Measurable Stochastics for Brane Calculus[☆]

Giorgio Bacci, Marino Miculan

Department of Mathematics and Computer Science, University of Udine, Italy

Abstract

The main aim of this work is to give a stochastic extension of the Brane Calculus, along the lines of recent work by Cardelli and Mardare [12]. In this approach, the semantics of a process is a measure of the stochastic distribution of possible derivations. To this end, we first introduce a compositional, finitely branching labelled transition system for Brane Calculus; interestingly, the associated strong bisimulation is a congruence. Then, we give a stochastic semantics to Brane systems by defining them as Markov processes over the measurable space generated by terms up-to syntactic congruence, and where the measures are indexed by the actions of this new LTS. Finally, we provide an SOS presentation of this stochastic semantics, which is compositional and syntax-driven, and moreover the induced rate bisimilarity is a congruence.

Keywords: Brane Calculus, Structural Operational Semantics, Stochastic Semantics, Markov Processes, Rate Bisimilarity, Systems Biology

1. Introduction

A fundamental issue in Systems Biology is modelling the membrane interaction machinery. A cell is constructed by thousands of nested biological membranes, which can be considered as mobile containers, both coordinating the activity of the cell and transporting material within the cell. For instance, most functions of the Golgi apparatus (like protein sorting) are implemented by membrane interactions, but also viral infections, T-cells phagocytosis, etc.

Several models of membranes have been proposed in the literature [22, 31, 2]. Among them, the Brane Calculus (BC) [11] has been arisen as a good model focusing on abstract membrane interactions, still being sound with respect to biological constraints (e.g. bitonality). A process of this calculus represents a system of nested membranes, carrying their active components *on* membranes, not inside them. This reflects the biological evidence that functional molecules (proteins) are embedded in membranes, with consistent orientation.

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Email addresses: `giorgio.bacci@uniud.it` (Giorgio Bacci), `marino.miculan@uniud.it` (Marino Miculan)

Membranes interact according to three reaction rules, corresponding to phagocytosis, endo/exocytosis, and pinocytosis.

In the original definition of the Brane Calculus [11], reaction rules do not consider quantitative aspects like rates, volumes, etc. However, it is important to address these aspects, e.g. for implementing stochastic simulations, for connecting Brane Calculus with quantitative models at lower abstraction levels (such as stochastic π-calculus and κ-calculus for protein interactions), and of course for comparing with experimental observations.

In this paper, we introduce a stochastic semantics for the Brane Calculus. Clearly, a "stochastic Brane Calculus" can be obtained just by adding rates to reaction rules, similarly to what have been done for BioAmbients in [9, 7]. However, the resulting "pointwise" rated reduction semantics is not satisfactory for several reasons. First, it is not compositional: reaction rates of a process are not defined on the syntactic structure of the process, in terms of the rates of its components. Second, stochastic reaction rules are not easy to deal with in the presence of large populations of agents (as it is often the case in biological systems), because we have to count a large number of occurrences for calculating the effective reaction rates. Third, this approach does not generalize easily to other quantitative aspects (e.g. volumes).

To overcome these issues, in this paper we adopt a novel approach recently introduced by Cardelli and Mardare [12] (similar ideas have been proposed for probabilistic automata [21, 32], and Markov processes [20, 8, 25]). The main point of this approach is that the semantics of a process is a *measure* of the stochastic distribution of the possible outcomes. Thus, processes form a *measurable space*, and each process is given an action-indexed family of measures on this space. For an action a, the measure μ_a associated to a process P specifies for each measurable set S of processes, the rate $\mu_a(S) \in \mathbb{R}^+$ of a-transitions from P to (elements of) S. The resulting structures, called *Markov processes* (MPs), are not continuous-time Markov chains because each transition is from a state to a possibly infinite class of states (closed to the congruence relation over processes) and consequently cannot be described in a pointwise style. An advantage of this approach is that we can apply results from measure theory for solving otherwise difficult issues, like instance-counting problems; moreover, process measures are defined compositionally, and can be characterized also by means of operational semantics in GSOS form. Finally, other measurable aspects of processes can be dealt with along the same lines.

In order to apply the approach of [12] to Brane Calculus, we have to solve some problems; in particular, we need a finitely branching, compositional *labelled transition system* (LTS) for Brane Calculus. Defining such a LTS is easy for simple calculi like CCS, but it is much more difficult for a calculus intended to model agent mobility, like BC. The main difficulty is to describe precisely how a system can interact with the surrounding environment. The first labelled transition system for the Brane Calculus has been given in [3], but it is neither structural nor finitely branching, and hence not adequate for our purposes.

This problem has been overcome in [1], where we have introduced the first finitely branching SOS for Brane Calculus. In that work, we identified labels and transitions bearing in mind the so-called *IPO construction* [24], as done in [30, 6] for Mobile Ambients. However, an effect of this methodology is that transitions may yield *higher-order* terms, i.e., in a transition $P \xrightarrow{a} Q$ the target Q may be a system with "holes" (like π-calculus' "abstractions"), waiting to be instantiated with terms, of even other abstractions. This fact has had several consequences on the definition of syntax, bisimulation, and the Markov kernel itself: all these notions had to be accommodated in order to deal with higher-order terms.

In this paper we show that in the case of Brane Calculus, higher-order terms are an unnecessary complication. We present here an operational semantics in GSOS format, which refines the SOS given in [1], avoiding higher-order terms (replaced by *tuples* of terms), and simplifying further the labels (which will not contain processes or membranes anymore).

The theory of Stochastic Brane Calculus will benefit from this simplification in many aspects. First, we can maintain the original syntax and semantics of the calculus (which we recall in Section 2), without modifications. Secondly, also the compositional labelled transition system for BC, which we present in Section 3, is simpler, with transitions of the form $P \xrightarrow{\alpha} \langle P_1, \ldots, P_n \rangle$ with tuples in place of abstractions. This kind of transitions are called *sorted*, because the label determine the sorts in the resulting tuple. We prove that this SOS is adequate with respect to the usual reduction semantics, but moreover, the bisimilarity naturally induced by this labelled transition system turns out to be a congruence; this is important because it allows for compositional reasoning.

This compositional LTS is the starting point for defining the stochastic semantics for the Brane Calculus. To this end, we need first to introduce *sorted Markov processes* and bisimulation on them (Section 4); sorted Markov processes can be seen as the stochastic counterparts of sorted transition systems. Equipped with this theory, in Section 5 we will endow terms of Brane Calculus with a Markov kernel, which is consistent with the non-stochastic semantics (that is, a process has a transition iff the rate of that transition is not null).

After that a correct Markov kernel for Brane Calculus has been defined, we can look for a simpler presentation of the semantics of Markov processes. In Section 6 we present an SOS system capturing the Markov kernel over processes: the stochastic bisimilarity induced by this SOS corresponds to the Markov bisimilarity defined in Section 5. Therefore, this semantics can be fruitfully used for simulations, or for verifying system equivalences.

Some concluding remarks and directions for further work are in Section 7. The notions from measure theory we use in this work are recalled in Appendix Appendix A.

2. Brane Calculus

In this section we recall Cardelli's Brane Calculus [11] focusing on its basic version (without communication primitives, complexes and replication).

First, let us fix the notation we will use hereafter. Let S be a set of *sorts* (or "types"), ranged over by s, t, and T a set of S-sorted terms; for $t \in S$, $T_t \subseteq T$

denotes the set of terms of sort t. For A a set of symbols, A^* denotes the set of finite words (or lists) over A, and $\langle a_1, \ldots, a_n \rangle$ denotes a word in A^*. For a word $\langle t_1, \ldots, t_n \rangle$ in S^*, we define $T_{\langle t_1,\ldots,t_n\rangle} \triangleq T_{t_1} \times \cdots \times T_{t_n}$.

Syntax. The sorts and the set $\mathbb{B}$ of terms of Brane Calculus are the following:

$$
\begin{array}{ll}
\text{Sorts} :: S & t ::= \mathsf{sys} \ \mid \ \mathsf{mem} \\
\text{Membranes} :: \mathbb{B}_{\mathsf{mem}} & \sigma, \tau ::= \mathbf{0} \ \mid \ \sigma|\tau \ \mid \ \circlearrowright_n.\sigma \ \mid \ \circlearrowright_n^{\perp}(\tau).\sigma \ \mid \\
 & \qquad \circlearrowleft_n.\sigma \ \mid \ \circlearrowleft_n^{\perp}.\sigma \mid \circledcirc_n(\tau).\sigma \\
\text{Systems} :: \mathbb{B}_{\mathsf{sys}} & P, Q ::= \diamond \ \mid \ P \circ Q \ \mid \ \sigma \llparenthesis P \rrparenthesis
\end{array}
$$

The subscripted names n are taken from a countable set Λ. By convention we shall use M, N, ... to denote generic Brane Calculus terms in $\mathbb{B}$.

A *membrane* can be either the empty membrane $\mathbf{0}$, or the parallel composition of two membranes $\sigma|\tau$, or the action-prefixed membrane $\epsilon.\sigma$. Actions are: *phagocytosis* $\circlearrowright$, *exocytosis* $\circlearrowleft$, and *pinocytosis* $\circledcirc$. Each action but pinocytosis comes with a matching co-action, indicated by the superscript $^{\perp}$.

A *system* can be either the empty system $\diamond$, or the parallel composition $P \circ Q$, or the system nested within a membrane $\sigma \llparenthesis P \rrparenthesis$. Notice that, differently from [11], pino actions are indexed by names in Λ. In [11], names are meant only to pair up an action with its corresponding co-action, hence a pino action does not need to be indexed by any name. Actually, names can be thought of as an abstract representation of particular protein conformational shapes; hence, each name can correspond to a different biological behaviour, possibly with different kinetic performances. Therefore, if we want to observe also kinetic properties of processes, it is important to keep track of names in pino actions. We will come back on this in Sections 3 and 6 (Examples 3.8 and 6.17).

Terms can be rearranged according to a structural congruence relation; the intended meaning is that two congruent terms actually denote the same system. Structural congruence $\equiv$ is the smallest equivalence relation over $\mathbb{B}$ which satisfies the axioms and rules listed below.

$$
P \circ Q \equiv Q \circ P \qquad P \circ (Q \circ R) \equiv (P \circ Q) \circ R \qquad P \circ \diamond \equiv P
$$

$$
\sigma|\tau \equiv \tau|\sigma \qquad \sigma|(\tau|\rho) \equiv (\sigma|\tau)|\rho \qquad \sigma|\mathbf{0} \equiv \sigma
$$

$$
\mathbf{0}\llparenthesis \diamond \rrparenthesis \equiv \diamond \qquad \frac{P \equiv Q}{P \circ R \equiv Q \circ R} \qquad \frac{\sigma \equiv \tau}{\sigma|\rho \equiv \tau|\rho} \qquad \frac{P \equiv Q \quad \sigma \equiv \tau}{\sigma\llparenthesis P \rrparenthesis \equiv \tau \llparenthesis Q \rrparenthesis}
$$

$$
\frac{\alpha \in \{\circlearrowright_n, \circlearrowleft_n, \circlearrowleft_n^{\perp}\}_{n\in\Lambda} \quad \sigma \equiv \tau}{\alpha.\sigma \equiv \alpha.\tau} \qquad \frac{\beta \in \{\circlearrowright_n^{\perp}, \circledcirc_n\}_{n\in\Lambda} \quad \rho \equiv \nu \quad \sigma \equiv \tau}{\beta(\rho).\sigma \equiv \beta(\nu).\tau}
$$

Differently from [11], we allow to rearrange also the sub-membranes contained in co-phago and pino actions (by means of the last inference rule above).

Reduction Semantics. The dynamic behaviour of Brane Calculus is specified by means of a reduction semantics, defined over a reduction relation ("reaction")

$$\frac{}{\circlearrowleft_n^{\perp}(\rho).\tau|\tau_0\llparenthesis Q\rrparenthesis \circ \circlearrowleft_n.\sigma|\sigma_0\llparenthesis P\rrparenthesis \Rightarrow \tau|\tau_0\llparenthesis \rho\llparenthesis \sigma|\sigma_0\llparenthesis P\rrparenthesis\rrparenthesis \circ Q\rrparenthesis}\ (\text{red-phago})$$

$$\frac{}{\circlearrowleft_n^{\perp}.\tau|\tau_0\llparenthesis \circlearrowleft_n.\sigma|\sigma_0\llparenthesis P\rrparenthesis \circ Q\rrparenthesis \Rightarrow \sigma|\sigma_0|\tau|\tau_0\llparenthesis Q\rrparenthesis \circ P}\ (\text{red-exo})$$

$$\frac{}{\circledcirc(\rho).\sigma|\sigma_0\llparenthesis P\rrparenthesis \Rightarrow \sigma|\sigma_0\llparenthesis \rho\llparenthesis \diamond\rrparenthesis \circ P\rrparenthesis}\ (\text{red-pino}) \qquad \frac{P \Rightarrow Q}{\sigma\llparenthesis P\rrparenthesis \Rightarrow \sigma\llparenthesis Q\rrparenthesis}\ (\text{red-loc})$$

$$\frac{P \Rightarrow Q}{P \circ R \Rightarrow Q \circ R}\ (\text{red-comp}) \qquad \frac{P \equiv P' \quad P' \Rightarrow Q' \quad Q' \equiv Q}{P \Rightarrow Q}\ (\text{red-equiv})$$

Table 1: Reduction semantics for the Brane Calculus.

$\Rightarrow \subseteq \mathbb{B}_{\mathsf{sys}} \times \mathbb{B}_{\mathsf{sys}}$, whose rules are listed in Table 1. Notice that the presence of (red-phago/exo/pino) and (red-equiv) makes this not a *structural* presentation, since these rules are not primitive recursive in the syntax (i.e., structural recursive) as required by the SOS format (see [28] for a gentle introduction and [29] for further details about the origins and motivations of SOS).

3. A compositional GSOS for Brane Calculus

In this section we introduce a structural operational semantics for the Brane Calculus. The approach proposed here improves the semantics of [1] in several aspects: (i) labels are not Brane Calculus terms; (ii) we avoid "λ-abstractions", using instead *tuples* of simple terms; (iii) the associated strong bisimulation is simpler, as we do not need to close it by instantiation of the λ-abstractions. Moreover, we prove that this bisimulation is *compositional*, which means that it respects the algebraic structure of terms, i.e., it is a congruence.

The novelty of our approach derives from the format of the transitions. Transitions are not of the form $M \xrightarrow{a} M'$, as usual in process algebras, because in our case the continuation can be composed by several processes, scattered into different locations as a result of the rearrangement of the nesting structure. For this reason we use transitions of the form $M \xrightarrow{a} \langle M_1, M_2, \ldots, M_n\rangle$, where each component M_i represents a process in a different location. The number n of continuations, and their sorts, are uniquely determined by the action a.

As an example, let us consider the transition $\circlearrowleft_n.\sigma\llparenthesis P\rrparenthesis|Q \xrightarrow{\mathrm{ph}_n} \langle\sigma\llparenthesis P\rrparenthesis, Q\rangle$. The first component, $\sigma\llparenthesis P\rrparenthesis$, is the part of the system that has been phagocytized, hence ready to be moved into another (not yet known) compartment; the second component, Q, is the part of the system that has not been moved, and hence resides in a different location with respect to $\sigma\llparenthesis P\rrparenthesis$. We say that ph_n has *arity* $\mathsf{sys} \to \langle\mathsf{sys}, \mathsf{sys}\rangle$, because it labels transitions of the form $P \xrightarrow{\mathrm{ph}_n} \langle P', P''\rangle$ where the source P is a system and the continuation $\langle P', P''\rangle$ is a pair of systems.

In order to formalize this idea, we introduce the following generalization of the usual notions of labelled transition relation and system.

Definition 3.1 (Sorted labelled transition relation). *Let S be a set of sorts. An* action arity *is an element of $S \times S^*$. An action arity $\langle t, t_1, \dots, t_n\rangle$ is denoted as $t \to \langle t_1, \dots, t_n\rangle$. A set of action labels* with arities *is a set A of labels with a function $ar\colon A \to S \times S^*$.*

Let T be a set of S-sorted terms. Given $a \in A$, a sorted a-labelled transition relation *is a relation $\xrightarrow{a} \subseteq T_{ar(a)}$.*

Notice that if $ar(a) = t \to \langle t_1, \dots, t_n\rangle$, then $T_{ar(a)} = T_t \times T_{t_1} \times \cdots \times T_{t_n}$. Given $M \in T_t$ and $M_i \in T_{t_i}$ for $1 \le i \le n$, the fact "$(M, M_1, \dots, M_n) \in \xrightarrow{a}$" will be denoted by $M \xrightarrow{a} \langle M_1, \dots, M_n\rangle$.

Definition 3.2 (Sorted labelled transition system). *Let A be a set of labels with arity function $ar\colon A \to S \times S^*$. An* (sorted) A-transition system *is a pair $\mathcal{S} = (T, \{\xrightarrow{a}\}_{a\in A})$, where T is a set of S-sorted terms and for all $a \in A$, $\xrightarrow{a} \subseteq T_{ar(a)}$ is a sorted transition relation.*

Clearly, traditional labelled transition relations and systems are particular cases of these definitions, i.e. when $S = \{*\}$ and $ar(a) = * \to \langle * \rangle$ for all $a \in A$.

The set of action labels for the Brane Calculus will be denoted by $\mathbb{A}$ and can be partitioned with respect to the source sort, as follows:

$$
\begin{aligned}
\mathbb{A}_{\mathsf{sys}} \triangleq\ & \{id : \mathsf{sys} \to \langle \mathsf{sys} \rangle\} \cup \{\mathrm{ph}_n : \mathsf{sys} \to \langle \mathsf{sys}, \mathsf{sys}\rangle \mid n \in \Lambda\} \cup \\
& \{\mathrm{ph}^{\perp}_n : \mathsf{sys} \to \langle \mathsf{mem}, \mathsf{mem}, \mathsf{sys}, \mathsf{sys}\rangle \mid n \in \Lambda\} \cup \\
& \{\mathrm{ex}_n : \mathsf{sys} \to \langle \mathsf{mem}, \mathsf{sys}, \mathsf{sys}\rangle \mid n \in \Lambda\} \\
\mathbb{A}_{\mathsf{mem}} \triangleq\ & \{\circlearrowright_n, \circlearrowleft_n, \circlearrowleft^{\perp}_n : \mathsf{mem} \to \langle \mathsf{mem}\rangle \mid n \in \Lambda\} \cup \\
& \{\circlearrowright^{\perp}_n, \circledcirc_n : \mathsf{mem} \to \langle \mathsf{mem}, \mathsf{mem}\rangle \mid n \in \Lambda\}
\end{aligned}
$$

The transition system specification (TSS) for the Brane Calculus is in Table 2, and it is organized into two parts: rules for membranes and rules for systems.

The rules devoted to membrane terms are of two sorts: prefix and parallel rules. All rules are quite standard, apart from ($\circlearrowright^{\perp}$-pref) and ($\circledcirc$-pref) which decouple the argument of the actions from the membrane.

The rules for system terms are more interesting. The rule ($\circlearrowright$) describes how a phago transition at the level of membrane terms is lifted at the level of system terms; the same applies to rules ($\circlearrowright^{\perp}$), ($\circlearrowright^{\perp}$), ($\circlearrowleft$), and (id$\circledcirc$). Rules for composition, that is, (L$\circ\circlearrowright$), (L$\circ\circlearrowright^{\perp}$), (L$\circ\circlearrowleft$) and their symmetric right counterparts, extend a transition to the composition of systems. The rules (L-id$\circlearrowright$) and (R-id$\circlearrowright$) show how phago and co-phago transitions synchronize in order to cause the actual phagocytosis reaction. The rule (id$\circlearrowleft$) behaves similarly, but in this case the synchronization is between a system transition and a membrane transition. Finally, (id-loc), (L$\circ$id), and (R$\circ$id) are contextual rules, and allow to focus on the reacting parts of the system.

As a remark, we notice that this structural operational semantics formalizes what was implicitly stated by the reduction semantic in Section 2, that is, reactions that happen at the level of systems are caused only by actions on

$$\frac{}{\circlearrowleft_n.\sigma \xrightarrow{\circlearrowleft_n} \langle\sigma\rangle}\;(\circlearrowleft\text{-pref}) \qquad \frac{}{\circlearrowleft_n^{\perp}(\rho).\sigma \xrightarrow{\circlearrowleft_n^{\perp}} \langle\sigma,\rho\rangle}\;(\circlearrowleft^{\perp}\text{-pref})$$

$$\frac{}{\circledcirc_n(\rho).\sigma \xrightarrow{\circledcirc_n} \langle\sigma,\rho\rangle}\;(\circledcirc\text{-pref}) \qquad \frac{}{\circlearrowright_n.\sigma \xrightarrow{\circlearrowright_n} \langle\sigma\rangle}\;(\circlearrowright\text{-pref}) \qquad \frac{}{\circlearrowright_n^{\perp}.\sigma \xrightarrow{\circlearrowright_n^{\perp}} \langle\sigma\rangle}\;(\circlearrowright^{\perp}\text{-pref})$$

$$\frac{\sigma \xrightarrow{\alpha} \langle\sigma'\rangle}{\sigma|\tau \xrightarrow{\alpha} \langle\sigma'|\tau\rangle}\;(\text{L-par}) \qquad \frac{\tau \xrightarrow{\alpha} \langle\tau'\rangle}{\sigma|\tau \xrightarrow{\alpha} \langle\sigma|\tau'\rangle}\;(\text{R-par}) \qquad (\alpha \in \{\circlearrowleft_n, \circlearrowright_n, \circlearrowright_n^{\perp}\}_{n\in\Lambda})$$

$$\frac{\sigma \xrightarrow{\beta} \langle\sigma',\rho\rangle}{\sigma|\tau \xrightarrow{\beta} \langle\sigma'|\tau,\rho\rangle}\;(\text{L()-par}) \qquad \frac{\tau \xrightarrow{\beta} \langle\tau',\rho\rangle}{\sigma|\tau \xrightarrow{\beta} \langle\sigma|\tau',\rho\rangle}\;(\text{R()-par}) \qquad (\beta \in \{\circlearrowleft_n^{\perp}, \circledcirc_n\}_{n\in\Lambda})$$

$$\frac{\sigma \xrightarrow{\circlearrowleft_n} \langle\sigma'\rangle}{\sigma\llparenthesis P\rrparenthesis \xrightarrow{\mathrm{ph}_n} \langle\sigma'\llparenthesis P\rrparenthesis, \diamond\rangle}\;(\circlearrowleft) \qquad \frac{P \xrightarrow{\mathrm{ph}_n} \langle P', P''\rangle}{P\circ Q \xrightarrow{\mathrm{ph}_n} \langle P', P''\circ Q\rangle}\;(\text{L}\circ\circlearrowleft)$$

$$\frac{\sigma \xrightarrow{\circlearrowleft_n^{\perp}} \langle\sigma',\rho\rangle}{\sigma\llparenthesis P\rrparenthesis \xrightarrow{\mathrm{ph}_n^{\perp}} \langle\sigma',\rho,P,\diamond\rangle}\;(\circlearrowleft^{\perp}) \qquad \frac{P \xrightarrow{\mathrm{ph}_n^{\perp}} \langle\sigma,\rho,P',P''\rangle}{P\circ Q \xrightarrow{\mathrm{ph}_n^{\perp}} \langle\sigma,\rho,P',P''\circ Q\rangle}\;(\text{L}\circ\circlearrowleft^{\perp})$$

$$\frac{\sigma \xrightarrow{\circlearrowright_n} \langle\sigma'\rangle}{\sigma\llparenthesis P\rrparenthesis \xrightarrow{\mathrm{ex}_n} \langle\sigma',P,\diamond\rangle}\;(\circlearrowright) \qquad \frac{P \xrightarrow{\mathrm{ex}_n} \langle\sigma,P',P''\rangle}{P\circ Q \xrightarrow{\mathrm{ex}_n} \langle\sigma,P',P''\circ Q\rangle}\;(\text{L}\circ\circlearrowright)$$

$$\frac{P \xrightarrow{\mathrm{ph}_n} \langle P',P''\rangle \qquad Q \xrightarrow{\mathrm{ph}_n^{\perp}} \langle\tau,\rho,Q',Q''\rangle}{P\circ Q \xrightarrow{id} \langle P''\circ\tau\llparenthesis\rho\llparenthesis P'\rrparenthesis\circ Q'\rrparenthesis\circ Q''\rangle}\;(\text{L-id}\circlearrowleft)$$

$$\frac{P \xrightarrow{\mathrm{ex}_n} \langle\sigma,P',P''\rangle \qquad \tau \xrightarrow{\circlearrowright_n^{\perp}} \langle\tau'\rangle}{\tau\llparenthesis P\rrparenthesis \xrightarrow{id} \langle\sigma|\tau'\llparenthesis P''\rrparenthesis\circ P'\rangle}\;(\text{id}\circlearrowright) \qquad \frac{\sigma \xrightarrow{\circledcirc_n} \langle\sigma',\rho\rangle}{\sigma\llparenthesis P\rrparenthesis \xrightarrow{id} \langle\sigma'\llparenthesis\rho\llparenthesis\diamond\rrparenthesis\circ P\rrparenthesis\rangle}\;(\text{id}\circledcirc)$$

$$\frac{P \xrightarrow{id} \langle P'\rangle}{\sigma\llparenthesis P\rrparenthesis \xrightarrow{id} \langle\sigma\llparenthesis P'\rrparenthesis\rangle}\;(\text{id-loc}) \qquad \frac{P \xrightarrow{id} \langle P'\rangle}{P\circ Q \xrightarrow{id} \langle P'\circ Q\rangle}\;(\text{L}\circ\text{id})$$

Table 2: GSOS for Brane Calculus. Symmetric rules (R$\circ\circlearrowleft$), (R$\circ\circlearrowleft^{\perp}$), (R$\circ\circlearrowright$), (R$\circ$id), and (R-id$\circlearrowleft$) are omitted.

membranes. This dependency is not mutual, indeed systems transitions do not occur as premises in any rule for membrane terms.

An example of derivation of labelled transition is shown in Figure 1: the derivation leads to a simplified version of the (red-phago) reaction of Table 1.

The following proposition states that this SOS is adequate with respect to the reduction semantics given in Section 2.

Proposition 3.3. *For $P, Q \in \mathbb{B}_{\mathsf{sys}}$, the following hold:*

1. *If $P \xrightarrow{id} \langle Q\rangle$ then $P \Rightarrow Q$.*
2. *If $P \Rightarrow Q$ then $P \xrightarrow{id} \langle Q'\rangle$ for some $Q' \equiv Q$.*

$$
\dfrac{
  \dfrac{\dfrac{}{\text{ᘊ}_n.\sigma \xrightarrow{\text{ᘊ}_n} \langle \sigma \rangle}\,(\text{ᘊ}\text{-pref})}{\text{ᘊ}_n.\sigma \llparenthesis P \rrparenthesis \xrightarrow{\mathrm{ph}_n} \langle \sigma \llparenthesis P \rrparenthesis, \diamond \rangle}\,(\text{ᘊ})
  \qquad
  \dfrac{\dfrac{}{\text{ᘊ}^{\perp}_n(\rho).\tau \xrightarrow{\text{ᘊ}^{\perp}_n} \langle \tau, \text{ᘊ} \rangle}\,(\text{ᘊ}^{\perp}\text{-pref})}{\text{ᘊ}^{\perp}_n(\rho).\tau \llparenthesis Q \rrparenthesis \xrightarrow{\mathrm{ph}^{\perp}_n} \langle \tau, \rho, Q, \diamond \rangle}\,(\text{ᘊ}^{\perp})
}{
  \text{ᘊ}_n.\sigma \llparenthesis P \rrparenthesis \circ \text{ᘊ}^{\perp}_n(\rho).\tau \llparenthesis Q \rrparenthesis \xrightarrow{id} \langle \diamond \circ \tau \llparenthesis \rho \llparenthesis \sigma \llparenthesis P \rrparenthesis \rrparenthesis \circ Q \rrparenthesis \circ \diamond \rangle
}\,(\text{L-id}\text{ᘊ})
$$

Figure 1: Derivation for the simplified (red-phago) reaction.

The $\mathbb{A}$-transition relations for the Brane Calculus are compatible with structural congruence in the following sense:

Lemma 3.4. *Let $M, N \in \mathbb{B}$. If $M \xrightarrow{a} \langle M_1, \dots, M_n \rangle$ and $M \equiv N$ then there exist $N_1, \dots, N_n \in \mathbb{B}$ such that $N_i \equiv M_i$, for $1 \le i \le n$, and $N \xrightarrow{a} \langle N_1, \dots, N_n \rangle$.*

The sorted $\mathbb{A}$-transition system $(\mathbb{B}, \{\xrightarrow{a}\}_{a \in \mathbb{A}})$ is finitely branching, that is, for every $M \in \mathbb{B}$ there are only finitely many $a \in \mathbb{A}$ and $\langle M_1, \dots, M_n \rangle$ such that $M \xrightarrow{a} \langle M_1, \dots, M_n \rangle$. This can be readily proven by induction on the structure of M, observing that only finitely many rules can be applied.

Lemma 3.5. $(\mathbb{B}, \{\xrightarrow{a}\}_{a \in \mathbb{A}})$ *is finitely branching.*

Sorted bisimilarity. As usual, this labelled transition system induces a bisimilarity relation on terms. We show now that this bisimilarity is an equivalence relation (which is not obvious, due to the nonstandard form of the transitions) and, moreover, that it is consistent with respect to structural congruence. First, let us define formally strong bisimilarity over sorted transition systems. (This definition corresponds to the usual one in category theory, for a suitable behaviour endofunctor corresponding to sorted transition systems.)

Definition 3.6 (Strong bisimilarity). *Let $\mathcal{S} = (T, \{\xrightarrow{a}\}_{a \in A})$ be a sorted A-transition system. A binary relation $\mathcal{R} \subseteq T \times T$ over terms is a* bisimulation *iff whenever $(M, N) \in \mathcal{R}$ and $a \in A$:*

- *if $M \xrightarrow{a} \langle M_1, \dots, M_n \rangle$ then there is a transition $N \xrightarrow{a} \langle N_1, \dots, N_n \rangle$ such that for each $1 \le i \le n$, $(M_i, N_i) \in \mathcal{R}$;*
- *if $N \xrightarrow{a} \langle N_1, \dots, N_n \rangle$ then there is a transition $M \xrightarrow{a} \langle M_1, \dots, M_n \rangle$ such that for each $1 \le i \le n$, $(M_i, N_i) \in \mathcal{R}$.*

Two terms M and N are strong bisimilar*, written $M \sim N$, iff there is a bisimulation that relates them.*

The next proposition states some properties about $\sim$ which hold for general sorted labelled transition systems.

Proposition 3.7. *The following statements about strong bisimilarity hold:*

1. *$\sim$ is an equivalence relation;*
2. *$\sim$ is the largest bisimulation relation;*

3. $M \sim N$ *iff for each action* $a \in \mathbb{A}$

   – *if* $M \xrightarrow{a} \langle M_1, \ldots, M_n \rangle$ *then there is a transition* $N \xrightarrow{a} \langle N_1, \ldots, N_n \rangle$ *such that for each* $1 \leq i \leq n$, $M_i \sim N_i$*;*
   – *if* $N \xrightarrow{a} \langle N_1, \ldots, N_n \rangle$ *then there is a transition* $M \xrightarrow{a} \langle M_1, \ldots, M_n \rangle$ *such that for each* $1 \leq i \leq n$, $M_i \sim N_i$*;*

4. *if* $M \sim N$ *then* M *and* N *are of the same sort.*

Before beginning to explore the properties of strong bisimilarity in the specific setting of Brane Calculus, let us see some simple examples.

Example 3.8. *Let* $n, m \in \Lambda$. *Consider the system terms* $P = \circledcirc_n(\rho).\sigma \llparenthesis R \rrparenthesis$ *and* $Q = \circledcirc_m(\rho).\sigma \llparenthesis R \rrparenthesis$. *We prove that* $P \sim Q$.

Let us consider the binary relation $\mathcal{R} = \Delta_{\mathbb{B}} \cup \mathcal{P}$ *where* $\Delta_{\mathbb{B}}$ *denotes the identity relation over* $\mathbb{B}$ *and* $\mathcal{P}$ *is defined by*

$$\mathcal{P} = \{(\circledcirc_{n'}(\rho').\sigma' \llparenthesis R' \rrparenthesis, \circledcirc_{m'}(\rho').\sigma' \llparenthesis R' \rrparenthesis) \mid \sigma', \rho', R' \in \mathbb{B} \text{ and } n', m' \in \Lambda\}$$

It is easy to see that $(P, Q) \in \mathcal{R}$, *hence to prove* $P \sim Q$ *it suffices to show that* $\mathcal{R}$ *is a bisimulation. Suppose* $(P', Q') \in \mathcal{P}$ *(the case* $(P', Q') \in \Delta_{\mathbb{B}}$ *is trivial), hence* $P' = \circledcirc_{n'}(\rho').\sigma' \llparenthesis R' \rrparenthesis$ *and* $Q' = \circledcirc_{m'}(\rho').\sigma' \llparenthesis R' \rrparenthesis$ *for some* $\sigma', \rho', R' \in \mathbb{B}$ *and* $n', m' \in \Lambda$. *Assume* $P' \xrightarrow{a} \langle P'_1, \ldots, P'_k \rangle$. *By a structural analysis on* P', *the transition admits two possible forms depending on the last rule applied to derive it. We consider the two cases separately:*

Rule (id-loc): $a = id$, $k = 1$ *and* $P'_1 = \circledcirc_{n'}(\rho').\sigma' \llparenthesis R'' \rrparenthesis$ *for some* R'' *such that* $R' \xrightarrow{id} \langle R'' \rangle$. *Using the transition* $R' \xrightarrow{id} \langle R'' \rangle$ *as premise in rule (id-loc) we can infer that* $Q' \xrightarrow{id} \langle Q'_1 \rangle$, *where* $Q'_1 = \circledcirc_{m'}(\rho').\sigma' \llparenthesis R'' \rrparenthesis$. *It is easy to see that* $(P'_1, Q'_1) \in \mathcal{R}$.

Rule (id$\circledcirc$): $a = id$, $k = 1$ *and* $P'_1 = \sigma' \llparenthesis \rho' \llparenthesis \diamond \rrparenthesis \circ R' \rrparenthesis$. *From axiom ($\circledcirc$-pref) we infer the transition* $\circledcirc_{m'}(\rho').\sigma \xrightarrow{\circledcirc_{m'}} \langle \sigma', \rho' \rangle$ *and using it as premise in rule (id$\circledcirc$) we can derive the transition* $Q' \xrightarrow{id} \langle Q'_1 \rangle$, *where* $Q'_1 = \sigma' \llparenthesis \rho' \llparenthesis \diamond \rrparenthesis \circ R' \rrparenthesis$. *Since* $P'_1 = Q'_1$, *we have that* $(P'_1, Q'_1) \in \mathcal{R}$.

The same argument holds assuming $Q' \xrightarrow{a} \langle Q'_1, \ldots, Q'_k \rangle$, *hence we are done.*

The above example explains the reason why the original formulation of Brane Calculus [11] does not consider names for pino actions: in a non-stochastic setting, all pino actions are observationally equivalent (they simply "happen"). On the other hand, in a stochastic setting we can observe also the rate which these reactions take place at; therefore, we have to distinguish among different kinds of pino actions, hence the names.

Example 3.9. *Consider the Brane Calculus systems* $P = \circledcirc_n(\sigma) | \circledcirc_n(\sigma) \llparenthesis R \rrparenthesis$ *and* $Q = \circledcirc_n(\sigma).\circledcirc_n(\sigma) \llparenthesis R \rrparenthesis$. *We prove that* $P \sim Q$.

Let us consider the binary relation $\mathcal{R} = \Delta_{\mathbb{B}} \cup \mathcal{R}'$ where $\Delta_{\mathbb{B}}$ denotes the identity relation over $\mathbb{B}$ and $\mathcal{R}'$ is defined by

$$\mathcal{R}' = \{(\circledcirc_{n'}(\sigma')|\circledcirc_{n'}(\sigma')\llparenthesis R' \rrparenthesis, \circledcirc_{n'}(\sigma').\circledcirc_{n'}(\sigma')\llparenthesis R' \rrparenthesis) \mid \sigma', R' \in \mathbb{B} \text{ and } n' \in \Lambda\}$$

One should readily notice that $(P, Q) \in \mathcal{R}$, hence it suffices to prove that $\mathcal{R}$ is a bisimulation. Suppose $(P', Q') \in \mathcal{R}'$ (the case $(P', Q') \in \Delta_{\mathbb{B}}$ is trivial), hence $P' = \circledcirc_{n'}(\sigma')|\circledcirc_{n'}(\sigma')\llparenthesis R' \rrparenthesis$ and $Q' = \circledcirc_{n'}(\sigma').\circledcirc_{n'}(\sigma')\llparenthesis R' \rrparenthesis$ for some $\sigma', R' \in \mathbb{B}$ and $n' \in \Lambda$. Assume $P' \xrightarrow{a} \langle P'_1, \ldots, P'_k \rangle$. By a structural analysis on P', the transition can be inferred applying either the rule (id-loc) or the rule (id$\circledcirc$). We consider the two cases separately:

Rule (id-loc): *$a = id$, $k = 1$ and $P'_1 = \circledcirc_{n'}(\sigma')|\circledcirc_{n'}(\sigma')\llparenthesis R'' \rrparenthesis$ for some R'' such that $R' \xrightarrow{id} \langle R'' \rangle$. Using the transition $R' \xrightarrow{id} \langle R'' \rangle$ as premise in rule (id-loc) we can infer $Q' \xrightarrow{id} \langle Q'_1 \rangle$, where $Q'_1 = \circledcirc_{n'}(\sigma').\circledcirc_{n'}(\sigma')\llparenthesis R'' \rrparenthesis$. It is easy to see that $(P'_1, Q'_1) \in \mathcal{R}$.*

Rule (id$\circledcirc$): *$a = id$, $k = 1$ and $P'_1 = \circledcirc_{n'}(\sigma')\llparenthesis \sigma' \llparenthesis \diamond \rrparenthesis \circ R' \rrparenthesis$. From the axiom ($\circledcirc$-pref) we can infer the transition $\circledcirc_{n'}(\sigma').\circledcirc_{n'}(\sigma') \xrightarrow{\circledcirc_{n'}} \langle \circledcirc_{n'}(\sigma'), \sigma' \rangle$, which can be used as premise in rule (id$\circledcirc$) to derive $Q' \xrightarrow{id} \langle Q'_1 \rangle$, where $Q'_1 = \circledcirc_{n'}(\sigma')\llparenthesis \sigma' \llparenthesis \diamond \rrparenthesis \circ R' \rrparenthesis$. Since $P'_1 = Q'_1$, we have $(P'_1, Q'_1) \in \mathcal{R}$.*

A similar argument holds assuming $Q' \xrightarrow{a} \langle Q'_1, \ldots, Q'_k \rangle$. The only difference is when the derivation ends with an application of rule (id$\circledcirc$); in that case, in order to construct the transition for P', we have to apply also the rule (L()-par).

Next we show that bisimilarity respects structural congruence, i.e. $\equiv \subseteq \sim$.

Lemma 3.10. *If $M \equiv N$ then $M \sim N$.*

Proof. It suffices to show that $\equiv$ is a strong bisimulation. The proof is by induction on the derivation of $\equiv$. □

Actually a stronger result holds: as shown by Examples 3.8 and 3.9, structural equivalence does not coincide with strong bisimulation, and in particular $\sim$ equates more terms than $\equiv$, that is $\equiv \subsetneq \sim$.

We conclude this section with the important result that $\sim$ is a congruence, i.e., it behaves well with respect to the algebraic structure of terms.

Theorem 3.11 (Congruence). *Let $\sigma, \tau, \rho, \rho' \in \mathbb{B}_{\mathsf{mem}}$ and $P, Q, R \in \mathbb{B}_{\mathsf{sys}}$. Assume that $\sigma \sim \tau$, $\rho \sim \rho'$, and $P \sim Q$, then the following statements hold:*

1. *$\alpha.\sigma \sim \alpha.\tau$ for each $\alpha \in \{\circlearrowright_n, \circlearrowleft_n, \circlearrowright_n^{\perp} \mid n \in \Lambda\}$,*
2. *$\beta(\rho).\sigma \sim \beta(\rho').\tau$ for each $\beta \in \{\circlearrowright_n^{\perp}, \circledcirc_n \mid n \in \Lambda\}$.*
3. *$\sigma|\rho \sim \tau|\rho$ and $\rho|\sigma \sim \rho|\tau$,*
4. *$\rho\llparenthesis P \rrparenthesis \sim \rho\llparenthesis Q \rrparenthesis$,*
5. *$P \circ R \sim Q \circ R$ and $R \circ P \sim R \circ Q$.*

Remark 3.12. The category theory *cognoscenti* will notice that the rules in Table 2 adhere the *abstract GSOS* specification of [33], for some suitable behaviour functor. Many results above (e.g. Theorem 3.11) could be obtained more directly within that theoretical framework, but we leave this discussion to future work.

4. Sorted Markov kernels, processes and rate bisimulation

In this section we develop a brief theory of *sorted Markov kernels* and processes, generalizing similar constructions in [12], which in turn are based on an equivalence between Markov process and Harsanyi type spaces [19]. This theory will be needed in the next section for giving the stochastic semantics of Brane Calculus. We assume the reader to be familiar with basic notions from measure theory; for a brief summary, see Appendix Appendix A.

We start introducing the notation used hereafter. As usual $\mathbb{R}$ denotes set of real numbers, $\mathbb{R}^+$ the set of positive real numbers with zero, and $\mathbb{R}^+_\infty$ its extension with ∞, assumed to be strictly greater than all $r \in \mathbb{R}^+$. Let $\{A_i\}_{i=1}^n$ be a finite family of nonempty sets, we call *rectangle* any subset $R \subseteq A_1 \times \cdots \times A_n$ of the form $R = R_1 \times \cdots \times R_n$, where $R_i \subseteq A_i$ for all $1 \leq i \leq n$.

Given a measurable space (M, Σ), the elements of Σ are called *measurable sets* and M the *support-set*. Let $\{(M_i, \Sigma_i)\}_{i=1}^n$ be a finite family of measurable spaces, we call *measurable rectangles* any rectangle in $\{\Sigma_i\}_{i=1}^n$, and the collection of all such rectangles is denoted by $\coprod_{i=1}^n \Sigma_i$. The *product σ-algebra* of $\{\Sigma_i\}_{i=1}^n$, denoted by $\bigotimes_{i=1}^n \Sigma_i$, is the smallest σ-algebra generated by measurable rectangles, and $(\prod_{i=1}^n M_i, \bigotimes_{i=1}^n \Sigma_i)$ denotes the *product measurable space* of $\{(M_i, \Sigma_i)\}_{i=1}^n$. Given two measurable spaces (M, Σ) and (N, Θ) a mapping $f\colon M \to N$ is *measurable* if for any $\mathcal{N} \in \Theta$, $f^{-1}(\mathcal{N}) \in \Sigma$.

Let $\Delta(M, \Sigma)$ be the class of measures $\mu\colon \Sigma \to \mathbb{R}^+_\infty$ on (M, Σ). $\Delta(M, \Sigma)$ can be organized into a measurable space where its σ-algebra is the one generated by the sets $\{\mu \in \Delta(M, \Sigma) \mid \mu(S) \geq r\}$ for $S \in \Sigma$ and $r > 0$. From $\Delta(M, \Sigma)$ we distinguish the *null measure* ω for which $\omega(\mathcal{M}) = 0$ for all $\mathcal{M} \in \Sigma$.

Let us introduce *sorted Markov processes* and stochastic bisimulation on them. These structures are the stochastic counterparts of sorted transition systems and strong bisimulation on them. We propose a definition of Markov kernel that extends that of [12] to measurable spaces over sorted sets. In particular, notice that if (T, Σ) is a measurable space over a sorted set T, (T_t, Σ_t) is a well defined measurable space, for some sort t and $\Sigma_t = \{\mathcal{M}_t \mid \mathcal{M} \in \Sigma\}$.

Definition 4.1 (Sorted Markov processes). *Let S be a set of sorts, (T, Σ) be a measurable space over a set of S-sorted terms T, and A a set of action labels with arity function $ar\colon A \to S \times S^*$. A* (sorted) A-Markov kernel *is a tuple $\mathcal{M} = (T, \Sigma, \{\theta_a\}_{a \in A})$, where for all $a \in A$, given $ar(a) = t \to \langle t_1, \dots t_n \rangle$,*

$$\theta_a\colon T_t \to \Delta(T_{\langle t_1, \dots t_n \rangle}, \textstyle\bigotimes_{i=1}^n \Sigma_{t_i})$$

is a measurable function, said Markov a-transition function. *An* A-Markov process *of $\mathcal{M}$ with $M \in T$ as* initial state, *written $(\mathcal{M}, M)$, is the tuple (T, Σ, θ, M).*

The adjective "Markovian" is usually employed in the probabilistic setting; here it indicates that the transitions depend entirely on the present state and not on the past history of the system. Interactions among processes are represented as in process algebras: the labels in A represent all possible interactions of processes with the environment. If $a \in A$ is a label, $ar(a) = t \to \langle t_1, \dots, t_n \rangle$, $M \in T_t$ is the current state of the system, and $\mathcal{M} \in \bigotimes_{i=1}^n \Sigma_{t_i}$ is a measurable set of $(T_{\langle t_1,\dots,t_n\rangle}, \bigotimes_{i=1}^n \Sigma_{t_i})$, the function $\theta_a(M)$ is a measure and $\theta_a(M)(\mathcal{M}) \in \mathbb{R}^+$ represents the *rate* of an exponentially distributed random variable characterizing the duration of an a-transition from M to arbitrary elements in $\mathcal{M}$.

We can now introduce a notion of rate bisimulation between Markov processes. Given a binary relation $\mathcal{R} \subseteq T \times T$ and two subsets $X, Y \subseteq T$, the pair (X, Y) is said $\mathcal{R}$-*closed* if and only if

$$\mathcal{R} \cap (X \times T) = \mathcal{R} \cap (T \times Y)\,.$$

Lemma 4.2. *Let $\mathcal{R}, \mathcal{R}' \subseteq T \times T$ such that $\mathcal{R}' \subseteq \mathcal{R}$. If (X, Y) is $\mathcal{R}$-closed, then (X, Y) is also $\mathcal{R}'$-closed.*

Lemma 4.3. *For $\mathcal{R} \in T \times T$ an equivalence relation, if (X, Y) are $\mathcal{R}$-closed then $X = Y$, moreover X is a reunion of $\mathcal{R}$-equivalence classes.*

For a measurable space (T, Σ) and a binary relation $\mathcal{R} \subseteq T \times T$, we define the set $\Sigma(\mathcal{R}) \triangleq \{(X, Y) \mid (X, Y)\ \mathcal{R}\text{-closed and } X, Y \in \Sigma\}$ as the collection of measurable $\mathcal{R}$-closed pairs of measurable sets in Σ.

Definition 4.4 (Stochastic bisimulation). *Let T be a set of S-sorted terms and $\mathcal{M} = (T, \Sigma, \{\theta_a\}_{a \in A})$ be an A-Markov kernel. A binary relation $\mathcal{R} \subseteq T \times T$ is a* rate bisimulation *iff whenever $(M, N) \in \mathcal{R}$, $a \in A$, $ar(a) = t \to \langle t_1, \dots, t_n \rangle$, and $(\mathcal{M}_i, \mathcal{N}_i) \in \Sigma_{t_i}(\mathcal{R})$ for $1 \le i \le n$:*

$$\theta_a(M)(\mathcal{M}_1 \times \cdots \times \mathcal{M}_n) = \theta_a(N)(\mathcal{N}_1 \times \cdots \times \mathcal{N}_n)\,.$$

Two Markov processes $(\mathcal{M}, M)$ and $(\mathcal{M}, N)$ are stochastic bisimilar*, written $M \sim_{\mathcal{M}} N$, if they are related by a rate bisimulation.*

Restricted to the case of simple A-Markov kernel $(M, \Sigma, \{\theta_a\}_{a \in A})$ (i.e. when $S = \{*\}$ and $ar(a) = * \to \langle * \rangle$ for all $a \in A$) and equivalence relations $R \subseteq T \times T$, Definition 4.4 coincides with the definition of rate bisimulation by Cardelli and Mardare in [12]; moreover it generalizes the definition of probabilistic bisimulation by Larsen and Skou [23] into the stochastic setting.

Remark 4.5. Definition 4.4 could be generalized further, in order to relate arbitrary A-Markov kernels $(M, \Sigma, \{\alpha_a\}_{a \in A})$ and $(N, \Theta, \{\beta_a\}_{a \in A})$, following [4, 16]. In fact, bisimulations as in Definition 4.4 do not correspond to the coalgebraic bisimulations arising from a stochastic behaviour functor in a suitable category of measurable spaces; still, these two approaches yield the same bisimilarity [16].

A natural but not trivial question is whether stochastic bisimilarity is an equivalence relation. This is proved in the next proposition.

Proposition 4.6. *Let $\mathcal{M} = (T, \Sigma, \{\theta_a\}_{a\in A})$ be a sorted A-Markov kernel, then $\sim_{\mathcal{M}}$ is an equivalence relation.*

Proof. Symmetry is trivial: it is easy to check that if $\mathcal{R}$ is a rate bisimulation, then so is $\mathcal{R}^{-1} = \{(N, M) \mid (M, N) \in \mathcal{R}\}$.

For reflexivity, we have to prove that the identity relation Δ_T is a rate bisimulation, i.e., we need to prove that for all $(M, N) \in \Delta_T$, $a \in A$ with $ar(a) = t \to \langle t_1, \dots, t_n\rangle$, and $(\mathcal{M}_i, \mathcal{N}_i) \in \Sigma_{t_i}(\Delta_T)$ for $1 \leq i \leq n$:

$$\theta_a(M)(\mathcal{M}_1 \times \cdots \times \mathcal{M}_n) = \theta_a(M)(\mathcal{N}_1 \times \cdots \times \mathcal{N}_n)\,. \tag{1}$$

But for all $(\mathcal{M}_i, \mathcal{N}_i) \in \Sigma_{t_i}(\Delta_T)$, it is $\mathcal{M}_i = \mathcal{N}_i$, because Δ_T is an equivalence and by Lemma 4.3; hence equation (1) trivially holds.

There remains to prove transitivity. To this end, it suffices to show that, given $\mathcal{R}_1$ and $\mathcal{R}_2$ rate bisimulations, there exists a rate bisimulation $\mathcal{R}$ that contains the composition relation of $\mathcal{R}_1$ and $\mathcal{R}_2$, i.e.,

$$\mathcal{R}_1;\mathcal{R}_2 \triangleq \{(M, O) \mid (M, N) \in \mathcal{R}_1 \text{ and } (N, O) \in \mathcal{R}_2 \text{ for some } N \in T\}\,,$$

Let $\mathcal{R}$ be the (unique) smallest equivalence relation containing $\mathcal{R}_1 \cup \mathcal{R}_2$; this can be defined as $\mathcal{R} = \Delta_T \cup \bigcup_{m\in\mathbb{N}} \mathcal{S}_m$, where

$$\mathcal{S}_0 \triangleq \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_1^{-1} \cup \mathcal{R}_2^{-1} \qquad\qquad \mathcal{S}_{m+1} \triangleq \mathcal{S}_m;\mathcal{S}_m\,.$$

It is easy to see that $\mathcal{R}_1;\mathcal{R}_2 \subseteq \mathcal{R}$; we are left to show that $\mathcal{R}$ is indeed a rate bisimulation. By Lemma 4.3, it suffices to prove that for all $a \in A$, where $ar(a) = t \to \langle t_1, \dots, t_n\rangle$, and $(\mathcal{C}_i, \mathcal{C}_i) \in \Sigma_{t_i}(\mathcal{R})$ for $1 \leq i \leq n$:

$$\text{for all } (M, N) \in \mathcal{R}: \quad \theta_a(M)(\mathcal{C}_1 \times \cdots \times \mathcal{C}_n) = \theta_a(N)(\mathcal{C}_1 \times \cdots \times \mathcal{C}_n)\,. \tag{$*$}$$

Now, if $(M, N) \in \mathcal{R}$, then $(M, N) \in \Delta_T$ or $(M, N) \in \mathcal{S}_m$ for some $m \geq 0$. If $(M, N) \in \Delta_T$ then $M = N$ hence equation $(*)$ trivially holds. We show now, by induction on $m \geq 0$, that for all $(M, N) \in \mathcal{S}_m$, equation $(*)$ holds.

Base case ($m = 0$): for all $(M, N) \in \mathcal{R}_j$ ($j = 1, 2$), equation $(*)$ holds since, by Lemma 4.2 and $\mathcal{R}_j \subseteq \mathcal{R}$, $(\mathcal{C}_i, \mathcal{C}_i) \in \Sigma_{t_i}(\mathcal{R}_j)$, for all $1 \leq i \leq n$, and by the hypothesis that $\mathcal{R}_j$ is a rate bisimulation. For all $(M, N) \in \mathcal{R}_j^{-1}$ ($j = 1, 2$) we have that $(N, M) \in \mathcal{R}_j$, hence $(*)$ holds too.

Inductive case ($m + 1$): for $m \geq 0$, the inductive hypothesis is that

$$\text{for all } (M', N') \in \mathcal{S}_m: \quad \theta_a(M')(\mathcal{C}_1 \times \cdots \times \mathcal{C}_n) = \theta_a(N')(\mathcal{C}_1 \times \cdots \times \mathcal{C}_n)\,. \tag{2}$$

Then, it is easy to see that equation $(*)$ holds for all $(M, N) \in \mathcal{S}_{m+1}$: by definition, there exists some $O \in T$ such that $(M, O) \in \mathcal{S}_m$ and $(O, N) \in \mathcal{S}_m$, and hence the following are two applications of equation (2):

$$\theta_a(M)(\mathcal{C}_1 \times \cdots \times \mathcal{C}_n) = \theta_a(O)(\mathcal{C}_1 \times \cdots \times \mathcal{C}_n) = \theta_a(N)(\mathcal{C}_1 \times \cdots \times \mathcal{C}_n)\,. \quad \square$$

Remark 4.7. We would like to stress that transitivity of $\sim_{\mathcal{M}}$ is not obvious. In [25, 18] bisimilarity is expressed as a span of *zigzag morphisms* between (probabilistic) labelled Markov processes, and in order to obtain transitivity they restrict to analytic spaces (it is not known yet whether bisimilarity is transitive for generic measurable spaces). Subsequently, a dual notion called *event bisimulation* or *probabilistic co-congruence*, ensuring transitivity for general measurable spaces, was proposed independently by Danos et al. [15] and by Bartels et al. [5], and recently redeveloped in [13, 14]. One may think to recast those definitions into the stochastic setting, and define a kind of *event stochastic bisimilarity.* However, this is out of the scope of this paper, and will be left as future work.

Proposition 4.8. *Let $\mathcal{M} = (T, \Sigma, \{\theta_a\}_{a\in A})$ be a sorted A-Markov kernel, then $\sim_{\mathcal{M}}$ is the largest bisimulation relation.*

Proof. We aim at showing that $\sim_{\mathcal{M}}$ is the largest rate bisimulation over T. By definition we have:

$$\sim_{\mathcal{M}}= \bigcup\{\mathcal{R} \subseteq T \times T \mid \mathcal{R} \text{ is a rate bisimulation}\}\,. \tag{3}$$

This yields immediately that each bisimulation is included in $\sim_{\mathcal{M}}$. Let us denote by $\mathcal{R}_\cup$ the right-hand side of equation (3). We are left to show that $\mathcal{R}_\cup$ is a rate bisimulation, i.e., we need to prove that for all $(M, N) \in \mathcal{R}_\cup$, $a \in A$ with $ar(a) = t \to \langle t_1, \dots, t_n\rangle$, and $(\mathcal{M}_i, \mathcal{N}_i) \in \Sigma_{t_i}(\mathcal{R}_\cup)$ for $1 \le i \le n$:

$$\theta_a(M)(\mathcal{M}_1 \times \dots \times \mathcal{M}_n) = \theta_a(N)(\mathcal{N}_1 \times \dots \times \mathcal{N}_n)\,.$$

Let $(M', N') \in \mathcal{R}_\cup$. By definition there exists a rate bisimulation $\mathcal{R}$ such that $(M', N') \in \mathcal{R}$. By Lemma 4.2, for all pairs $(\mathcal{M}_i, \mathcal{N}_i) \in \Sigma_{t_i}(\mathcal{R}_\cup)$ we have $(\mathcal{M}_i, \mathcal{N}_i) \in \Sigma_{t_i}(\mathcal{R})$, and since $\mathcal{R}$ is a rate bisimulation we are done. ☐

Note that the proof above states also that a relation $\mathcal{R}_\cup$ that consists of reunions of rate bisimulations relations (i.e. $\mathcal{R}_\cup = \bigcup_{i\in I}\{\mathcal{R}_i\}$ such that for $i \in I$, $\mathcal{R}_i$ is a rate bisimulation) is itself a rate bisimulation relation.

It turns out that stochastic bisimilarity can be characterized as follows:

Proposition 4.9. *Let $\mathcal{M} = (T, \Sigma, \{\theta_a\}_{a\in A})$ be a sorted A-Markov kernel, and $M, N \in T_t$, then $M \sim_{\mathcal{M}} N$ iff for all $a \in A$ with $ar(a) = t \to \langle t_1, \dots, t_n\rangle$, and $(\mathcal{C}_i, \mathcal{C}_i) \in \Sigma_{t_i}(\sim_{\mathcal{M}})$ for $1 \le i \le n$, $\theta_a(M)(\mathcal{C}_1 \times \dots \times \mathcal{C}_n) = \theta_a(N)(\mathcal{C}_1 \times \dots \times \mathcal{C}_n)$.*

Proof. The implication from left to right is an immediate consequence of the fact that $\sim_{\mathcal{M}}$ is an equivalence relation (by Proposition 4.6) and that $\sim_{\mathcal{M}}$ is a rate bisimulation (by Proposition 4.8). We are left to prove the implication from right to left. To this end, assume $M, N \in T_t$ having the following property:

$$\text{for all } a \in A,\ ar(a) = t \to \langle t_1, \dots, t_n\rangle, \text{ and } (\mathcal{C}_i, \mathcal{C}_i) \in \Sigma_{t_i}(\sim_{\mathcal{M}}), \text{ for } 1 \le i \le n,$$
$$\theta_a(M)(\mathcal{C}_1 \times \dots \times \mathcal{C}_n) = \theta_a(N)(\mathcal{C}_1 \times \dots \times \mathcal{C}_n)\,.$$

Let us call it $(*)$. We shall prove that $M \sim_{\mathcal{M}} N$ showing a rate bisimulation $\mathcal{R}$ such that $(M, N) \in \mathcal{R}$. Let $\mathcal{R}$ be the smallest equivalence relation that contains (M, N) and $\sim_{\mathcal{M}}$; this can be defined as $\mathcal{R} = \Delta_T \cup \bigcup_{m \in \mathbb{N}} \mathcal{S}_m$, where

$$\mathcal{S}_0 \triangleq \{(M, N), (N, M)\} \cup \sim_{\mathcal{M}} \qquad\qquad \mathcal{S}_{m+1} \triangleq \mathcal{S}_m ; \mathcal{S}_m \,.$$

(";" denotes relation composition). By Lemma 4.3, it suffices to prove that for all $a \in A$, with $ar(a) = t \to \langle t_1, \ldots, t_n \rangle$, and $(\mathcal{C}'_i, \mathcal{C}'_i) \in \Sigma_{t_i}(\mathcal{R})$ for $1 \leq i \leq n$:

$$\text{for all } (M', N') \in \mathcal{R} : \quad \theta_a(M')(\mathcal{C}'_1 \times \cdots \times \mathcal{C}'_n) = \theta_a(N')(\mathcal{C}'_1 \times \cdots \times \mathcal{C}'_n)\,. \quad (4)$$

If $(M', N') \in \mathcal{R}$, then $(M', N') \in \Delta_T$ or $(M', N') \in \mathcal{S}_m$ for some $m \geq 0$. If $(M', N') \in \Delta_T$ then $M' = N'$ hence equation (4) trivially holds. We show now, by induction on $m \geq 0$, that for all $(M', N') \in \mathcal{S}_m$, equation (4) holds.

Base case ($m = 0$): for all $(M', N') \in \sim_{\mathcal{M}}$, equation (4) holds since by Lemma 4.2 and $\sim_{\mathcal{M}} \subseteq \mathcal{R}$, $(\mathcal{C}'_i, \mathcal{C}'_i) \in \Sigma(\sim_{\mathcal{M}})$, for all $1 \leq i \leq n$, and by Proposition 4.8, $\sim_{\mathcal{M}}$ is a rate bisimulation relation. If $M' = M$ (resp. $M' = N$) and $N' = N$ (resp. $N' = M$), then property $(*)$ holds. Again, by Lemma 4.2 and $\sim_{\mathcal{M}} \subseteq \mathcal{R}$, we have $(\mathcal{C}'_i, \mathcal{C}'_i) \in \Sigma(\sim_{\mathcal{M}})$, hence equation (4) holds trivially.

Inductive case ($m + 1$): for $m \geq 0$, the inductive hypothesis is that

$$\text{for all } (M'', N'') \in \mathcal{S}_m : \quad \theta_a(M'')(\mathcal{C}'_1 \times \cdots \times \mathcal{C}'_n) = \theta_a(N'')(\mathcal{C}'_1 \times \cdots \times \mathcal{C}'_n)\,. \quad (5)$$

Then, it is easy to see that equation (4) holds for all $(M', N') \in \mathcal{S}_{m+1}$: by definition, there exists some $O \in T$ such that $(M', O) \in \mathcal{S}_m$ and $(O, N') \in \mathcal{S}_m$, and hence the following are two applications of equation (5):

$$\theta_a(M')(\mathcal{C}'_1 \times \cdots \times \mathcal{C}'_n) = \theta_a(O)(\mathcal{C}'_1 \times \cdots \times \mathcal{C}'_n) = \theta_a(N')(\mathcal{C}'_1 \times \cdots \times \mathcal{C}'_n)\,. \quad \square$$

5. A Stochastic Semantics for Brane Calculus

In this section we present a stochastic semantics for the Brane Calculus, showing how it can be organized as a sorted A-Markov kernel.

To ease the reading in the following we will use the notation $\Delta_a(T, \Sigma)$ to denote the set $\Delta(T_{\langle t_1, \ldots t_n \rangle}, \bigotimes_{i=1}^{n} \Sigma_{t_i})$, for $ar(a) = t \to \langle t_1, \ldots, t_n \rangle$. Let $\mathbb{B}/_{\equiv}$ be the set of $\equiv$-equivalence classes on $\mathbb{B}$. For $M \in \mathbb{B}$, we denote by $[M]_{\equiv}$ the $\equiv$-equivalence class of M.

Definition 5.1 (Measurable space of terms). *The* measurable space of terms $(\mathbb{B}, \Pi)$ *is given by the measurable space over* $\mathbb{B}$ *where* Π *is the* σ*-algebra generated by* $\mathbb{B}/_{\equiv}$.

Notice that $\mathbb{B}/_{\equiv}$ is a denumerable partition of $\mathbb{B}$, hence it is a base (a generator such that all its elements are disjoint) for Π. Any element of Π can be obtained by a countable union of elements of the base, i.e., for all $\mathcal{M} \in \Pi$ there exist $\{M_i\}_{i \in I}$, for some countable I, such that $\mathcal{M} = \bigcup_{i \in I} [M_i]_{\equiv}$. As a consequence, in order to generate the whole Π we can simply compute all these unions, without the need of any closure by complement.

A similar argument holds for the product space $(\mathbb{B}_{\langle t_1,\dots,t_n\rangle}, \bigotimes_{i=1}^n \Pi_{t_i})$, where $t_i \in \{\mathsf{mem}, \mathsf{sys}\}$ $(1 \le i \le n)$; indeed $\bigotimes_{i=1}^n \Pi_{t_i}$ can be generated from the base $\mathbb{B}_{\langle t_1,\dots,t_n\rangle}/_{\equiv_{\langle t_1,\dots,t_n\rangle}}$, where $\equiv_{\langle t_1,\dots,t_n\rangle} \subseteq \mathbb{B}_{\langle t_1,\dots,t_n\rangle} \times \mathbb{B}_{\langle t_1,\dots,t_n\rangle}$ is defined by

$$\langle M_1,\dots,M_n\rangle \equiv_{\langle t_1,\dots,t_n\rangle} \langle N_1,\dots,N_n\rangle \quad \text{iff} \quad M_i \equiv N_i, \text{ for all } 1 \le i \le n\,,$$

which can be easily checked to be an equivalence relation. $\equiv_{\langle t_1,\dots,t_n\rangle}$-equivalence classes are rectangles, i.e. $[\langle M_1,\dots,M_n\rangle]_{\equiv_{\langle t_1,\dots,t_n\rangle}} = [M_1]_\equiv \times \cdots \times [M_n]_\equiv$, therefore the product measure $\bigotimes_{i=1}^n \Pi_{t_i}$ is well defined. For sake of simplicity in the following we write $[\langle M_1,\dots,M_n\rangle]_\equiv$ in place of $[\langle M_1,\dots,M_n\rangle]_{\equiv_{\langle t_1,\dots,t_n\rangle}}$, and $\mathbb{B}_{\langle t_1,\dots,t_n\rangle}/_\equiv$ in place of $\mathbb{B}_{\langle t_1,\dots,t_n\rangle}/_{\equiv_{\langle t_1,\dots,t_n\rangle}}$.

The definition of the Markov kernel for the Brane Calculus will be guided by the rules given in Table 2, and in particular we will use the same set of action labels $\mathbb{A}$ (with the same arity function). Except for the silent action id, each label is subscripted by a name $n \in \Lambda$ that distinguishes actions of the same kind. With each name (actually, with each action) we associate a basic execution rate determining the average duration of the atomic reaction. In a biological context this corresponds to the average rate of a particular reaction, which can be determined experimentally. Formally, we define a weight function $\iota \colon \Lambda \to \mathbb{R}_{>0}$ associating a strictly positive rate with each name.

Now, we aim to define a Markov a-transition θ_a, for each action $a \in \mathbb{A}$; this will conclude the construction of the $\mathbb{A}$-Markov kernel for $(\mathbb{B}, \Pi)$. To this end, it is useful to give some operations on measurable sets. For arbitrary $\mathcal{P}, \mathcal{Q} \in \Pi_{\mathsf{sys}}$, and $\mathcal{S}, \mathcal{T} \in \Pi_{\mathsf{mem}}$, we define

$$\mathcal{S}|\mathcal{T} \triangleq \bigcup\{[\sigma|\tau]_\equiv \mid \sigma \in \mathcal{S}, \tau \in \mathcal{T}\} \qquad \mathcal{S}_{|\tau} \triangleq \bigcup\{[\sigma]_\equiv \mid \sigma|\tau \in \mathcal{S}\}$$

$$\mathcal{P} \circ \mathcal{Q} \triangleq \bigcup\{[P \circ Q]_\equiv \mid P \in \mathcal{P}, Q \in \mathcal{Q}\} \qquad \mathcal{P}_{\circ Q} \triangleq \bigcup\{[P]_\equiv \mid P \circ Q \in \mathcal{P}\}$$

$$\mathcal{S}(\!(\mathcal{P})\!) \triangleq \bigcup\{[\sigma(\!(P)\!)]_\equiv \mid \sigma \in \mathcal{S}, P \in \mathcal{P}\} \qquad \mathcal{P}_{\sigma(\!(\,)\!)} \triangleq \bigcup\{[P]_\equiv \mid \sigma(\!(P)\!) \in \mathcal{P}\}$$

For $a \in \mathbb{A}$, such that $ar(a) = t \to \langle t_1,\dots,t_n\rangle$, and $M \in \mathbb{B}_t$, we define the measure $\theta_a(M)$ by induction on the structure of M. It suffices to define it only on elements of the base $\mathbb{B}_{\langle t_1,\dots,t_n\rangle}/_\equiv$: the definition extends to generic measurable sets in $\bigotimes_{i=1}^n \Pi_{t_i}$ in the canonic way.

Actions $\circlearrowright, \circlearrowleft, \circlearrowleft^\perp$: For arbitrary $n, m \in \Lambda$, $\epsilon, \alpha \in \{\circlearrowright, \circlearrowleft, \circlearrowleft^\perp\}$, $\beta \in \{\circlearrowright^\perp, \circledcirc\}$, and $X \in \mathbb{B}_{\mathsf{mem}}/_\equiv$, we define $\theta_{\epsilon_n} : \mathbb{B}_{\mathsf{mem}} \to \Delta_{\epsilon_n}(\mathbb{B}, \Pi)$ by

$$\theta_{\epsilon_n}(\mathbf{0})(X) = 0$$

$$\theta_{\epsilon_n}(\alpha_m.\sigma)(X) = \begin{cases} \iota(n) & \text{if } \alpha_m = \epsilon_n \text{ and } \sigma \in X \\ 0 & \text{otherwise} \end{cases}$$

$$\theta_{\epsilon_n}(\beta_m(\tau).\sigma)(X) = 0$$

$$\theta_{\epsilon_n}(\sigma|\tau)(X) = \theta_{\epsilon_n}(\sigma)(X_{|\tau}) + \theta_{\epsilon_n}(\tau)(X_{|\sigma})$$

Actions $\circlearrowright^{\perp}, \circledcirc$**:** For arbitrary $n, m \in \Lambda$, $\alpha \in \{\circlearrowright, \circlearrowleft, \circlearrowleft^{\perp}\}$, $\epsilon, \beta \in \{\circlearrowright^{\perp}, \circledcirc\}$, and $X, Y \in \mathbb{B}_{\mathsf{mem}}/_{\equiv}$, we define $\theta_{\epsilon_n} : \mathbb{B}_{\mathsf{mem}} \to \Delta_{\epsilon_n}(\mathbb{B}, \Pi)$ by

$$
\begin{aligned}
&\theta_{\epsilon_n}(\mathbf{0})(X \times Y) = 0 \\
&\theta_{\epsilon_n}(\alpha_m.\sigma)(X \times Y) = 0 \\
&\theta_{\epsilon_n}(\beta_m(\tau).\sigma)(X \times Y) = \begin{cases} \iota(n) & \text{if } \beta_m = \epsilon_n,\ \sigma \in X, \text{ and } \tau \in Y \\ 0 & \text{otherwise} \end{cases} \\
&\theta_{\epsilon_n}(\sigma|\tau)(\mathcal{M})(X \times Y) = \theta_{\epsilon_n}(\sigma)(X_{|\tau} \times Y) + \theta_{\epsilon_n}(\tau)(X_{|\sigma} \times Y)
\end{aligned}
$$

Action ph_n**:** For arbitrary $n \in \Lambda$, $X, Y \in \mathbb{B}_{\mathsf{sys}}/_{\equiv}$, we define the function $\theta_{\mathrm{ph}_n} : \mathbb{B}_{\mathsf{sys}} \to \Delta_{\mathrm{ph}_n}(\mathbb{B}, \Pi)$ by

$$
\begin{aligned}
&\theta_{\mathrm{ph}_n}(\diamond)(X \times Y) = 0 \\
&\theta_{\mathrm{ph}_n}(\sigma \llparenthesis P \rrparenthesis)(X \times Y) = \begin{cases} \theta_{\circlearrowright_n}(\sigma)([\sigma']_{\equiv}) & \text{if } \sigma' \llparenthesis P \rrparenthesis \in X, \text{ and } \diamond \in Y \\ 0 & \text{otherwise} \end{cases} \\
&\theta_{\mathrm{ph}_n}(P \circ Q)(X \times Y) = \theta_{\mathrm{ph}_n}(P)(X \times Y_{\circ Q}) + \theta_{\mathrm{ph}_n}(Q)(X \times Y_{\circ P})
\end{aligned}
$$

Action $\mathrm{ph}_n^{\perp}$**:** For arbitrary $n \in \Lambda$, $X, Y \in \mathbb{B}_{\mathsf{mem}}/_{\equiv}$ and $Z, W \in \mathbb{B}_{\mathsf{sys}}/_{\equiv}$, we define $\theta_{\mathrm{ph}_n^{\perp}} : \mathbb{B}_{\mathsf{sys}} \to \Delta_{\mathrm{ph}_n^{\perp}}(\mathbb{B}, \Pi)$ by

$$
\begin{aligned}
&\theta_{\mathrm{ph}_n^{\perp}}(\diamond)(X \times Y \times Z \times W) = 0 \\
&\theta_{\mathrm{ph}_n^{\perp}}(\sigma \llparenthesis P \rrparenthesis)(X \times Y \times Z \times W) = \begin{cases} \theta_{\circlearrowright_n^{\perp}}(\sigma)(X \times Y) & \text{if } P \in Z \text{ and } \diamond \in W \\ 0 & \text{otherwise} \end{cases} \\
&\theta_{\mathrm{ph}_n^{\perp}}(P \circ Q)(X \times Y \times Z \times W) = \theta_{\mathrm{ph}_n^{\perp}}(P)(X \times Y \times Z \times W_{\circ Q}) + \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \theta_{\mathrm{ph}_n^{\perp}}(Q)(X \times Y \times Z \times W_{\circ P})
\end{aligned}
$$

Action ex_n**:** For arbitrary $n \in \Lambda$, $X \in \mathbb{B}_{\mathsf{mem}}/_{\equiv}$, $Y, Z \in \mathbb{B}_{\mathsf{sys}}/_{\equiv}$, we define the function $\theta_{\mathrm{ex}_n} : \mathbb{B}_{\mathsf{sys}} \to \Delta_{\mathrm{ex}_n}(\mathbb{B}, \Pi)$ by

$$
\begin{aligned}
&\theta_{\mathrm{ex}_n}(\diamond)(X \times Y \times Z) = 0 \\
&\theta_{\mathrm{ex}_n}(\sigma \llparenthesis P \rrparenthesis)(X \times Y \times Z) = \begin{cases} \theta_{\circlearrowleft_n}(\sigma)(X) & \text{if } P \in Y \text{ and } \diamond \in Z \\ 0 & \text{otherwise} \end{cases} \\
&\theta_{\mathrm{ex}_n}(P \circ Q)(X \times Y \times Z) = \theta_{\mathrm{ex}_n}(P)(X \times Y \times Z_{\circ Q}) + \theta_{\mathrm{ex}_n}(Q)(X \times Y \times Z_{\circ P})
\end{aligned}
$$

Action id**:** For $X \in \mathbb{B}_{\mathsf{sys}}/_{\equiv}$, the function $\theta_{id} \colon \mathbb{B}_{\mathsf{sys}} \to \Delta_{id}(\mathbb{B}, \Pi)$ is defined by

$$\theta_{id}(\diamond)(X) = 0$$

$$\begin{aligned}
\theta_{id}(\sigma\llparenthesis P\rrparenthesis)(X) = \theta_{id}(P)(X_{\sigma\llparenthesis\rrparenthesis}) + \sum_{X'\llparenthesis X''\llparenthesis[\diamond]_{\equiv}\rrparenthesis\circ[P]_{\equiv}\rrparenthesis = X}^{n\in\Lambda} \theta_{\circledcirc_n}(\sigma)(X' \times X'') + \\
\sum_{X'|X''\llparenthesis Y''\rrparenthesis\circ Y' = X}^{n\in\Lambda} \frac{\theta_{\mathrm{ex}_n}(P)(X' \times Y' \times Y'') \cdot \theta_{\circlearrowright_n^{\perp}}(\sigma)(X'')}{\iota(n)}
\end{aligned}$$

$$\begin{aligned}
\theta_{id}(P \circ Q)(X) = \theta_{id}(P)(X_{\circ Q}) + \theta_{id}(Q)(X_{\circ P}) + \\
\sum_{X_1\llparenthesis X_2\llparenthesis Y_1\rrparenthesis\circ Z_1\rrparenthesis\circ Y_2\circ Z_2 = X}^{n\in\Lambda} \frac{\theta_{\mathrm{ph}_n}(P)(Y_1 \times Y_2) \cdot \theta_{\mathrm{ph}_n^{\perp}}(Q)(X_1 \times X_2 \times Z_1 \times Z_2)}{\iota(n)} + \\
\sum_{X_1\llparenthesis X_2\llparenthesis Z_1\rrparenthesis\circ Y_1\rrparenthesis\circ Z_2\circ Y_2 = X}^{n\in\Lambda} \frac{\theta_{\mathrm{ph}_n^{\perp}}(P)(X_1 \times X_2 \times Y_1 \times Y_2) \cdot \theta_{\mathrm{ph}_n}(Q)(Z_1 \times Z_2)}{\iota(n)}
\end{aligned}$$

Intuitively, each summand agrees with a rule in Table 2. For example, the last summand in $\theta_{id}(P \circ Q)$ corresponds to the (R-id$\circlearrowright$) rule. Similarly, if for a term M there are no a-transitions, $\theta_a(M)$ is the null measure.

Example 5.2. *Let* $P = \circlearrowright_n.\sigma\llparenthesis\diamond\rrparenthesis$ *and* $Q = \circlearrowright_n^{\perp}(\rho).\tau\llparenthesis\diamond\rrparenthesis$*, and assume* $\iota(n) = r$*, for* $n \in \Lambda$*. For* $X, Y \in \mathbb{B}_{\mathsf{sys}}/_{\equiv}$ *and* $Z, W \in \mathbb{B}_{\mathsf{mem}}/_{\equiv}$*, and* $m \in \Lambda$ *we have*

$$\theta_{\mathrm{ph}_m}(P \circ Q)(X \times Y) = \begin{cases} r & \textit{if } m = n,\ \sigma\llparenthesis\diamond\rrparenthesis \in X,\ \textit{and } Q \in Y \\ 0 & \textit{otherwise} \end{cases}$$

$$\theta_{\mathrm{ph}_m^{\perp}}(P \circ Q)(Z \times W \times X \times Y) = \begin{cases} r & \textit{if } m = n,\ \diamond \in X,\ P \in Y, \\ & \tau \in Z,\ \textit{and } \rho \in W \\ 0 & \textit{otherwise} \end{cases}$$

$$\theta_{id}(P \circ Q)(X) = \begin{cases} r & \textit{if } \tau\llparenthesis\rho\llparenthesis\sigma\llparenthesis\diamond\rrparenthesis\rrparenthesis\rrparenthesis \in X \\ 0 & \textit{otherwise} \end{cases}$$

$$\theta_{\mathrm{ex}_m}(P \circ Q)(Z \times X \times Y) = 0\,.$$

Notice that, for each non-null measurable set there is a compatible (up-to $\equiv$*) transition, namely,* $P \circ Q \xrightarrow{\mathrm{ph}_n} \langle \sigma\llparenthesis\diamond\rrparenthesis, \diamond \circ Q\rangle$*,* $P \circ Q \xrightarrow{\mathrm{ph}_n^{\perp}} \langle \tau, \rho, \diamond, P \circ \diamond\rangle$ *and* $P \circ Q \xrightarrow{id} \langle \diamond \circ \tau\llparenthesis\rho\llparenthesis\sigma\llparenthesis\diamond\rrparenthesis\rrparenthesis \circ \diamond\rrparenthesis\rangle$*.*

There is a formal correspondence between the LTS and the $\mathbb{A}$-Markov kernel.

Proposition 5.3. *Let* $a \in \mathbb{A}$ *be such that* $ar(a) = t \to \langle t_1, \ldots, t_n\rangle$ *and* $M \in \mathbb{B}_t$*:*

1. *if* $\theta_a(M)(\mathcal{M}_1 \times \cdots \times \mathcal{M}_n) > 0$*, then, for all* $1 \le i \le n$*, there exist* $M_i \in \mathcal{M}_i$*, such that* $M \xrightarrow{a} \langle M_1, \ldots, M_n\rangle$*,*

2. *if* $M \xrightarrow{a} \langle M_1, \ldots, M_n \rangle$*, then, for all* $1 \le i \le n$*, there exist measurable sets* $\mathcal{M}_i \in \Pi_{t_i}$ *such that* $M_i \in \mathcal{M}_i$ *and* $\theta_a(M)(\mathcal{M}_1 \times \cdots \times \mathcal{M}_n) > 0$.

In the proposition above, (1) can be proven by induction on the structure of the term M, while the proof for (2) is by induction on the height of the derivation of $M \xrightarrow{a} \langle M_1, \ldots, M_n \rangle$.

A direct consequence of Proposition 5.3 is the following.

Corollary 5.4. $M \xrightarrow{a} \langle M_1, \ldots, M_n \rangle$ *iff* $\theta_a(M)([\langle M_1, \ldots, M_n \rangle]_\equiv) > 0$.

The $\mathbb{A}$-transition functions $\{\theta_a\}_{a \in \mathbb{A}}$ of the Markov kernel for Brane Calculus are compatible with structural congruence in the following sense:

Proposition 5.5. *For* $a \in \mathbb{A}$ *and* $M, N \in \mathbb{B}$*, if* $M \equiv N$*, then* $\theta_a(M) = \theta_a(N)$.

This Proposition is crucial in the proof of the next theorem, which states that $(\mathbb{B}, \Pi, \{\theta_a\}_{a \in \mathbb{A}})$ is a $\mathbb{A}$-Markov kernel.

Theorem 5.6 (Markov kernel). $\mathcal{B} \triangleq (\mathbb{B}, \Pi, \{\theta_a\}_{a \in \mathbb{A}})$ *is a* $\mathbb{A}$*-Markov kernel.*

Proof. First, it is easy to check that for each $a \in \mathbb{A}$ and $M \in \mathbb{B}$, $\theta_a(M)$ is a measure in $\Delta_a(\mathbb{B}, \Pi)$, by construction. Then, we prove that θ_a is also a measurable function. Let $ar(a) = t \to \langle t_1, \ldots, t_n \rangle$; for $S \in \bigotimes_{i=1}^n \Pi_{t_i}$ and $r > 0$, we denote by $U^a_{S,r} \triangleq \{\mu \in \Delta_a(\mathbb{B}, \Pi) \mid \mu(S) \ge r\}$ an element of the generator of measures space. We have to prove that $\theta_a^{-1}(U^a_{S,r})$ is a measurable set, that is, an element of Π. To this end, it suffices to prove that $\theta_a^{-1}(U^a_{S,r})$ is given by (countable) unions of $\equiv$-equivalence classes. This is equivalent to prove that for any $M, M' \in \mathbb{B}$ such that $M \equiv M'$, if $M \in \theta_a^{-1}(U^a_{S,r})$ then $M' \in \theta_a^{-1}(U^a_{S,r})$, and indeed this holds by Proposition 5.5. □

A consequence of Theorem 5.6 is that for each $M \in \mathbb{B}$, $(\mathcal{B}, M)$ is a Markov process, hence we can define a stochastic bisimulation for Brane Calculus simply as the stochastic bisimulation $\sim_\mathcal{B}$ over Markov processes.

We conclude this section observing that for $M \in \mathbb{B}_t$ and $a \in \mathbb{A}$, such that $ar(a) = t \to \langle t_1, \ldots, t_n \rangle$, the measure space $(\mathbb{B}_{\langle t_1, \ldots, t_n \rangle}, \bigotimes_{i=1}^n \Pi_{t_i}, \theta_a(M))$ is finite, hence each stochastic transition has a finite rate associated with.

Proposition 5.7. *For* $a \in \mathbb{A}$ *such that* $ar(a) = t \to \langle t_1, \ldots, t_n \rangle$ *and* $M \in \mathbb{B}_t$*, the measure space* $(\mathbb{B}_{\langle t_1, \ldots, t_n \rangle}, \bigotimes_{i=1}^n \Pi_{t_i}, \theta_a(M))$ *is finite.*

Proof. For each $a \in \mathbb{A}$, such that $ar(a) = t \to \langle t_1, \ldots, t_n \rangle$, and $M \in \mathbb{B}_t$ it has to be shown that $\theta_a(M)(\mathbb{B}_{\langle t_1, \ldots, t_n \rangle}) \le r$, for some $r \in \mathbb{R}^+$. This can be done by induction on the structure M. The only non trivial case is when $a = id$, where one must notice that the infinite summations involved in the definition have only a finite number of nonzero summands. This is can be easily proved by contradiction using Corollary 5.4 and Lemma 3.5. □

6. Stochastic Structural Operational Semantics and Bisimulation

In this section we introduce the stochastic structural operational semantics for the Brane Calculus, in order to define a behavioral equivalence on system terms that coincides with their bisimulation as Markov processes on $(\mathbb{B}, \Pi, \{\theta_a\}_{a\in\mathbb{A}})$. Following the pattern of [12], this semantics is directly induced from the definition of the set $\{\theta_a\}_{a\in\mathbb{A}}$ of Markov $\mathbb{A}$-transition functions. In order to maintain "the spirit" of process algebras, Cardelli and Mardare replace the classic "pointwise" rules of the form $P \xrightarrow{a,r} P'$ with rules of the form $P \to \mu$, where μ is an indexed class of measures on the measurable space of processes. Let us see how this construction can be applied in the case of Brane Calculus.

For simplifying the presentation of semantics rules, we introduce some constants and operations over indexed families of measures. For a set (of labels) A, let us denote by $\Delta^A(\mathbb{B}, \Pi)$ the set $\prod_{a\in\mathbb{A}} \Delta_a(\mathbb{B}, \Pi)$ of A-indexed families of measures over $(\mathbb{B}, \Pi)$. Given a family of measures $\mu \in \Delta^A(\mathbb{B}, \Pi)$ and $a \in A$, the a-component of μ will be denoted as $\mu_a \in \Delta_a(\mathbb{B}, \Pi)$.

Null: Let $\omega_{\mathsf{mem}} \in \Delta^{\mathbb{A}_{\mathsf{mem}}}(\mathbb{B}, \Pi)$ be the constantly zero measure, i.e., for all $a{\in}\mathbb{A}_{\mathsf{mem}}$ such that $ar(a) = t \to \langle t_1, \ldots, t_n\rangle$ and $\mathcal{M} \in \bigotimes_{i=1}^n \Pi_{t_i}$: $(\omega_{\mathsf{mem}})_a(\mathcal{M}){=}0$.

Prefix: For arbitrary $n \in \Lambda$, $\alpha \in \{\circlearrowright, \circlearrowleft, \circlearrowleft^\perp\}$, and $\beta \in \{\circlearrowright^\perp, \circledcirc\}$, let the constants $[\alpha_n]_\sigma, [\beta_n]_\sigma^\tau \in \Delta^{\mathbb{A}_{\mathsf{mem}}}(\mathbb{B}, \Pi)$ be defined, for arbitrary $X, Y \in \mathbb{B}_{\mathsf{mem}}/_\equiv$, by

$$
\begin{aligned}
([\alpha_n]_\sigma)_{\alpha_m}(X) &= \begin{cases} \iota(n) & \text{if } n = m \text{ and } \sigma \in X \\ 0 & \text{otherwise} \end{cases} \\
([\alpha_n]_\sigma)_{\beta_m}(X \times Y) &= 0 \\
([\beta_n]_\sigma^\tau)_{\beta_m}(X \times Y) &= \begin{cases} \iota(n) & \text{if } n = m \text{ and } \sigma \in X,\ \tau \in Y \\ 0 & \text{otherwise} \end{cases} \\
([\beta_n]_\sigma)_{\alpha_m}(X) &= 0
\end{aligned}
$$

Parallel: For $\mu, \mu' \in \Delta^{\mathbb{A}_{\mathsf{mem}}}(\mathbb{B}, \Pi)$, let $\mu\ {}_\sigma\!\oplus_\tau\ \mu' \in \Delta^{\mathbb{A}_{\mathsf{mem}}}(\mathbb{B}, \Pi)$ be defined, for $n \in \Lambda$, $\alpha \in \{\circlearrowright, \circlearrowleft, \circlearrowleft^\perp\}$, $\beta \in \{\circlearrowright^\perp, \circlearrowleft^\perp\}$, and $X, Y \in \mathbb{B}_{\mathsf{mem}}/_\equiv$, by

$$
\begin{aligned}
(\mu\ {}_\sigma\!\oplus_\tau\ \mu')_{\alpha_n}(X) &= \mu_{\alpha_n}(X_{|\tau}) + \mu'_{\alpha_n}(X_{|\sigma}) \\
(\mu\ {}_\sigma\!\oplus_\tau\ \mu')_{\beta_n}(X \times Y) &= (\mu)_{\beta_n}(X_{|\tau} \times Y) + (\mu')_{\beta_n}(X_{|\sigma} \times Y)
\end{aligned}
$$

Void: Let $\omega_{\mathsf{sys}} \in \Delta^{\mathbb{A}_{\mathsf{sys}}}(\mathbb{B}, \Pi)$ be defined by $(\omega_{\mathsf{sys}})_a(\mathcal{M}) = 0$ for any $a \in \mathbb{A}_{\mathsf{sys}}$, such that $ar(a) = t \to \langle t_1, \ldots, t_n\rangle$, and $\mathcal{M} \in \bigotimes_{i=1}^n \Pi_{t_i}$.

Nesting: For $\nu \in \Delta^{\mathbb{A}_{\mathsf{mem}}}(\mathbb{B}, \Pi)$ and $\mu \in \Delta^{\mathbb{A}_{\mathsf{sys}}}(\mathbb{B}, \Pi)$, let $\mu\ @_P^\sigma\ \nu \in \Delta^{\mathbb{A}_{\mathsf{sys}}}(\mathbb{B}, \Pi)$

be defined, for $X, Y \in \mathbb{B}_{\mathsf{mem}}/_{\equiv}$ and $Z, W \in \mathbb{B}_{\mathsf{sys}}/_{\equiv}$, by

$$(\mu\ @^{\sigma}_{P}\ \nu)_{\mathrm{ph}_n}(Z \times W) = \begin{cases} \nu_{\circlearrowleft_n}([\sigma']_{\equiv}) & \text{if } \sigma' \llparenthesis P \rrparenthesis \in Z \text{ and } \diamond \in W \\ 0 & \text{otherwise} \end{cases}$$

$$(\mu\ @^{\sigma}_{P}\ \nu)_{\mathrm{ph}^{\perp}_n}(X \times Y \times Z \times W) = \begin{cases} \nu_{\circlearrowleft^{\perp}_n}(X \times Y) & \text{if } P \in Z \text{ and } \diamond \in W \\ 0 & \text{otherwise} \end{cases}$$

$$(\mu\ @^{\sigma}_{P}\ \nu)_{\mathrm{ex}_n}(X, Z, W) = \begin{cases} \nu_{\circlearrowleft_n}(X) & \text{if } P \in Z \text{ and } \diamond \in W \\ 0 & \text{otherwise} \end{cases}$$

$$(\mu\ @^{\sigma}_{P}\ \nu)_{id}(X) = \mu_{id}(X_{\sigma \llparenthesis \rrparenthesis}) + \sum_{X' \llparenthesis X'' \llparenthesis [\diamond]_{\equiv} \rrparenthesis \circ [P]_{\equiv} \rrparenthesis = X}^{n \in \Lambda} \nu_{\circledcirc_n}(X' \times X'') +$$
$$\sum_{X' | X'' \llparenthesis Y'' \rrparenthesis \circ Y' = X}^{n \in \Lambda} \frac{\mu_{\mathrm{ex}_n}(X' \times Y' \times Y'') \cdot \nu_{\circlearrowleft^{\perp}_n}(X'')}{\iota(n)}$$

Composition: For $\mu, \mu' \in \Delta^{\mathbb{A}_{\mathsf{sys}}}(\mathbb{B}, \Pi)$, let $\mu\ {}_{P}\!\otimes_{Q}\ \mu' \in \Delta^{\mathbb{A}_{\mathsf{sys}}}(\mathbb{B}, \Pi)$ be defined, for $X, Y \in \mathbb{B}_{\mathsf{mem}}/_{\equiv}$ and $Z, W \in \mathbb{B}_{\mathsf{sys}}/_{\equiv}$, by

$$(\mu\ {}_{P}\!\otimes_{Q}\ \mu')_{\mathrm{ph}_n}(Z \times W) = \mu_{\mathrm{ph}_n}(Z \times W_{\circ Q}) + \mu'_{\mathrm{ph}_n}(Z \times W_{\circ P})$$

$$(\mu\ {}_{P}\!\otimes_{Q}\ \mu')_{\mathrm{ph}^{\perp}_n}(X \times Y \times Z \times W) = \mu_{\mathrm{ph}^{\perp}_n}(X \times Y \times Z \times W_{\circ Q}) +$$
$$\mu'_{\mathrm{ph}^{\perp}_n}(X \times Y \times Z \times W_{\circ P})$$

$$(\mu\ {}_{P}\!\otimes_{Q}\ \mu')_{\mathrm{ph}^{\perp}_n}(X \times Z \times W) = \mu_{\mathrm{ex}_n}(X \times Z \times W_{\circ Q}) +$$
$$\mu'_{\mathrm{ex}_n}(X \times Z \times W_{\circ P})$$

$$(\mu\ {}_{P}\!\otimes_{Q}\ \mu')_{id}(X) = \mu_{id}(X_{\circ Q}) + \mu'_{id}(X_{\circ P}) +$$
$$\sum_{X_1 \llparenthesis X_2 \llparenthesis Y_1 \rrparenthesis \circ Z_1 \rrparenthesis \circ Y_2 \circ Z_2 = X}^{n \in \Lambda} \frac{\mu_{\mathrm{ph}_n}(Y_1 \times Y_2) \cdot \mu'_{\mathrm{ph}^{\perp}_n}(X_1 \times X_2 \times Z_1 \times Z_2)}{\iota(n)} +$$
$$\sum_{X_1 \llparenthesis X_2 \llparenthesis Z_1 \rrparenthesis \circ Y_1 \rrparenthesis \circ Z_2 \circ Y_2 = X}^{n \in \Lambda} \frac{\mu_{\mathrm{ph}^{\perp}_n}(X_1 \times X_2 \times Y_1 \times Y_2) \cdot \mu'_{\mathrm{ph}_n}(Z_1 \times Z_2)}{\iota(n)}$$

The next two lemmata prove that the definitions of ${}_{\sigma}\Phi_{\tau}$, ${}_{P}\!\otimes_{Q}$, and $@^{\sigma}_{P}$, for arbitrary $\sigma, \tau \in \mathbb{B}_{\mathsf{mem}}$ and $P, Q \in \mathbb{B}_{\mathsf{sys}}$ are correct; they also state some basic properties of these operators.

Lemma 6.1. *The following statements hold.*

1. *For arbitrary* $\sigma, \tau, \rho \in \mathbb{B}_{\mathsf{mem}}$ *and* $\mu', \mu'', \mu''' \in \Delta^{\mathbb{A}_{\mathsf{mem}}}(\mathbb{B}, \Pi)$*:*
   (a) $\mu'\ {}_{\sigma}\Phi_{\tau}\ \mu'' = \mu''\ {}_{\tau}\Phi_{\sigma}\ \mu'$,
   (b) $(\mu'\ {}_{\sigma}\Phi_{\tau}\ \mu'')\ {}_{\sigma|\tau}\Phi_{\rho}\ \mu''' = \mu'\ {}_{\sigma}\Phi_{\tau|\rho}\ (\mu''\ {}_{\tau}\Phi_{\rho}\ \mu''')$,
   (c) $\mu'\ {}_{\sigma}\Phi_{\mathbf{0}}\ \omega_{\mathsf{mem}} = \mu'$.

$$\frac{}{\mathbf{0} \to \omega_{\mathsf{mem}}}\ (\text{zero}) \qquad \frac{\alpha \in \{\circlearrowright_n, \circlearrowleft_n, \circlearrowright_n^{\perp} \mid n \in \Lambda\}}{\alpha.\sigma \to [\alpha]_\sigma}\ (\text{pref})$$

$$\frac{\beta \in \{\circlearrowright_n^{\perp}, \circledcirc_n \mid n \in \Lambda\}}{\beta(\tau).\sigma \to [\beta]_\sigma^\tau}\ (\text{pref-arg}) \qquad \frac{\sigma \to \mu' \qquad \tau \to \mu''}{\sigma|\tau \to \mu' \ {}_{\sigma}\Phi_{\tau}\ \mu''}\ (\text{par})$$

$$\frac{}{\diamond \to \omega_{\mathsf{sys}}}\ (\text{void}) \qquad \frac{\sigma \to \nu \qquad P \to \mu}{\sigma(\!|P|\!) \to \mu \ @_P^\sigma\ \nu}\ (\text{loc}) \qquad \frac{P \to \mu' \qquad Q \to \mu''}{P \circ Q \to \mu' \ {}_{P}\!\otimes_{Q}\ \mu''}\ (\text{comp})$$

Table 3: Stochastic structural operational semantics for Brane Calculus

2. *For arbitrary* $P, Q, R \in \mathbb{B}_{\mathsf{sys}}$ *and* $\mu', \mu'', \mu''' \in \Delta^{\mathbb{A}_{\mathsf{sys}}}(\mathbb{B}, \Pi)$*:*
   (a) $\mu' \ {}_{P}\!\otimes_{Q}\ \mu'' = \mu'' \ {}_{Q}\!\otimes_{P}\ \mu'$,
   (b) $(\mu' \ {}_{P}\!\otimes_{Q}\ \mu'') \ {}_{P|Q}\!\otimes_{R}\ \mu''' = \mu' \ {}_{P}\!\otimes_{Q|R}\ (\mu'' \ {}_{Q}\!\otimes_{R}\ \mu''')$,
   (c) $\mu' \ {}_{P}\!\otimes_{\diamond}\ \omega_{\mathsf{sys}} = \mu'$.

3. $\omega_{\mathsf{sys}} \ @_\diamond^{\mathbf{0}}\ \omega_{\mathsf{mem}} = \omega_{\mathsf{sys}}$.

Lemma 6.2. *The following statements hold.*

1. *For arbitrary* $\sigma, \sigma', \tau, \tau' \in \mathbb{B}_{\mathsf{mem}}$ *and* $\mu', \mu'' \in \Delta^{\mathbb{A}_{\mathsf{mem}}}(\mathbb{B}, \Pi)$*:*
   (a) *for* $\alpha \in \{\circlearrowright_n, \circlearrowleft_n, \circlearrowright_n^{\perp} \mid n \in \Lambda\}$, $\sigma \equiv \sigma'$ *implies* $[\alpha]_\sigma = [\alpha]_{\sigma'}$,
   (b) *for* $\beta \in \{\circlearrowright_n^{\perp}, \circledcirc_n \mid n \in \Lambda\}$, $\sigma \equiv \sigma'$ *and* $\tau \equiv \tau'$ *imply* $[\beta]_\sigma^\tau = [\beta]_{\sigma'}^{\tau'}$,
   (c) $\sigma \equiv \sigma'$ *and* $\tau \equiv \tau'$ *imply* $\mu' \ {}_{\sigma}\Phi_{\tau}\ \mu'' = \mu' \ {}_{\sigma'}\Phi_{\tau'}\ \mu''$.

2. *For arbitrary* $P, P', Q, Q' \in \mathbb{B}_{\mathsf{sys}}$, $\sigma, \tau \in \mathbb{B}_{\mathsf{mem}}$, $\mu, \mu', \mu'' \in \Delta^{\mathbb{A}_{\mathsf{sys}}}(\mathbb{B}, \Pi)$, *and* $\nu \in \Delta^{\mathbb{A}_{\mathsf{mem}}}(\mathbb{B}, \Pi)$*:*
   (a) $P \equiv Q$ *and* $\sigma \equiv \tau$ *imply* $\mu \ @_P^\sigma\ \nu = \mu \ @_Q^\tau\ \nu$,
   (b) $P \equiv P'$ *and* $Q \equiv Q'$ *imply* $\mu' \ {}_{P}\!\otimes_{Q}\ \mu'' = \mu' \ {}_{P'}\!\otimes_{Q'}\ \mu''$.

The rules of the operational semantics are listed in Table 3. The operational semantics associates with each membrane a family of measures in $\Delta^{\mathbb{A}_{\mathsf{mem}}}(\mathbb{B}, \Pi)$, and with each system a family of measures in $\Delta^{\mathbb{A}_{\mathsf{sys}}}(\mathbb{B}, \Pi)$.

The next lemma states that the stochastic transition relation $\to$ (and hence the operational semantics) is well-defined and consistent, that is, for each process we have exactly one family of measures of its continuations.

Lemma 6.3 (Uniqueness). *For* $a \in \mathbb{A}$ *such that* $ar(a) = t \to \langle t_1, \ldots, t_n \rangle$, *and* $M \in \mathbb{B}_t$, *there exists a unique* $\mu \in \Delta^{\mathbb{A}_t}(\mathbb{B}, \Pi)$ *such that* $M \to \mu$.

Proof. It suffices to show that $M \to \mu$ has a unique derivation, and this can be proved by induction on the structure of M observing that for each algebraic constructor only one rule can be applied. $\square$

A consequence of Lemmata 6.1 and 6.2 is that operational semantics does not distinguish structurally equivalent terms:

Lemma 6.4. *If $M \equiv N$ and $M \to \mu$, then $N \to \mu$.*

The converse does not hold in general, that is, $M \to \mu$, $N \to \mu$ does not imply that $M \equiv N$. Next we show two counterexamples.

Counterexample 6.5. *Let $P = \mathbf{0}\llparenthesis \circlearrowleft_n \llparenthesis \diamond \rrparenthesis \rrparenthesis$ and $Q = \diamond$, for some $n \in \Lambda$, then*

$$\begin{aligned} & P \to \left(\mu_1 = (\omega_{\mathsf{sys}} \mathbin{@}_{\diamond}^{\circlearrowleft_n} [\circlearrowleft_n]_{\mathbf{0}}) \mathbin{@}^{\mathbf{0}}_{\circlearrowleft_n \llparenthesis \diamond \rrparenthesis} \omega_{\mathsf{mem}}\right), \\ & Q \to \left(\mu_2 = \omega_{\mathsf{sys}}\right). \end{aligned}$$

We show that $\mu_1 = \mu_2$. Let $\mu_1' = \omega_{\mathsf{sys}} \mathbin{@}_{\diamond}^{\circlearrowleft_n} [\circlearrowleft_n]_{\mathbf{0}}$. For all $a \in \mathbb{A}_{\mathsf{sys}} \setminus \{id\}$ is easy to see that $(\mu_1)_a = (\mu_2)_a$, since $\mu_1 = \mu_1' \mathbin{@}^{\mathbf{0}}_{\circlearrowleft_n \llparenthesis \diamond \rrparenthesis} \omega_{\mathsf{mem}}$ and its value depends only on ω_{mem}. It remains to prove that $(\mu_1)_{id} = (\mu_2)_{id}$. By definition we have:

$$\begin{aligned} (\mu_1)_{id}(X) = {} & (\omega_{\mathsf{sys}} \mathbin{@}_{\diamond}^{\circlearrowleft_n} [\circlearrowleft_n]_{\mathbf{0}})_{id}(X_{\mathbf{0}\llparenthesis\rrparenthesis}) + \\ & \sum_{X'\llparenthesis X''\llparenthesis[\diamond]_{\equiv}\rrparenthesis \circ [\circlearrowleft_n \llparenthesis \diamond \rrparenthesis]_{\equiv} \rrparenthesis = X}^{m\in\Lambda} (\omega_{\mathsf{mem}})_{\circledcirc_m}(X' \times X'') + \\ & \sum_{X'|X''\llparenthesis Y''\rrparenthesis \circ Y' = X}^{m\in\Lambda} \frac{(\mu_1')_{\mathrm{ex}_m}(X' \times Y' \times Y'') \cdot (\omega_{\mathsf{mem}})_{\circlearrowleft^{\perp}_m}(X'')}{\iota(m)}. \end{aligned}$$

The last two summands are always equal to zero, because they depend only on ω_{mem}. Therefore it suffices to verify that $(\omega_{\mathsf{sys}} \mathbin{@}_{\diamond}^{\circlearrowleft_n} [\circlearrowleft_n]_{\mathbf{0}})_{id}(X_{\mathbf{0}\llparenthesis\rrparenthesis}) = 0$. By definition it is easy it verify that $(\omega_{\mathsf{sys}} \mathbin{@}_{\diamond}^{\circlearrowleft_n} [\circlearrowleft_n]_{\mathbf{0}})_{id}$ depends only on $(\omega_{\mathsf{sys}})_{id}$, $([\circlearrowleft_n]_{\mathbf{0}})_{\circledcirc_m}$, and $(\omega_{\mathsf{sys}})_{\mathrm{ex}_m}$, for $m \in \Lambda$, which are by construction always null, hence $\mu_1 = \mu_2$.

Counterexample 6.6. *Let $P = \circlearrowleft^{\perp}_n \llparenthesis \diamond \rrparenthesis$ and $Q = \diamond$, for some $n \in \Lambda$, then*

$$\begin{aligned} & P \to \left(\mu_1 = \omega_{\mathsf{sys}} \mathbin{@}_{\diamond}^{\circlearrowleft^{\perp}_n} [\circlearrowleft^{\perp}_n]_{\mathbf{0}}\right), \\ & Q \to \left(\mu_2 = \omega_{\mathsf{sys}}\right). \end{aligned}$$

We show that $\mu_1 = \mu_2$. Since $([\circlearrowleft^{\perp}_n]_{\mathbf{0}})_a = (\omega_{\mathsf{mem}})_a$, for all $a \in \mathbb{A}_{\mathsf{mem}} \setminus \{\circlearrowleft^{\perp}_n\}$, it is easy to see that $(\mu_1)_a = (\mu_2)_a$, for all $a \in \mathbb{A}_{\mathsf{sys}} \setminus \{id\}$. The only case left to prove is $(\mu_1)_{id} = (\mu_2)_{id}$. By definition

$$\begin{aligned} (\mu_1)_{id}(X) = {} & (\omega_{\mathsf{sys}})_{id}(X_{\circlearrowleft^{\perp}_n \llparenthesis\rrparenthesis}) + \\ & \sum_{X'\llparenthesis X''\llparenthesis[\diamond]_{\equiv}\rrparenthesis \circ [\diamond]_{\equiv} \rrparenthesis = X}^{m\in\Lambda} ([\circlearrowleft^{\perp}_n]_{\mathbf{0}})_{\circledcirc_m}(X' \times X'') + \\ & \sum_{X'|X''\llparenthesis Y''\rrparenthesis \circ Y' = X}^{m\in\Lambda} \frac{(\omega_{\mathsf{sys}})_{\mathrm{ex}_m}(X' \times Y' \times Y'') \cdot ([\circlearrowleft^{\perp}_n]_{\mathbf{0}})_{\circlearrowleft^{\perp}_m}(X'')}{\iota(n)}. \end{aligned}$$

The first and the last summand are always zero since they depend only on ω_{sys}. The second summand equals zero since $([\circlearrowleft^{\perp}_n]_{\mathbf{0}})_{\circledcirc_m} = (\omega_{\mathsf{mem}})_{\circledcirc_m}$, for all $m \in \Lambda$.

This operational semantics can be used to define various "more traditional" pointwise semantics, as e.g.:

$$M \xrightarrow{a,r} \langle M_1, \ldots, M_n \rangle \qquad \text{iff} \qquad M \to \mu \;\text{ and }\; \mu_a([\langle M_1, \ldots, M_n \rangle]_{\equiv}) = r$$

Let us see some simple examples, and how the property of σ-additivity of measures is exploited to correctly sum up rates.

Example 6.7. *Let $P = \circlearrowright_n.\sigma \llparenthesis R \rrparenthesis$ and $\iota(n) = r$, we show that*

$$P \xrightarrow{\mathrm{ph}_n, r} \langle \sigma \llparenthesis R \rrparenthesis, \diamond \rangle \,.$$

Assume $P \to \mu$. By definition we have to prove that $\mu_{\mathrm{ph}_n}([\langle \sigma \llparenthesis R \rrparenthesis, \diamond \rangle]_{\equiv}) = r$. By a structural analysis on P, for $R \to \mu'$, we have $\mu = \mu' \ @^{R}_{\circlearrowright_n.\sigma} \ [\circlearrowright_n]_{\sigma}$. By a straightforward application of the operator definitions we have:

$$\mu_{\mathrm{ph}_n}([\langle \sigma \llparenthesis R \rrparenthesis, \diamond \rangle]_{\equiv}) = [\circlearrowright_n]_{\sigma}([\sigma]_{\equiv}) = \iota(n) = r \,.$$

Example 6.8. *Let $P = \circlearrowright_n.\sigma \llparenthesis R \rrparenthesis$ and $\iota(n) = r$, we show that*

$$P \circ P \xrightarrow{\mathrm{ph}_n, 2r} \langle \sigma \llparenthesis R \rrparenthesis, P \rangle \,.$$

Assume $P \circ P \to \nu$. We have to prove that $\nu_{\mathrm{ph}_n}([\langle \sigma \llparenthesis R \rrparenthesis, P \rangle]_{\equiv}) = 2r$. By Example 6.7, we have $P \to \mu$, where $R \to \mu'$ and $\mu = \mu' \ @^{R}_{\circlearrowright_n.\sigma} \ [\circlearrowright_n]_{\sigma}$, therefore $\nu = \mu \ {}_{P}{\otimes}_{P} \ \mu$. Again, by Example 6.7, we obtain:

$$\nu_{\mathrm{ph}_n}([\langle \sigma \llparenthesis R \rrparenthesis, P \rangle]_{\equiv}) = 2 \cdot \mu_{\mathrm{ph}_n}([\langle \sigma \llparenthesis R \rrparenthesis, \diamond \rangle]_{\equiv}) = 2r \,.$$

We now see how the stochastic structural operational semantics induces the $\mathbb{A}$-Markov kernel $(\mathbb{B}, \Pi, \{\theta\}_{a \in \mathbb{A}})$ for the Brane Calculus. This motivates a new characterization of rate bisimulation that is defined upon the transitions $M \to \mu$ that can be derived using the rules of the stochastic SOS of Table 3.

Theorem 6.9. *Let $\mathcal{B} = (\mathbb{B}, \Pi, \{\theta\}_{a \in \mathbb{A}})$ be the $\mathbb{A}$-Markov kernel for the Brane Calculus. Then, for all $M \in \mathbb{B}_t$, $\mu \in \Delta^{\mathbb{A}_t}(\mathbb{B}, \Pi)$:*

$$M \to \mu \text{ if and only if for all } a \in \mathbb{A}_t : \ \theta_a(M) = \mu_a.$$

Proof. The the two directions can be proven by induction on the structure of M. Note that the correspondence result holds by construction, indeed the operators over families of measures that are used in the rules of Table 3 are defined following the definition of θ_a on each algebraic construct. $\square$

Call a family of measures $\mu \in \Delta^{\mathbb{A}}(T, \Sigma)$ *finitely supported* if for all $a \in \mathbb{A}$, the set $\{\mu_a \in \Delta_a(T, \Sigma) \mid \mu_a \neq \omega\}$ is finite.

Proposition 6.10 (Finiteness). *For $a \in \mathbb{A}$ such that $ar(a) = t \to \langle t_1, \ldots, t_n \rangle$, and $M \in \mathbb{B}_t$, if $M \to \mu$, then μ is finitely supported and the measure space $(\mathbb{B}_{\langle t_1, \ldots, t_n \rangle}, \bigotimes_{i=1}^{n} \Pi_{t_i}, \mu_a)$ is finite.*

Proof. In order to prove that μ is finitely supported it is convenient to proceed by contradiction applying Theorem 6.9, Corollary 5.4 and Lemma 3.5: if μ is not finitely supported, then $M \xrightarrow{a} M'$ for infinite M', and hence M (and the SOS in Table 2) would be not finitely branching. Finally, in order to prove that the measure space $(\mathbb{B}_{\langle t_1,\ldots,t_n\rangle}, \bigotimes_{i=1}^{n} \Pi_{t_i}, \mu_a)$ is finite, it suffices to apply Theorem 6.9 and Proposition 5.7. $\square$

A direct consequence of Theorem 6.9 is that if our SOS assigns to different Brane Calculus terms the same family of measures, then they are stochastic bisimilar with respect to the bisimulation over Markov processes:

Corollary 6.11. *If $M \to \mu$ and $N \to \mu$, then $M \sim_{\mathcal{B}} N$.*

Moreover, Theorem 6.9 guarantees that we can safely specialize the definition of rate bisimulation on $\mathcal{B}$ as we do in Definition 6.12.

Definition 6.12 (Stochastic bisimulation for Brane Calculus). *A* rate bisimulation *over Brane Calculus terms is a relation $\mathcal{R} \subseteq \mathbb{B} \times \mathbb{B}$ such that whenever $(M, N) \in \mathcal{R}$, $a \in \mathbb{A}$, $ar(a) = t \to \langle t_1, \ldots, t_n\rangle$, $M \to \mu$, $N \to \mu'$ and $(\mathcal{M}_i, \mathcal{N}_i) \in \Pi_{t_i}(\mathcal{R})$ for $1 \le i \le n$:*

$$\mu_a(\mathcal{M}_1 \times \cdots \times \mathcal{M}_n) = \mu'_a(\mathcal{N}_1 \times \cdots \times \mathcal{N}_n)$$

Two terms $M, N \in \mathbb{B}$ are stochastic bisimilar, written $M \approx N$, if they are related by a rate bisimulation.

Stochastic bisimilarity between Brane Calculus terms, $\approx \subseteq \mathbb{B} \times \mathbb{B}$, satisfies the general properties of bisimilarity between Markov processes:

Proposition 6.13. *The following statements about $\approx$ hold:*

1. *$\approx$ is an equivalence relation,*
2. *$\approx$ is the largest rate bisimulation over Brane Calculus terms,*
3. *Let $M, N \in \mathbb{B}_t$, $M \to \mu$ and $N \to \nu$, then $M \approx N$ iff for all $a \in \mathbb{A}$, such that $ar(a) = t \to \langle t_1, \ldots, t_n\rangle$, and $(\mathcal{C}_i, \mathcal{C}_i) \in \Pi_{t_i}(\approx)$:*

$$\mu_a(\mathcal{C}_1 \times \cdots \times \mathcal{C}_n) = \nu_a(\mathcal{C}_1 \times \cdots \times \mathcal{C}_n)$$

4. *if $M \approx N$, then M and N are of the same sort.*

Proof. They can be proven applying Theorem 6.9, and Propositions 4.6, 4.8, and 4.9, respectively (note that, although not mentioned before, Lemma 6.3 is essential in order to prove reflexivity). Statement (4) is a direct consequence of Definition 6.12 and it holds in general for all rate bisimulation relations. $\square$

Stochastic bisimilarity behaves well with respect to structural equivalence:

Proposition 6.14. *If $M \equiv N$, then $M \approx N$.*

This is a direct consequence of Lemma 6.4. Note that the converse does not hold, that is, $M \approx N$ does not imply $M \equiv N$ (see Counterexamples 6.5 and 6.6, the same we used for Lemma 6.4, since Theorem 6.9 holds). This is a good property for $\approx$, because it states that stochastic bisimulation is strictly larger than structural equivalence, hence it equates more terms than $\equiv$.

An interesting fact about stochastic bisimilarity is that is it also a non-stochastic strong bisimulation.

Proposition 6.15. *$\approx$ is a (non-stochastic) strong bisimulation.*

Proof. Since by Proposition 6.14(1) $\approx$ is symmetric, it is sufficient to prove that if $M \approx N$ and $M \xrightarrow{a} \langle M_1, \dots, M_n \rangle$, then there is a transition $N \xrightarrow{a} \langle N_1, \dots, N_n \rangle$ such that $M_i \approx N_i$ for all $1 \le i \le n$.

Assume $M \approx N$ and $M \xrightarrow{a} \langle M_1, \dots, M_n \rangle$, for some $ar(a) = t \to \langle t_1, \dots, t_n \rangle$, and let $M \to \mu$ and $N \to \nu$. By Corollary 5.4 $M' \xrightarrow{a} \langle M_1, \dots, M_n \rangle$ we have $\theta_a(M')([M_1]_\equiv \times \dots \times [M_n]_\equiv) > 0$, hence $\mu_a([M_1]_\equiv \times \dots \times [M_n]_\equiv) > 0$, by Theorem 6.9. By Proposition 6.14, it holds that $[M_i]_\equiv \subseteq [M_i]_\approx$, and moreover that $([M_i]_\approx, [M_i]_\approx) \in \Pi_{t_i}(\approx)$, for all $1 \le i \le n$. By Proposition 6.13(3) we have

$$\mu_a([M_1]_\approx \times \dots \times [M_n]_\approx) = \nu_a([M_1]_\approx \times \dots \times [M_n]_\approx) \,. \tag{6}$$

From $\mu_a([M_1]_\equiv \times \dots \times [M_n]_\equiv) > 0$ and σ-additivity, $\mu_a([M_1]_\approx \times \dots \times [M_n]_\approx) > 0$, hence, by equation 6, $\nu_a([M_1]_\approx \times \dots \times [M_1]_\approx) > 0$. By σ-additivity, there exist $N_i \in \mathbb{B}$, for all $1 \le i \le n$, such that $N_i \approx M_i$ and $\nu_a([N_1]_\equiv \times \dots \times [N_n]_\equiv) > 0$. By Theorem 6.9 and Corollary 5.4, from $\nu_a([N_1]_\equiv \times \dots \times [N_n]_\equiv) > 0$ we obtain that $N \xrightarrow{a} \langle N_1, \dots, N_n \rangle$, hence we are done. $\square$

A direct consequence of Proposition 6.15 is that stochastic bisimilarity implies non-stochastic strong bisimilarity:

Corollary 6.16. *If $M \approx N$, then $M \sim N$.*

Note that the converse does not hold, that is, $M \sim N$ does not imply $M \approx N$. A counterexample is shown in Example 6.17.

Example 6.17. *Let $n, m \in \Lambda$ be such that $\iota(n) \neq \iota(m)$. Consider the systems $P = \circledcirc_n(\rho)(\!|R|\!)$ and $Q = \circledcirc_m(\rho)(\!|R|\!)$. We prove that $P \not\approx Q$.*

We proceed by contradiction, assuming $P \approx Q$. Let $\mathcal{C} = [\mathbf{0}(\!|\rho(\!|\diamond|\!) \circ R|\!)]_\approx$ then, by Proposition 6.14, it is easy to see that $(\mathcal{C}, \mathcal{C}) \in \Pi(\approx)$. By Proposition 6.13(3) we have that $\mu_{id}(\mathcal{C}) - \nu_{id}(\mathcal{C}) = 0$, assumed $P \to \mu$ and $Q \to \nu$ (note that by Proposition 6.10 the above subtraction is well defined). Exploiting the definition we obtain,

$$\begin{aligned} \mu_{id}(\mathcal{C}) - \nu_{id}(\mathcal{C}) &= ([\circledcirc_n]^\rho_\mathbf{0})_{\circledcirc_n}([\mathbf{0}]_\equiv \times [\rho]_\equiv) - ([\circledcirc_m]^\rho_\mathbf{0})_{\circledcirc_m}([\mathbf{0}]_\equiv \times [\rho]_\equiv) \\ &= \iota(n) - \iota(m) \end{aligned}$$

By hypothesis $\iota(n) \neq \iota(m)$, hence we have a contradiction, therefore $P \not\approx Q$.

This example concludes the discussion started in Section 2 about the importance of having names in pino actions, in the stochastic setting.

We have established many properties of $\approx$; in particular we showed it to be an equivalence relation in Proposition 6.13. But we have not yet shown that it has an essential property of equality, namely that we can “substitute equals for equals”. In other words, we have not shown it to be a congruence relation. It is important to prove that stochastic bisimilarity over the Brane Calculus terms is a congruence, because it means that the Markov processes associated with membranes or systems can be inspected compositionally.

Theorem 6.18 (Congruence). *Let $\sigma, \tau, \rho, \rho' \in \mathbb{B}_{\mathsf{mem}}$ and $P, Q, R \in \mathbb{B}_{\mathsf{sys}}$. Assume that $\sigma \approx \tau$, $\rho \approx \rho'$ and $P \approx Q$, then the following statements hold:*

1. $\alpha.\sigma \approx \alpha.\tau$ *for each* $\alpha \in \{\circlearrowright_n, \circlearrowleft_n, \circlearrowright_n^{\perp} \mid n \in \Lambda\}$,
2. $\beta(\rho).\sigma \approx \beta(\rho').\tau$ *for each* $\beta \in \{\circlearrowright_n^{\perp}, \circledcirc_n \mid n \in \Lambda\}$.
3. $\sigma|\rho \approx \tau|\rho$ *and* $\rho|\sigma \approx \rho|\tau$,
4. $\rho \llparenthesis P \rrparenthesis \approx \rho \llparenthesis Q \rrparenthesis$,
5. $P \circ R \approx Q \circ R$ *and* $R \circ P \approx R \circ Q$.

Proof. Let $\alpha \in \{\circlearrowright_n, \circlearrowleft_n, \circlearrowright_n^{\perp} \mid n \in \Lambda\}$, and $\beta \in \{\circlearrowright_n^{\perp}, \circledcirc_n \mid n \in \Lambda\}$. Assume $\sigma, \tau, \rho, \rho' \in \mathbb{B}_{\mathsf{mem}}$ and $P, Q, R \in \mathbb{B}_{\mathsf{sys}}$, such that $\sigma \approx \tau$, $\rho \approx \rho'$, and $P \approx Q$. We provide a rate bisimulation $\mathcal{R} \subseteq \mathbb{B} \times \mathbb{B}$ such that $(\alpha.\sigma, \alpha.\tau) \in \mathcal{R}$, $(\beta(\rho).\sigma, \beta(\rho').\tau) \in \mathcal{R}$, $(\sigma|\rho, \tau|\rho) \in \mathcal{R}$, $(\rho|\sigma, \rho|\tau) \in \mathcal{R}$, $(\rho\llparenthesis\sigma\rrparenthesis, \rho\llparenthesis\tau\rrparenthesis) \in \mathcal{R}$, $(P \circ R, Q \circ R) \in \mathcal{R}$, and $(R \circ Q, R \circ Q) \in \mathcal{R}$. Let $\mathcal{R} \triangleq \bigcup_{m \in \mathbb{N}} \mathcal{R}_m$ be defined, for $j \in \{1, 2\}$, $\sigma_j, \tau_j \in \mathbb{B}_{\mathsf{mem}}$ and, $P_j, Q_j \in \mathbb{B}_{\mathsf{sys}}$, by:

$$
\begin{aligned}
\mathcal{R}_0 &= \approx \\
\mathcal{R}_{m+1} &= \mathcal{R}_m \cup \{(\alpha.\sigma_1, \alpha.\tau_1) \mid (\sigma_1, \tau_1) \in \mathcal{R}_m\} \cup \\
&\qquad \{(\beta(\sigma_1).\sigma_2, \beta(\tau_1).\tau_2) \mid (\sigma_j, \tau_j) \in \mathcal{R}_m,\ \text{for } j \in \{1, 2\}\} \\
&\qquad \{(\sigma_1|\sigma_2, \tau_1|\tau_2) \mid (\sigma_j, \tau_j) \in \mathcal{R}_m,\ \text{for } j \in \{1, 2\}\} \cup \\
&\qquad \{(\sigma_1\llparenthesis P_1\rrparenthesis, \sigma_2\llparenthesis P_2\rrparenthesis) \mid (\sigma_1, \sigma_2) \in \mathcal{R}_m,\ \text{and } (P_1, P_2) \in \mathcal{R}_m\} \cup \\
&\qquad \{(P_1 \circ P_2, Q_1 \circ Q_2) \mid (P_j, Q_j) \in \mathcal{R}_m,\ \text{for } j \in \{1, 2\}\}\,.
\end{aligned}
$$

Clearly, $(\alpha.\sigma, \alpha.\tau) \in \mathcal{R}$, $(\beta(\rho).\sigma, \beta(\rho).\tau) \in \mathcal{R}$, $(\sigma|\rho, \tau|\rho) \in \mathcal{R}$, $(\rho|\sigma, \rho|\tau) \in \mathcal{R}$, $(\rho\llparenthesis\sigma\rrparenthesis, \rho\llparenthesis\tau\rrparenthesis) \in \mathcal{R}$, $(P \circ R, Q \circ R) \in \mathcal{R}$, and $(R \circ Q, R \circ Q) \in \mathcal{R}$. Moreover it can be proven, by induction on $m \geq 0$, that $\mathcal{R}_m$ is an equivalence.

We prove now that, for all $m \geq 0$, $\mathcal{R}_m$ is a rate bisimulation, and hence also $\mathcal{R}$ is so. We proceed by induction on $m \geq 0$.

Base case ($m = 0$): it trivially holds by Proposition 6.13(2).

Inductive case ($m + 1$): the inductive hypothesis is that $\mathcal{R}_m$ is a rate bisimulation. Since $\mathcal{R}_{m+1}$ is an equivalence, by Lemma 4.3, it suffices to show that, for any $(M, N) \in \mathcal{R}_{m+1}$ with $M \to \mu$ and $N \to \nu$, $a \in \mathbb{A}$ such that $ar(a) = t \to \langle t_1, \dots, t_n \rangle$, and $(\mathcal{C}_i, \mathcal{C}_i) \in \Pi_{t_i}(\mathcal{R}_{m+1})$, for $1 \leq i \leq n$,

$$
\mu_a(\mathcal{C}_1 \times \cdots \times \mathcal{C}_n) = \nu_a(\mathcal{C}_1 \times \cdots \times \mathcal{C}_n)\,. \tag{7}
$$

Notice first that each syntactic operator can be "lifted" to $\mathcal{R}_m$-closed pairs, that is, the corresponding operations over $\mathcal{R}_m$-closed sets preserve $\mathcal{R}_m$-closure. Formally, the following properties hold, for $k \geq 0$:

(a) if $(\sigma', \tau') \in \mathcal{R}_k$ and $(\mathcal{S}, \mathcal{S}) \in \Pi_{\mathsf{mem}}(\mathcal{R}_k)$, then $(\mathcal{S}_{|\sigma'}, \mathcal{S}_{|\tau'}) \in \Pi_{\mathsf{mem}}(\mathcal{R}_k)$;

(b) if $(P', Q') \in \mathcal{R}_k$ and $(\mathcal{P}, \mathcal{P}) \in \Pi_{\mathsf{sys}}(\mathcal{R}_k)$, then $(\mathcal{P}_{\circ P'}, \mathcal{P}_{\circ Q'}) \in \Pi_{\mathsf{sys}}(\mathcal{R}_k)$;

(c) if $(\sigma', \tau') \in \mathcal{R}_k$ and $(\mathcal{P}, \mathcal{P}) \in \Pi_{\mathsf{sys}}(\mathcal{R}_k)$, then $(\mathcal{P}_{\sigma' \llparenthesis\rrparenthesis}, \mathcal{P}_{\tau' \llparenthesis\rrparenthesis}) \in \Pi_{\mathsf{sys}}(\mathcal{R}_k)$;

(d) if $(\mathcal{S}|\mathcal{T}, \mathcal{S}|\mathcal{T}) \in \Pi_{\mathsf{mem}}(\mathcal{R}_k)$, then, for all $h \leq k$,

$$(\mathcal{S}, \mathcal{S}) \in \Pi_{\mathsf{mem}}(\mathcal{R}_h) \qquad \text{and} \qquad (\mathcal{T}, \mathcal{T}) \in \Pi_{\mathsf{mem}}(\mathcal{R}_h)\,;$$

(e) if $(\mathcal{P} \circ \mathcal{Q}, \mathcal{P} \circ \mathcal{Q}) \in \Pi_{\mathsf{mem}}(\mathcal{R}_k)$, then, for all $h \leq k$,

$$(\mathcal{P}, \mathcal{P}) \in \Pi_{\mathsf{mem}}(\mathcal{R}_h) \qquad \text{and} \qquad (\mathcal{Q}, \mathcal{Q}) \in \Pi_{\mathsf{mem}}(\mathcal{R}_h)\,;$$

(f) if $(\mathcal{S}\llparenthesis\mathcal{P}\rrparenthesis, \mathcal{S}\llparenthesis\mathcal{P}\rrparenthesis) \in \Pi_{\mathsf{sys}}(\mathcal{R}_k)$, then, for all $h \leq k$,

$$(\mathcal{S}, \mathcal{S}) \in \Pi_{\mathsf{mem}}(\mathcal{R}_k) \qquad \text{and} \qquad (\mathcal{P}, \mathcal{P}) \in \Pi_{\mathsf{sys}}(\mathcal{R}_k)\,.$$

Now we are ready to prove equation (7). We proceed case by case.

Case $(M, N) \in \mathcal{R}_m$**:** it holds by inductive hypothesis and by Lemma 4.2, since by construction $\mathcal{R}_m \subseteq \mathcal{R}_{m+1}$.

Case $M = \alpha.\sigma_1$**,** $N = \alpha.\tau_1$**:** hence $(\sigma_1, \tau_1) \in \mathcal{R}_m$, and $\mu = [\alpha]_{\sigma_1}$ and $\nu = [\alpha]_{\tau_1}$. By definition, if $a \neq \alpha$, then $([\alpha]_{\sigma_1})_a(\mathcal{C}_1) = 0 = ([\alpha]_{\tau_1})_a(\mathcal{C}_1)$. Therefore, we are left to prove $([\alpha]_{\sigma_1})_\alpha(\mathcal{C}_1) = ([\alpha]_{\tau_1})_\alpha(\mathcal{C}_1)$. Assume $n \in \Lambda$ be the subscripted name in α. By definition, $([\alpha]_{\sigma_1})_\alpha(\mathcal{C}_1) = \iota(n)$ iff $\sigma_1 \in \mathcal{C}_1$ and $([\alpha]_{\tau_1})_\alpha(\mathcal{C}_1) = \iota(n)$ iff $\tau_1 \in \mathcal{C}_1$. But $(\sigma_1, \tau_1) \in \mathcal{R}_m$, hence, by $(\mathcal{C}_1, \mathcal{C}_1) \in \Pi_{\mathsf{mem}}(\mathcal{R}_{m+1})$ and $\mathcal{R}_m \subseteq \mathcal{R}_{m+1}$, $\sigma_1 \in \mathcal{C}_1$ iff $\tau_1 \in \mathcal{C}_1$. Therefore $([\alpha]_{\sigma_1})_\alpha(\mathcal{C}_1) = ([\alpha]_{\tau_1})_\alpha(\mathcal{C}_1)$.

Case $M = \beta(\sigma_1).\sigma_1$**,** $N = \beta(\tau_1).\tau_2$**:** can be treated similarly.

Case $M = \sigma_1|\sigma_2$**,** $N = \tau_1|\tau_2$**:** hence $(\sigma_j, \tau_j) \in \mathcal{R}_m$, for $j \in \{1, 2\}$. Assume now $\sigma_j \to \mu_j$ and $\tau_j \to \nu_j$, for $j \in \{1, 2\}$, thus $\mu = \mu_1 \;{}_{\sigma_1}\!\obar_{\sigma_2}\; \mu_2$ and $\nu = \nu_1 \;{}_{\tau_1}\!\obar_{\tau_2}\; \nu_2$. Let $a \in \{\circlearrowleft_n, \circledcirc_n \mid n \in \Lambda\}$ (the other cases are treated similarly):

$$\begin{aligned}
(\mu_1 \;{}_{\sigma_1}\!\obar_{\sigma_2}\; \mu_2)_a(\mathcal{C}_1 \times \mathcal{C}_2) &= (\mu_1)_a((\mathcal{C}_1)_{|\sigma_2} \times \mathcal{C}_2) + (\mu_2)_a((\mathcal{C}_1)_{|\sigma_1} \times \mathcal{C}_2) \\
&\hspace{14em} \text{(by definition)} \\
&= (\nu_1)_a((\mathcal{C}_1)_{|\tau_2} \times \mathcal{C}_2) + (\nu_2)_a((\mathcal{C}_1)_{|\tau_1} \times \mathcal{C}_2) \\
&\hspace{8em} \text{(by inductive hypothesis and (a))} \\
&= (\nu_1 \;{}_{\tau_1}\!\obar_{\tau_2}\; \nu_2)_a(\mathcal{C}_1 \times \mathcal{C}_2)\,. \qquad \text{(by definition)}
\end{aligned}$$

Case $M = \sigma_1 \llparenthesis P_1 \rrparenthesis$**,** $N = \sigma_2 \llparenthesis P_2 \rrparenthesis$**:** hence $(\sigma_1, \sigma_2) \in \mathcal{R}_m$, $(P_1, P_2) \in \mathcal{R}_m$. Assume $\sigma_j \to \nu_j$ and $P_j \to \mu_j$, for $j \in \{1, 2\}$, thus $\mu = \mu_1 \mathbin{@}^{\sigma_1}_{P_1} \nu_1$ and $\nu = \mu_2 \mathbin{@}^{\sigma_2}_{P_2} \nu_2$.

Let $a = \mathrm{ph}_n$, for some $n \in \Lambda$. If $\diamond \notin \mathcal{C}_2$, the proof is simple, since by definition we have that, $(\mu_1 \mathbin{@}^{\sigma_1}_{P_1} \nu_1)_{\mathrm{ph}_n}(\mathcal{C}_1 \times \mathcal{C}_2) = 0 = (\mu_2 \mathbin{@}^{\sigma_2}_{P_2} \nu_2)_{\mathrm{ph}_n}(\mathcal{C}_1 \times \mathcal{C}_2)$, otherwise:

$$
\begin{aligned}
(\mu_1 \mathbin{@}^{\sigma_1}_{P_1} \nu_1)_{\mathrm{ph}_n}(\mathcal{C}_1 \times \mathcal{C}_2) &= \sum_{[\sigma' \llparenthesis P_1 \rrparenthesis]_\equiv \subseteq \mathcal{C}_1} (\nu_1)_{\mathfrak{d}_n}([\sigma']_\equiv) && \text{(by definition)} \\
&= \sum_{[\sigma' \llparenthesis P_2 \rrparenthesis]_\equiv \subseteq \mathcal{C}_1} (\nu_2)_{\mathfrak{d}_n}([\sigma']_\equiv) \\
& && \text{(by inductive hypothesis and (f))} \\
&= (\mu_2 \mathbin{@}^{\sigma_2}_{P_2} \nu_2)_{\mathrm{ph}_n}(\mathcal{C}_1 \times \mathcal{C}_2)\,. && \text{(by definition)}
\end{aligned}
$$

Let $a = \mathrm{ph}^\perp_n$, for some $n \in \Lambda$. Note that, by $\mathcal{R}_{m+1}$-closeness and $\mathcal{R}_m \subseteq \mathcal{R}_{m+1}$, $P_1 \in \mathcal{C}_3$ iff $P_2 \in \mathcal{C}_3$. If $P_1 \notin \mathcal{C}_3$ (hence $P_2 \notin \mathcal{C}_3$) or $\diamond \notin \mathcal{C}_4$, by definition $(\mu_1 \mathbin{@}^{\sigma_1}_{P_1} \nu_1)_{\mathrm{ph}^\perp_n}(\mathcal{C}_1 \times \cdots \times \mathcal{C}_4) = 0 = (\mu_2 \mathbin{@}^{\sigma_2}_{P_2} \nu_2)_{\mathrm{ph}^\perp_n}(\mathcal{C}_1 \times \cdots \times \mathcal{C}_4)$; otherwise,

$$
\begin{aligned}
(\mu_1 \mathbin{@}^{\sigma_1}_{P_1} \nu_1)_{\mathrm{ph}^\perp_n}(\mathcal{C}_1 \times \cdots \times \mathcal{C}_4) &= (\nu_1)_{\mathfrak{d}^\perp_n}(\mathcal{C}_1 \times \mathcal{C}_2) && \text{(by definition)} \\
&= (\nu_2)_{\mathfrak{d}^\perp_n}(\mathcal{C}_1 \times \mathcal{C}_2) && \text{(by inductive hypothesis)} \\
&= (\mu_2 \mathbin{@}^{\sigma_2}_{P_2} \nu_2)_{\mathrm{ph}^\perp_n}(\mathcal{C}_1 \times \cdots \times \mathcal{C}_4)\,. \\
& && \text{(by definition)}
\end{aligned}
$$

The case for $a = \mathrm{ex}_n$, for some $n \in \Lambda$, is analogous.

Let $a = id$. We prove $(\mu_1 \mathbin{@}^{\sigma_1}_{P_1} \nu_1)_{id}(\mathcal{C}_1) = (\mu_2 \mathbin{@}^{\sigma_2}_{P_2} \nu_2)_{id}(\mathcal{C}_1)$ noticing that

$$\mu_{id}(\mathcal{C}_{\sigma_1 \llparenthesis\rrparenthesis}) = \mu_{id}(\mathcal{C}_{\sigma_2 \llparenthesis\rrparenthesis})$$

holds by inductive hypothesis and (c);

$$\sum^{n \in \Lambda}_{X' \llparenthesis X'' \llparenthesis [\diamond]_\equiv \rrparenthesis \circ [P]_\equiv \rrparenthesis \subseteq \mathcal{C}_1} (\nu_1)_{\circledcirc_n}(X' \times X'') = \sum^{n \in \Lambda}_{X' \llparenthesis X'' \llparenthesis [\diamond]_\equiv \rrparenthesis \circ [Q]_\equiv \rrparenthesis \subseteq \mathcal{C}_1} (\nu_2)_{\circledcirc_n}(X' \times X'')$$

holds by inductive hypothesis, (f) and (e); and

$$
\begin{aligned}
&\sum^{n \in \Lambda}_{X' | X'' \llparenthesis Y'' \rrparenthesis \circ Y' \subseteq \mathcal{C}_1} \frac{(\mu_1)_{\mathrm{ex}_n}(X' \times Y' \times Y'') \cdot (\nu_1)_{\mathfrak{d}^\perp_n}(X'')}{\iota(n)} = \\
&\qquad \sum^{n \in \Lambda}_{X' | X'' \llparenthesis Y'' \rrparenthesis \circ Y' \subseteq \mathcal{C}_1} \frac{(\mu_2)_{\mathrm{ex}_n}(X' \times Y' \times Y'') \cdot (\nu_2)_{\mathfrak{d}^\perp_n}(X'')}{\iota(n)}
\end{aligned}
$$

holds by inductive hypothesis, (d), (f), and (e).

Case $M = P_1 \circ P_2$**,** $N = Q_1 \circ Q_2$**:** can be treated similarly. $\square$

We would like to remark that the proof technique we have used in Theorem 6.18 is quite general, since it can be applied also to other calculi, whenever stochastic bisimilarity is an equivalence and each syntactic constructor is “lifted” to $\mathcal{R}$-closed pairs (i.e., properties like (a)–(f) hold).

In addition to the examples proposed so far, we show another stochastic bisimilarity, which points out the benefits of the compositionality of $\approx$.

Example 6.19 (Garbage collection). *Let $\sigma, \tau \in \mathbb{B}_{\mathsf{mem}}$ be such that $\sigma \approx \tau$. We prove that, for $n, m \in \Lambda$,*

$$\circlearrowleft^{\perp}_{m} \llparenthesis \mathbf{0} \llparenthesis \circlearrowright_{n} \llparenthesis \circlearrowleft^{\perp}_{m} \llparenthesis \diamond \rrparenthesis \rrparenthesis \rrparenthesis \rrparenthesis \approx \diamond \, .$$

By Proposition 6.13(3), Counterexamples 6.5 and 6.6, we have $\mathbf{0} \llparenthesis \circlearrowright_{n} \llparenthesis \diamond \rrparenthesis \rrparenthesis \approx \diamond$ and $\circlearrowleft^{\perp}_{m} \llparenthesis \diamond \rrparenthesis \approx \diamond$, respectively. The prove follows trivially by repeated applications of Theorem 6.18. This equivalence asserts that the right-hand side term is actually inert, hence can be safely “garbage collected” from the system.

The apparent simplicity of this example points out the advantages given by a bisimulation which is a congruence: we can prove that two processes are equivalent by comparing their corresponding parts. In fact, one can check that proving this bisimilarity by means of direct calculation of measures would be much more cumbersome.

Remark 6.20. It is worthwhile to notice that the SOS in Table 3 is not properly in the abstract GSOS format as per [33]. Since we are working in the category of measurable spaces, the set of syntactic terms $\mathbb{B}$ has been endowed with the σ-algebra Π generated by structural congruence (Definition 5.1). This brings in an equational theory, and hence the object $(\mathbb{B}, \Pi)$ is not given by a freely generated monad. However, it is possible to show that whenever the LTS respects the equational theory of the congruence, the universal semantics for the freely generated terms factorizes through $(\mathbb{B}, \Pi)$, and hence we have again a fully abstract semantics.

7. Conclusions

In this paper we have presented a stochastic version of the Brane Calculus. Brane systems are interpreted as Markov processes over the measurable space generated by terms up-to syntactic congruence, and where the measures are indexed by the actions of the calculus. We have first introduced a compositional, finitely branching LTS for Brane Calculus. This new system is inspired by the one presented in [1], but simpler: we do not deal with higher-order processes, but by “tuples” of terms. To this end, we have introduced “sorted labelled transition systems” and corresponding bisimulations. Taking advantage of this compositional presentation, we have given a stochastic semantics to Brane systems by defining a suitable Markov kernel. Finally, we have provided an SOS presentation of this stochastic semantics, which is compositional and syntax-driven. We have proved that both the strong (i.e. non-stochastic) bisimulation and the stochastic bisimulation for Brane Calculus are congruences; this allows for compositional reasoning, both in the qualitative and in the quantitative setting.

Stochastic semantics for calculi of biological compartments (but not Brane Calculus) have been given in literature; see [9, 7] for stochastic versions of BioAmbients and [34] for a stochastic π-calculus with polyadic synchronization. However, these semantics are not structural but "pointwise", tailored for stochastic simulations using Gillespie algorithm. As shown in Section 6, a "pointwise" semantics can be readily obtained from the stochastic SOS given in this paper. An interesting future work is to investigate how these simulation algorithms and techniques can be adapted to our setting. For instance, in the stochastic abstract machines of [27, 26] a transition is performed in two steps: first, the stochastic rates are calculated over some data structures representing the "normal forms" of the current process, in a pointwise manner and taking care of not double counting instances; then, a particular transition is picked out and the state (i.e., the process) is changed. Using our approach we could simplify this mechanism by keeping track of the measures for the actual processes: the rate can be computed by composing these measures by σ-additivity, obtaining at the same time also the next state. Even more, the compositionality of our semantics allows to single out the differences between the actual state and the next chosen one.

There are several other directions for further work. First, we think that the notion of sorted labelled transition system can be successfully applied to other calculi which are only apparently higher-order, especially those regarding agent mobility like e.g. the Mobile Ambients.

Then, we can consider further constructs of the Brane Calculus, like "bind & release" and replication [11]. For the latter, we should add rules like $\frac{P \xrightarrow{\alpha} P'}{!P \xrightarrow{\alpha} P' | !P}$ to the LTS of Table 2; on the stochastic side, these rules would lead to a new case in the definition of θ_a in Section 5.

An interesting possibility is to extend the theory of stochastic measures with a notion of "approximate behaviour", in order to quantify how much two systems are bisimilar. This is quite important in biological contexts, where usually we can compare only with approximate data (e.g. coming from experiments).

Finally, we would like to apply the present approach to other measurable aspects; in particular, geometric (e.g. volumes), physic (e.g. pressure, temperature) and chemical aspects are of great interest in the biological domain.

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Appendix A. Some measure theory

Given a set M, a family Σ of subsets of M is called a *σ-algebra* if it contains M and is closed under complements and (infinite) countable unions:

1. $M \in \Sigma$;
2. $A \in \Sigma$ implies $A^c \in \Sigma$, where $A^c = M \setminus A$;
3. $\{A_i\}_{i\in\mathbb{N}} \subset \Sigma$ implies $\bigcup_{i\in\mathbb{N}} A_i \in \Sigma$.

Since $M \in \Sigma$ and $M^c = \emptyset$, $\emptyset \in \Sigma$, hence Σ is nonempty by definition. A σ-algebra is closed under countable set-theoretic operations: is closed under finite unions ($A, B \in \Sigma$ implies $A \cup B = A \cup B \cup \emptyset \cup \emptyset \cup \cdots \in \Sigma$), countable intersections (by DeMorgan's law $A \cap B = (A^c \cup B^c)^c$ in its finite and infinite version), and countable subtractions ($A, B \in \Sigma$ implies $A \setminus B = A \cap B^c \in \Sigma$).

Definition Appendix A.1 (Measurable Space). *Given a set M and a σ-algebra on M, the tuple (M, Σ) is called a* measurable space*, the elements of Σ* measurable sets*, and M the* support-set*.*

A set $\Omega \subseteq 2^M$ is a *generator for the σ-algebra* Σ on M if Σ is the closure of Ω under complement and countable union; we write $\sigma(\Omega) = \Sigma$ and say that Σ is generated by Ω. Note that the σ-algebra generated by a Ω is also the smallest σ-algebra containing Ω, that is, the intersection of all σ-algebras that contain Ω. In particular it holds that a completely arbitrary intersection of σ-algebras is a σ-algebra. A σ-algebra generated by Ω, denoted by $\sigma(\Omega)$, is minimal in the sense that if $\Omega \subset \Sigma$ and Σ is a σ-algebra, then $\sigma(\Omega) \subset \Sigma$. If Ω is a σ-algebra then obviously $\sigma(\Omega) = \Omega$; if Ω is empty or $\Omega = \{\emptyset\}$, or $\Omega = \{M\}$, then $\sigma(\Omega) = \{\emptyset, M\}$; if $\Omega \subset \Sigma$ and Σ is a σ-algebra, then $\sigma(\Omega) \subset \Sigma$. A generator Ω for Σ is a *base for* Σ if it has disjoin elements. Note that if Ω is a base for Σ, all measurable sets in Σ can be decomposed into countable unions of elements in Ω.

A *measure* on a measurable space (M, Σ) is a function $\mu\colon \Sigma \to \mathbb{R}^+_\infty$, where $\mathbb{R}^+_\infty$ denotes the extended positive real line, such that

1. $\mu(\emptyset) = 0$;
2. for any disjoint sequence $\{N_i\}_{i\in I} \subseteq \Sigma$ with $I \subseteq \mathbb{N}$, it holds

$$\mu(\textstyle\bigcup_{i\in I} N_i) = \sum_{i\in I} \mu(N_i)\,.$$

The triple (M, Σ, μ) is called a *measure space*. A measure space (M, Σ, μ) is called *finite* if $\mu(M)$ is a finite real number; it is called *σ-finite* if M can be decomposed into a countable union of measurable sets of finite measure, that is, $M = \bigcup_{i\in I} N_i$, for some $I \subseteq \mathbb{N}$ and $\mu(N_i) \in \mathbb{R}^+$ for each $i \in I$. A set in a measure space has *σ-finite measure* if it is a countable union of sets with finite measure. Specifying a measure includes specifying its domain. If μ is a measure on a measurable space (M, Σ) and Σ' is a σ-algebra contained in Σ, then the restriction μ' of μ to Σ' is also a measure, and in particular a measure on (M', Σ'), for some $M' \subseteq M$ such that Σ' is a σ-algebra on M'.

Given two measurable spaces and measures on them, one can obtain the product measurable space and the product measure on that space. Let (M_1, Σ_1) and (M_2, Σ_2) be measurable spaces, and μ_1 and μ_2 be measures on these spaces. Denote by $\Sigma_1 \otimes \Sigma_2$ the σ-algebra on the cartesian product $M_1 \times M_2$ generated by subsets of the form $B_1 \times B_2$, said rectangles, where $B_1 \in \Sigma_1$ and $B_2 \in \Sigma_2$. The *product measure* $\mu_1 \otimes \mu_2$ is defined to be the unique measure on the measurable space $(M_1 \times M_2, \Sigma_1 \otimes \Sigma_2)$ such that, for all $B_1 \in \Sigma_1$ and $B_2 \in \Sigma_2$

$$(\mu_1 \otimes \mu_2)(B_1 \times B_2) = \mu_1(B_1) \cdot \mu_2(B_2)$$

The existence of this measure is guaranteed by the Hahn-Kolmogorov theorem. The uniqueness of the product measure is guaranteed only in the case that both (M_1, Σ_1, μ_1) and (M_2, Σ_2, μ_2) are σ-finite.

Let $\Delta(M, \Sigma)$ be the family of measures on (M, Σ). It can be organized as a measurable space by considering the σ-algebra generated by the sets $\{\mu \in \Delta(M, \Sigma) : \mu(S) \geq r\}$, for arbitrary $S \in \Sigma$ and $r > 0$.

Given two measurable spaces (M, Σ) and (N, Θ) a mapping $f \colon M \to N$ is *measurable* if for any $T \in \Theta$, $f^{-1}(T) \in \Sigma$. Measurable functions are closed under composition: given $f \colon M \to N$ and $g \colon N \to O$ measurable functions then $g \circ f \colon M \to O$ is also measurable.

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