mathrm{Prob}[X]$ and $\nu \in \mathrm{Prob}[Y]$,*

$$\mu\,R^{\mathrm{stb}}\,\nu \quad \textit{iff} \quad \mu\,R^{\mathrm{wgt}}\,\nu.$$

Proof. Define the Borel function $f \colon X \to X$, $f(x) = x$. Since R is reflexive one has $x\,R\,f(x)$ and $f(x)\,R\,x$ for all $x \in X$. The claim thus follows from Theorems 3 and 4. □

4 Counterexample

We adapt an example presented in [Swa96] to obtain a counterexample for Claim 2. For this purpose let us first recall the result from [Swa96].

Proposition 7 ([Swa96])**.** *There are measurable spaces X_0, X_1, and X_2 as well as probability measures $\mu_{01} \in \mathrm{Prob}[X_0 \times X_1]$ and $\mu_{02} \in \mathrm{Prob}[X_0 \times X_2]$ such that the following two statements hold:*

1. $(\rho_0)_\sharp(\mu_{01}) = (\rho_0)_\sharp(\mu_{02})$.
2. *There is no probability measure $\mu_{012} \in \mathrm{Prob}[X_0 \times X_1 \times X_2]$ where $(\rho_{01})_\sharp(\mu_{012}) = \mu_{01}$ and $(\rho_{02})_\sharp(\mu_{012}) = \mu_{02}$.*

Here, ρ_0, ρ_{01}, and ρ_{02} denote the natural projections, i.e., $\rho_0(x_0, x_1, x_2) = x_1$, $\rho_{01}(x_0, x_1, x_2) = \langle x_0, x_1 \rangle$, and $\rho_{02}(x_0, x_1, x_2) = \langle x_0, x_2 \rangle$ for all $x_0 \in X_0$, $x_1 \in X_1$, and $x_2 \in X_2$

Note, Proposition 7 shows that the so-called *clueing lemma* presented in [Vil09] cannot be generalized to arbitrary measurable spaces.

We construct the counterexample for Claim 2. Let X_0, X_1, and X_2 as well as μ_{01} and μ_{02} be as in the proposition. Moreover, let ρ_0, ρ_{01}, and ρ_{02} be as before. Define

$$X = X_0 \times X_1, \quad Y = X_0 \times X_2, \quad \mu = \mu_{01}, \quad \text{and} \quad \nu = \mu_{02}$$

as well as

$$R = \{\langle\langle x_0, x_1\rangle, \langle y_0, y_2\rangle\rangle \in X \times Y \; ; \; x_0 = y_0\}.$$

Let us first observe that R is an quasi-equivalence in $X \times Y$. In what follows we show $\langle \mu, \nu \rangle \in R^{\mathrm{stb}}$, but $\langle \mu, \nu \rangle \notin R^{\mathrm{wgt}}$.

The claim $\langle \mu, \nu \rangle \in R^{\mathrm{stb}}$ follows from Proposition 7 (1) and the fact that for all measurable sets $E \subseteq X$ and $F \subseteq Y$ the following statements are equivalent:

- (E, F) is R-stable.
- There is a measurable set $M_0 \subseteq X_0$ such that $E = M_0 \times X_1$ and $F = M_0 \times X_2$.

This equivalence can be seen as follows. Obviously, for every measurable set $M_0 \subseteq X_0$ it holds that $(M_0 \times X_1, M_0 \times X_2)$ is R-stable. This justifies one implication. To see the reverse direction, assume that (E, F) is R-stable and define

$$M_E = \{x_0 \in X_0 \; ; \; \langle x_0, x_1 \rangle \in E \text{ for some } x_1 \in X_1\},$$
$$M_F = \{x_0 \in X_0 \; ; \; \langle x_0, x_2 \rangle \in F \text{ for some } x_2 \in X_2\}.$$

Relying on the assumption that (E, F) is R-stable, we obtain $M_E = M_F$ as well as $E = M_E \times X_1$ and $F = M_F \times X_2$. Since E and F are measurable in X and Y, respectively, we have that M_E is measurable in X_0.

It remains to show $\langle \mu, \nu \rangle \notin R^{\mathrm{wgt}}$. Towards a contradiciton assume there is a weight function W for (μ, R, ν). Define $f \colon X \times Y \to X_0 \times X_1 \times X_2$,

$$f(x_0, x_1, y_0, y_1) = \langle x_0, x_1, y_1 \rangle.$$

Trivally, f is measurable and hence we can safely define

$$\mu_{012} = f_\sharp(W).$$

It follows $(\rho_{01})_\sharp(\mu_{012}) = \mu_{01}$ and $(\rho_{02})_\sharp(\mu_{012}) = \mu_{02}$. However, this contradicts Proposition 7 (2). We conclude that there is no weight function for (μ, R, ν). Putting things togehter, Claim 2 does not hold for arbitrary measurable spaces.

Concluding remarks. Section 4 shows that Polish spaces are an apropriate assumption for Claim 2. However, in the setting of Polish spaces it would be interesting whether one can go beyond countably-separated relations (cf. Section 3).

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