Quantitative Equational Logic

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We develop a quantitative analogue of equational reasoning, which we call quantitative equational logic. The quantitative equations use, instead of classical equality, quantitative equalities, which are equalities indexed with nonnegative reals. Thus, $s =_\varepsilon t$ means that "s and t are points in a metric space and their distance is less than ε". Quantitative equalities will be used to encode behavioural distances, with ε being an upper bound on the measure of dissimilarity between two terms. We develop the metatheory of this subject. We define a notion of quantitative algebra, which is the quantitative analogue of universal algebra. We prove a completeness theorem for quantitative equational logic, and we show that we obtain monads on suitable categories of metric spaces. We present a set of examples where the free algebra of a quantitative equational theory corresponds to some well-known structure. These examples are: Hausdorff metrics from quantitative semilattices; p-Wasserstein metrics (hence also the Kantorovich metric), and the total variation metric.

CCS Concepts: • **Theory of computation → Equational logic and rewriting**.

Additional Key Words and Phrases: Metric spaces, Equational reasoning, Quantitative Algebras, Completeness

ACM Reference Format:
Giorgio Bacci, Radu Mardare, Prakash Panangaden, and Gordon Plotkin. 2026. Quantitative Equational Logic. *J. ACM* 1, 1 (January 2026), 52 pages. https://doi.org/10.1145/3818603

1 Introduction

This paper is a foundational paper on quantitative equational logic. The goal is to make available the power of equational reasoning in a quantitative setting. Equational reasoning lies at the heart of mathematics and is the cornerstone of algebra. In computer science, it provides a central tool for reasoning about program behaviour. However, with the rise of probabilistic systems and real-time systems, quantitative aspects of program behaviour have also become important. Metrics play a key role in quantitative reasoning and are more and more used in semantics, verification, and machine learning. The present work unifies the equational and the metric approaches and provides the power and flexibility of equational logic for quantitative reasoning.

The work began with a conference paper in LICS 2016 [56] and was subsequently continued by ourselves and several other researchers (a detailed summary of the relevant related papers is in Section 12). There are prior approaches to combining logic and quantitative reasoning. One such combines modal logic with quantitative extensions [34, 35, 50, 52], perhaps with fixed-point

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ACM 1557-735X/2026/1-ART
https://doi.org/10.1145/3818603

operators [49]. Another employs metrics measuring behavioural similarity [19, 82] (as previously advocated by Jou and Smolka [40]) using metric reasoning principles such as the Banach fixed-point theorem.

Equational reasoning has numerous applications in programming languages. One of the exciting themes of research in programming language theory is the algebraic study of computational phenomena initiated by Moggi [65, 66] where he showed how one can view notions of computation as monads. This allowed the incorporation of computational effects into a functional core in a compositional way. It became enormously influential and even led to monads being directly incorporated into programming languages like Haskell. A decade later Plotkin and Power [68, 69] began the study of computational effects from the point of view of equations and operations. From a categorical perspective, one is moving from monads to Lawvere theories (see the excellent historical survey by Hyland and Power for more details [37]).

There are several settings where quantitative reasoning is essential: numerical analysis, probabilistic programs, real-time programs, neural networks, reasoning about privacy, and several more. Some of these, like numerical analysis or neural networks, have spawned a whole subfield of computer science, and it is impossible to give just a few references that capture the extent of the subject. Neural networks have a vast literature. The subject makes contact with probabilistic reasoning as well as parts of analysis (see the textbook by Goodfellow et al. [28]). Gradient descent is the key driver of neural nets, and it has, of course, intimate connections with calculus and differential geometry and has led to the development of differentiable programming [1]. Indeed, the vast majority of modern AI is dominated by quantitative reasoning, with topics like stochastic differential equations and differential geometry playing an important part. Ever since the invention of differential privacy [20], reasoning about security and privacy has also evolved into a subject where quantitative analysis is the central reasoning tool [4]. Thus, there are strong motivations to investigate general techniques for quantitative reasoning.

For the present authors a powerful incentive is developing reasoning principles for probabilistic systems. Work in the semantics of probabilistic computation [39, 49, 50, 73, 74] began in the late 1970's. Early work on lambda-calculi for probabilistic programming is due to Saheb-Djahromi [74]. Claire Jones [38] developed a probabilistic λ-calculus in her thesis, gave an operational semantics and proved adequacy results. The fundamental work on probability monads is due to Lawvere [53][1] and Giry [27]. One can develop a probabilistic λ-calculus using this monad [70]. There has been very interesting work on models for higher-order probabilistic programming based on quasi-Borel spaces [31]; there are now several new developments based on quasi-Borel spaces. All this highlights the vital role played by probabilistic ideas and hence the need for reasoning principles to deal with such settings.

When reasoning about the correctness of probabilistic programs and protocols, one had to incorporate probability into the assertions, thus making quantitative reasoning essential. There has been much recent work on probabilistic reasoning spurred by interest from the machine learning community; see for example [14, 29, 30, 72] among many other research efforts on the theory and practice of probabilistic programming as it applies to machine learning and [23] for a recently developed probabilistic programming language for network applications.

Powerful tools have been built for verification of probabilistic systems [33, 47, 54], these tools are mostly based on modal logics augmented with quantitative parameters. In general, there are many quantitative parameters that appear in performance modelling, and techniques for reasoning about such systems incorporate ideas from process algebra and semantics [32, 79]. Our hope is

[1] Lawvere described the Kleisli category of what later was known as the Giry monad.

that a systematic equational formulation of such systems will allow for the development of new automated reasoning tools that can exploit the apparatus of automated equational reasoning [6].

The key new idea behind quantitative equational logic is to introduce equations annotated with non-negative real numbers written $=_\varepsilon$ to capture the notion of *approximate* equality. One should think of $s =_\varepsilon t$ as saying that s and t are "within ε of each other." This defines a quantitative analogue of equational logic with metric spaces playing a key role in the semantics. We do not emphasize category-theoretic connections here; instead, we concentrate on presenting the notion of quantitative equations as concretely as possible. The bulk of the paper is spent on some very pleasing examples and on general notions developed in the spirit of traditional universal algebra.

Our examples are all of the following form: we give a simple set of equations and define the algebras of the resulting theory. We then induce metrics on the free algebra and identify them with commonly defined metrics. Thus, for example, we show that the Hausdorff metric arises from a quantitative version of semilattices. We show that the total variation metric arises from an axiomatization of convexity in terms of barycentric axioms. We show that the famous Kantorovich metric [42–44][2] arises from a variant of the same axioms. In fact, the W_p metrics, which are generalizations of the Kantorovich metric [85], also arise from a variant of these same axioms. These metrics play a fundamental role in the study of probabilistic bisimulation [67, 81] and transport theory [85] and are important in machine learning [5] as well.

We present both finitary and infinitary versions of these constructions. In the finitary case, we work in the category of (extended) metric spaces and (for example) obtain the space of finitely-supported probability distributions with (for example) the Kantorovich distance as our free model over a metric space. In the infinitary case, we work in the category of complete separable metric spaces and show that the free model over such a space (the completion of the finitary one) yields the space of all Borel probability measures with the Kantorovich distance. For the example of the Hausdorff metric, we work with the category of extended metric spaces.

We mainly work with extended metrics, which take values in $[0, \infty]$. It is necessary to do this to make the construction of free algebras go through. Below, we nearly always discuss extended metrics: if we just say "metric space" we mean "extended metric space"; if we want to specifically talk about "ordinary" metric spaces we say so explicitly. The one exception to our use of extended metric spaces is when we consider barycentric algebras over complete metric spaces, where we use 1-bounded complete metric spaces for simplicity.

Here are some toy examples to illustrate the ideas. Consider the quantitative equation: $x =_\varepsilon y \vdash f(x) =_{\varepsilon/2} y$, where f is a function symbol and x, y are variables. The equation asserts that f shrinks the distance to y by a factor of 2. This innocent-looking theory is actually quite interesting. If we are working with ordinary complete metric spaces, *i.e.* without infinite distances, the only model of the equation is the one-point space. To see this, note that if we take x, y to be the same, we have $x =_0 x$ and thus $x =_0 f(x)$. Using the triangle inequality, we get $x =_{\varepsilon/2} y$; iterating this, we get a sequence of equations that essentially force x and y to be equal. These illustrate the role of metric and convergence ideas. The triangle inequality and convergence will be codified when we formalize quantitative equational logic in Section 3. Note that the previous argument does not work when we have extended metrics: if we take $\varepsilon = \infty$ then we cannot conclude that x equals y. Thus, in general, the models are metric spaces in which all distinct points are at infinite distance.

Another illustrative toy theory is $x =_\varepsilon y \vdash f(x) =_{\varepsilon/2} f(y)$. This says that f is a contractive function, and there are, of course, numerous examples of metric spaces with contractive functions,

[2]It is commonly called the Wasserstein metric or the Earth-mover's distance. It was invented by Kantorovich [42], but was also used by Vasershtein [83] much later and mistakenly attributed to him by Dobrushin; see the historical account in [84].

so the slight change in the equation allows many more models; indeed, there is nothing in the equation that restricts the metric space in any way.

Overview. The plan of the rest of the paper is as follows. Section 2 is background and notation on metric spaces, and we assume basic knowledge of measure theory and topology. Sections 3, 4, and 5 introduce the notions of quantitative equations, quantitative algebras, and algebraic semantics. Section 6 introduces the major examples that we will discuss in detail in Sections 8 and 10. We develop the theory of quantitative algebra in Section 7 and in Section 8, we have a detailed discussion of what the free algebras for our main examples look like. In Section 9 we develop the theory of algebras over complete metric spaces, and in Section 10 we work out what the free algebras of our example theories look like in the category of complete metric spaces. The material in this section has a rather more analysis-heavy character than the rest of the paper.

Section 11 contains the fundamental completeness theorems for quantitative equational logic; in addition to the main theorem there are a few variants. They are analogous to Birkhoff's theorem for ordinary equational logic. In fact we give three closely related completeness theorems. The first is analogous to Birkhoff's completeness theorem for equational theories; we also cover the case of Horn theories. The second is a similar theorem which arises when we restrict to rational indices for the equalities: in this case we can still characterize the same models. Finally we prove a completeness theorem for the case of models defined on complete metric spaces.

Section 12 is a survey of related work including a discussion of developments that appeared since the original conference paper in 2016. We end with some directions for future work in the concluding Section 13.

2 Background and notation

In this section, we collect some basic facts and summarize the notation. We will use both the real numbers and the real numbers with infinity adjoined. We write $\mathbf{R}_+$ for the set of nonnegative reals and $\overline{\mathbf{R}}_+$ for the set of nonnegative extended reals, *i.e.* $\mathbf{R}_+$ with infinity adjoined.

We assume that the reader is familiar with the basic notions of probability theory: σ-algebra, probability measure and related concepts. A *pseudometric* on a set X is a function $d : X \times X \longrightarrow \mathbf{R}_+$ satisfying the axioms:

$$\forall x \in X, d(x,x) = 0\,,$$
$$\forall x, y \in X, d(x,y) = d(y,x)\,,$$
$$\forall x, y, z \in X, d(x,y) \leq d(x,z) + d(z,y)\,.$$

If we require that $\forall x, y \in X, d(x,y) = 0$ implies $x = y$, we call it a *metric*. We will often say "metric" for brevity when it is clear from the context that we are talking about a pseudometric. A set equipped with a (pseudo)metric is called a (pseudo)metric space; we will write (X, d) for a metric space with its metric explicitly indicated when necessary. But if the metric is clear from context, we will often just write X and refer to it as the metric space. The same underlying set with different metrics is regarded as two different metric spaces.

Many of the metrics that arise in this paper can take on infinite values. We call such metrics *extended metrics*. We will avoid saying "extended pseudometrics" relying on context to make it clear whether it is a metric or a pseudometric.

We write **Met** for the category whose objects are metric spaces and the morphisms are non-expansive functions: $f\colon (X, d_X) \longrightarrow (Y, d_Y)$ where $\forall x, x' \in X, d_Y(f(x), f(x')) \leq d_X(x, x')$. We write **EMet** for the category of extended metric spaces: the objects are sets equipped with possibly extended metrics and the morphisms are nonexpansive maps as before. Clearly **Met** is a full subcategory of **EMet**.

Let (X, d) be an object in **EMet**. We define an equivalence relation $\sim$ on X by $x \sim y$ if $d(x, y) < \infty$. We call the equivalence classes the *components*[3] of X; each component is an ordinary metric space. One can leverage many facts about ordinary metric spaces to show properties of extended ones by working with the components.

Suppose one has a nonexpansive map $f : X \longrightarrow Y$, where X and Y are extended metric spaces. Suppose x_1, x_2 are two points in the same component of X, *i.e.* $d_X(x_1, x_2) < \infty$, then, by nonexpansiveness, $d_Y(f(x_1), f(x_2)) < \infty$. In other words, points in the same component of X are mapped to points in the same component of Y. Thus f can be decomposed into a collection of maps, each one between a component of X and a component of Y; these are maps between ordinary metric spaces and are each nonexpansive in the ordinary sense.

The category **EMet** has nicer categorical properties than **Met**: it is complete and cocomplete, whereas **Met** does not even have binary coproducts.

The notions of underlying topology, limit, Cauchy completion and being complete are essentially the same for extended metric spaces as for metric spaces. The σ-algebra generated by the open sets is called the *Borel σ-algebra* and a probability measure defined on this σ-algebra is called a *Borel probability measure.*

3 Quantitative equational theories

We start with an *algebraic similarity type (or signature)* Ω, which is a set of function symbols of finite arity. If n is the arity of the function $f \in \Omega$, we write $f : n \in \Omega$. We view constants as functions of arity 0.

Given a set X of *variables*, let $\hat{\Omega}X$ be the Ω-*algebra of terms* generated from X. Formally, the set of terms is defined as $\hat{\Omega}X = \bigcup_{i\in\mathbb{N}} \hat{\Omega}_i(X)$, where $\hat{\Omega}_i(X)$ denotes the set of terms of height at most $i \in \mathbb{N}$, inductively defined on i as follows:

$$\begin{aligned} \hat{\Omega}_0(X) &= X \\ \hat{\Omega}_{i+1}(X) &= \hat{\Omega}_i(X) \cup \bigcup_{f:n\in\Omega} \{f(t_1, \dots, t_n) \mid t_1, \dots, t_n \in \hat{\Omega}_i(X)\} \end{aligned}$$

The set $\hat{\Omega}X$ naturally carries the structure of an Ω-algebra. In fact, $\hat{\Omega}X$ is the free Ω-algebra generated from X. We refer to the elements of $\hat{\Omega}X$ as Ω-terms over X, or simply as terms, when Ω and X are clear.

We denote by $[X|\hat{\Omega}X]$ the set of *substitutions* $\sigma\colon X \longrightarrow \hat{\Omega}X$. Since $\hat{\Omega}X$ is the free Ω-algebra on X, substitutions of type $X \longrightarrow \hat{\Omega}X$ correspond exactly to Ω-homomorphisms of type $\hat{\Omega}X \longrightarrow \hat{\Omega}X$: each substitution extends uniquely to a homomorphism, and each such homomorphism restricts to a substitution. For convenience, we will often identify substitutions with their homomorphic extensions.

Quantitative equalities. A *quantitative equality* on $\hat{\Omega}X$, is a syntactic expression of the form

$$t =_\varepsilon s$$

for $t, s \in \hat{\Omega}X$ and $\varepsilon \in \mathbf{R}_+$ (*i.e.*, an equality between terms indexed by a non-negative real).

Let $\mathcal{E}(\hat{\Omega}X)$ be the set of quantitative equalities on $\hat{\Omega}X$, and denote by $\mathcal{E}(X)$ the subset of quantitative equalities $x =_\varepsilon y$ between variables $x, y \in X$. We refer to the elements in $\mathcal{E}(X)$ as *basic quantitative equalities.*

[3]Do not confuse this with the connected components of the underlying topological space.

Quantitative judgements. A *quantitative judgement* on $\hat{\Omega}X$ is a Horn clause involving quantitative equalities, *i.e.*, syntactic expression of the form

$$\{s_i =_{\varepsilon_i} t_i \mid i \in I\} \vdash s =_\varepsilon t,$$

where I is an index set (possibly empty), $s_i, t_i, s, t \in \hat{\Omega}X$ and $\varepsilon_i, \varepsilon \in \mathbf{R}_+$ for all $i \in I$. Let $\mathcal{J}(\hat{\Omega}X)$ be the set of quantitative judgements on $\hat{\Omega}X$.
If $\Gamma \vdash \phi \in \mathcal{J}(\hat{\Omega}X)$, where $\Gamma \subseteq \mathcal{E}(\hat{\Omega}X)$ and $\phi \in \mathcal{E}(\hat{\Omega}X)$, we refer to the elements of Γ as the *hypotheses* and to ϕ as the *conclusion* of the quantitative judgement.

When the set of hypotheses of a judgement is empty, we call the judgement *unconditional*.

If all the terms in a judgement contain no variables (they are built up from the constants in the signature), we call the judgement *closed*.

Quantitative equations. We use $\mathcal{J}(X)$ to identify the set of judgements of the form

$$\{x_i =_{\varepsilon_i} y_i \mid i \in I\} \vdash s =_\varepsilon t,$$

with $x_i, y_i \in X$, $s, t \in \hat{\Omega}X$ and $\varepsilon_i, \varepsilon \in \mathbf{R}_+$ for all $i \in I$. Note that the hypotheses must contain only basic equalities. We call the judgements in $\mathcal{J}(X)$ *quantitative equations*. This class of judgements will play a central role in our developments[4], especially in Section 9.

Unconditional judgements are a special case of quantitative equations. Note that, closed judgements, although they have no occurrences of variables, are not necessarily quantitative equations (only when they are unconditional).

When the set of hypotheses of a judgement is finite, we simplify the notation and, instead of

$$\{s_1 =_{\varepsilon_1} t_1, \dots, s_n =_{\varepsilon_n} t_n\} \vdash s' =_{\varepsilon'} t', \qquad\qquad \emptyset \vdash s =_\varepsilon t,$$

we write

$$s_1 =_{\varepsilon_1} t_1, \dots, s_n =_{\varepsilon_n} t_n \vdash s' =_{\varepsilon'} t', \qquad\qquad \vdash s =_\varepsilon t\,,$$

respectively.

We extend the application of substitutions to quantitative equalities and judgements, in the obvious way. For instance, if $\Gamma \subseteq \mathcal{E}(\hat{\Omega}X)$ is a set or quantitative equalities and $\sigma \in [X|\hat{\Omega}X]$ a substitution, we write

$$\sigma(\Gamma) = \{\sigma(t) =_\varepsilon \sigma(s) \mid t =_\varepsilon s \in \Gamma\}.$$

Equational and Horn theories. The quantitative judgements are used for reasoning, and to this end, we define the important concepts of deduction.

Definition 3.1 (Quantitative Horn theories). *Let Ω be a signature and X a set of variables. A quantitative Horn theory on $\hat{\Omega}X$ is a set $\mathcal{U}$ of judgements on $\hat{\Omega}X$ closed under the rules in Table 1.*

If $\mathcal{U}$ is a quantitative Horn theory and $U \subseteq \mathcal{U}$ is such that $\mathcal{U}$ is the smallest quantitative Horn theory containing U, then we say that U is a Horn axiomatization of $\mathcal{U}$; and the elements of U are the axioms of $\mathcal{U}$. We denote by $\overline{U}$ the quantitative Horn theory axiomatized by U.

Let U be a set of quantitative judgements. We say that $\Gamma \vdash \phi$ is *provable* from U if there is a sequence of applications of the closure rules in Table 1 starting from the elements of U that witnesses $\Gamma \vdash \phi \in \mathcal{U}$ — we call it a proof of $\Gamma \vdash \phi$ from U. Due to (Cont), these proofs may be infinite.

Clearly, if a quantitative Horn theory $\mathcal{U}$ is axiomatised by U, then any judgement in $\mathcal{U}$ is provable from the axioms in U.

[4]In [8, 57], these were called "basic quantitative judgements".

(Assumpt)	If $s =_\varepsilon t \in \Gamma$, then $\Gamma \vdash s =_\varepsilon t \in \mathcal{U}$
(Refl)	$\vdash s =_0 s \in \mathcal{U}$
(Symm)	If $\Gamma \vdash s =_\varepsilon t \in \mathcal{U}$, then $\Gamma \vdash t =_\varepsilon s \in \mathcal{U}$
(Triang)	If $\Gamma \vdash s =_{\varepsilon_1} u, \Gamma \vdash u =_{\varepsilon_2} t \in \mathcal{U}$, then $\Gamma \vdash s =_{\varepsilon_1+\varepsilon_2} t \in \mathcal{U}$
(Max)	If $\Gamma \vdash s =_{\varepsilon_1} t \in \mathcal{U}$, then for any $\varepsilon_2 > 0$, $\Gamma \vdash s =_{\varepsilon_1+\varepsilon_2} t \in \mathcal{U}$
(NExp)	If $\{\Gamma \vdash s_i =_\varepsilon t_i \mid i \leq n\} \subseteq \mathcal{U}$, then $\Gamma \vdash f(s_1, \dots, s_n) =_\varepsilon f(t_1, \dots, t_n) \in \mathcal{U}$
(CS)	If $\{\Gamma \vdash \sigma(u_i) =_{\varepsilon_i} \sigma(v_i) \mid i \in I\} \subseteq \mathcal{U}$ and $\{u_i =_{\varepsilon_i} v_i \mid i \in I\} \vdash s =_\varepsilon t \in \mathcal{U}$, then $\Gamma \vdash \sigma(s) =_\varepsilon \sigma(t) \in \mathcal{U}$
(Cont)	If $\{\Gamma \vdash s =_{\varepsilon'} t \mid \varepsilon' > \varepsilon\} \subseteq \mathcal{U}$, then $\Gamma \vdash s =_\varepsilon t \in \mathcal{U}$

Table 1. Closure rules for quantitative Horn theories. The rules are stated for arbitrary sets $\Gamma \subseteq \mathcal{E}(\hat{\Omega}X)$ of quantitative equalities, terms $s, t, s_i, t_i, u_i, v_i \in \hat{\Omega}X$, positive reals $\varepsilon, \varepsilon_i \in \mathbf{R}_+$, operations $f : n \in \Omega$, and substitutions $\sigma \in [X|\hat{\Omega}X]$.

Anticipating the intended semantics, an equality $s =_\varepsilon t$ expresses that s and t are two points in a metric space whose distance is at most ε. Under this interpretation, the rules in Table 1 capture familiar and expected properties.

(Assumpt), (Refl), and (Symm) behave exactly as in classical equational reasoning. Unlike the classical equality, $=_\varepsilon$ is not transitive, and transitivity is replaced by the *triangle inequality* (Triang). (Max) encodes the fact that the index ε in the equality is an upper bound of the distance. (NExp) expresses that all operations in the signature have non-expansive interpretation (this is the quantitative analogue of the classical congruence rule). (CS) combines the usual cut and substitution principles (see Lemma 3.2). Finally, (Cont) captures the fact that $\{t \mid s =_\varepsilon t\}$ is the closed ball with radius ε centred at s. Note that it is the only infinitary closure rule. In Example 5.5, we later show that (Cont) cannot be derived from the other rules in Table 1.

Lemma 3.2. *A set $\mathcal{U}$ of judgements is closed under the rule (CS) in Table 1 if and only if it is simultaneously closed under the rules (Cut) and (Subst) stated below, where $\sigma \in [X|\hat{\Omega}X]$:*

(Cut) *If $\{\Gamma \vdash s_i =_{\varepsilon_i} t_i \mid i \in I\} \subseteq \mathcal{U}$ and $\{s_i =_{\varepsilon_i} t_i \mid i \in I\} \vdash t =_\varepsilon s \in \mathcal{U}$, then $\Gamma \vdash t =_\varepsilon s \in \mathcal{U}$*

(Subst) *If $\Gamma \vdash t =_\varepsilon s \in \mathcal{U}$, then $\sigma(\Gamma) \vdash \sigma(t) =_\varepsilon \sigma(t) \in \mathcal{U}$*

Proof. We show that (Cut) and (Subst) are derivable from (CS). By instantiating (CS) with $u_i = s_i$, $v_i = t_i$, and σ as the identity, we obtain (Cut). The rule (Subst) is obtained by instantiating (CS) with $\Gamma = \sigma(\{u_i =_{\varepsilon_i} v_i \mid i \in I\})$.

Conversely, to show that (CS) is derivable from (Subst) and (Cut), first, apply the rule (Subst) to $\{u_i =_{\varepsilon_i} v_i \mid i \in I\} \vdash s =_\varepsilon t \in \mathcal{U}$ to obtain $\{\sigma(u_i) =_{\varepsilon_i} \sigma(v_i) \mid i \in I\} \vdash \sigma(s) =_\varepsilon \sigma(t) \in \mathcal{U}$. Next, apply (Cut) along with $\{\Gamma \vdash s_i =_{\varepsilon_i} t_i \mid i \in I\} \subseteq \mathcal{U}$, to derive $\Gamma \vdash \sigma(s) =_\varepsilon \sigma(t) \in \mathcal{U}$. ∎

It is also useful to define the concept of quantitative equational theory, which will not be sets of judgements, but of quantitative equations as defined above.

Definition 3.3 (Quantitative equational theories). *Let Ω be a signature and X a set of variables. A quantitative equational theory on $\hat{\Omega}X$ is a set $\mathcal{U}$ of quantitative equations on $\hat{\Omega}X$ closed under the rules in Table 1 stated under the restriction that Γ is always a set of basic quantitative equalities, i.e., $\Gamma \subseteq \mathcal{E}(X)$.*

Similarly to the case of Horn theories, if $\mathcal{U}$ is a quantitative equational theory and $U \subseteq \mathcal{U}$ is such that $\mathcal{U}$ is the smallest quantitative equational theory containing U, then we say that U is an equational axiomatization of $\mathcal{U}$; and the elements of U are the axioms of $\mathcal{U}$.

As a consequence of the definition, for any quantitative equation $\Gamma \vdash \phi \in \mathcal{U}$, we will always have at least one sequence of applications of the closure rules to the elements of U that witnesses $\Gamma \vdash \phi \in \mathcal{U}$ — we call it a proof of $\Gamma \vdash \phi$ from U. Due to (Cont), these proofs can be infinite.

For U a set of quantitative equations, we will abuse the notation and denote by $\overline{U}$ the quantitative theory axiomatized by U. This will not be problematic as the context will clarify its meaning. We also often omit the adjective "quantitative" when we refer to equational or Horn theories.

An equational/Horn theory $\mathcal{U}$ over $\hat{\Omega}X$ is *inconsistent* if

$$\vdash s =_0 t \in \mathcal{U} \text{ for all } s, t \in \hat{\Omega}X .$$

Notice that, when X contains two distinct variables $x, y \in X$, inconsistency can be characterized by $\vdash x =_0 y \in \mathcal{U}$, since from it, using (CS), we would be able to derive $\vdash s =_0 t \in \mathcal{U}$, for any $s, t \in \hat{\Omega}X$.

A theory $\mathcal{U}$ is *consistent* if it is not inconsistent. Note that, if a Horn theory $\mathcal{U}$ is inconsistent, then $\mathcal{U} = \mathcal{J}(\hat{\Omega}X)$. Similarly, $\mathcal{J}(X)$ is the only inconsistent equational theory.

Corollary 3.4. *Let Ω be a signature, X a set of variables, $\mathcal{U}$ a Horn/equational theory on $\hat{\Omega}X$, and $Y \subseteq X$. Let $\mathcal{V}$ be the restriction of $\mathcal{U}$ to $\hat{\Omega}Y$, that is*

$$\mathcal{V} = \{\{s_i =_{\varepsilon_i} t_i \mid i \in I\} \vdash s =_\varepsilon t \in \mathcal{U} \mid s_i, t_i, s, t \in \hat{\Omega}Y\} .$$

Then $\mathcal{V}$ is itself a Horn/equational theory.

4 Quantitative algebras

Given the notions of quantitative equation/judgement, we are ready to define algebras based on them. We mimic the usual presentation of universal algebra, see for example [15], while adapting the presentation to the quantitative settings. Thus, quantitative algebras are algebras over metric spaces, such that the metric structure interacts well with the algebraic structure: all the algebraic operators are nonexpansive with respect to the given metric.

Definition 4.1 (Quantitative Algebra). *Given a signature Ω, a* quantitative algebra over Ω *is a tuple $\mathcal{A} = (A, \Omega^{\mathcal{A}}, d^{\mathcal{A}})$, where*

- *$(A, \Omega^{\mathcal{A}})$ is an algebra over the signature Ω, meaning that each $f : n \in \Omega$ has a corresponding interpretation $f^{\mathcal{A}} \colon A^n \longrightarrow A$ in $\mathcal{A}$;*
- *$d^{\mathcal{A}} \colon A \times A \longrightarrow \overline{\mathbf{R}}_+$ is an extended metric on A such that all the operators in the signature are non-expansive: for any $f : n \in \Omega$ and $a_i, b_i \in A$,*

$$d^{\mathcal{A}}(f^{\mathcal{A}}(a_1, \dots, a_n), f^{\mathcal{A}}(b_1, \dots, b_n)) \leq \max_{i \leq n} d^{\mathcal{A}}(a_i, b_i) .$$

A quantitative algebra is *degenerate* if its carrier is empty or a singleton.

Note that what is usually called an Ω-algebra[5] $\mathcal{A} = (A, \Omega^{\mathcal{A}})$ is a particular case of quantitative algebra where the set A is endowed with the discrete metric assigning distinct elements $a, b \in A$ infinite distance ($d^{\mathcal{A}}(a, b) = \infty$ iff $a \neq b$). Indeed, all maps out of a discrete space are trivially non-expansive.

Definition 4.2 (Homomorphism of quantitative algebras). *Let $\mathcal{A} = (A, \Omega^{\mathcal{A}}, d^{\mathcal{A}})$ and $\mathcal{B} = (B, \Omega^{\mathcal{B}}, d^{\mathcal{B}})$ be quantitative Ω-algebras. A* homomorphism *from $\mathcal{A}$ to $\mathcal{B}$ is a* non-expansive Ω-homomorphism *$h \colon A \longrightarrow B$; i.e., such that*

[5] Or often a Σ-algebra.

- *for any $f : n \in \Omega$ and $a_i \in A$, $h(f^{\mathcal{A}}(a_1, \ldots, a_n)) = f^{\mathcal{B}}(h(a_1), \ldots, h(a_n))$;*
- *for arbitrary $a, b \in A$, $d^{\mathcal{B}}(h(a), h(b)) \leq d^{\mathcal{A}}(a, b)$.*

Note that the identity maps are homomorphisms and that homomorphisms are closed under composition, hence quantitative Ω-algebras and their homomorphisms form a category, denoted hereafter by $\mathbf{Q}[\Omega]$.

An *isomorphism* of quantitative algebras is a homomorphism $h\colon \mathcal{A} \longrightarrow \mathcal{B}$ that is bijective on the carrier sets and such that its inverse $h^{-1}\colon \mathcal{B} \longrightarrow \mathcal{A}$ is a homomorphism as well. In this case, h is an isometry of the underlying metric spaces.

Definition 4.3 (Subalgebras). *The quantitative algebra $\mathcal{B} = (B, \Omega^{\mathcal{B}}, d^{\mathcal{B}})$ is a subalgebra of the quantitative algebra $\mathcal{A} = (A, \Omega^{\mathcal{A}}, d^{\mathcal{A}})$, denoted by $\mathcal{B} \leq \mathcal{A}$, if*

- *$\mathcal{B}$ is an Ω-subalgebra of $\mathcal{A}$, i.e., $B \subseteq A$ and for arbitrary $f : n \in \Omega$ and $b_i \in B$, $f^{\mathcal{B}}(b_1, \ldots, b_n) = f^{\mathcal{A}}(b_1, \ldots, b_n)$;*
- *isometry[6]: for any $b, b' \in B$, $d^{\mathcal{B}}(b, b') = d^{\mathcal{A}}(b, b')$.*

If $C = (C, \Omega^{C}, d^{C})$ is another quantitative Ω-algebra, C is a subalgebra of $\mathcal{A}$ up-to-isomorphism, denoted by $C \precsim \mathcal{A}$, if there exists a subalgebra $\mathcal{B} \leq \mathcal{A}$ such that $\mathcal{B}$ is isomorphic to C.

5 Algebraic semantics for quantitative equations

We now turn to interpreting quantitative equations and judgements in a quantitative algebra, and to defining what it means for such an algebra to satisfy them.

Definition 5.1 (Assignment). *Given a quantitative algebra $\mathcal{A} = (A, \Omega^{\mathcal{A}}, d^{\mathcal{A}})$ of signature Ω and a set X of variables, an* assignment *on $\mathcal{A}$ is a function $\alpha\colon X \longrightarrow A$ which interprets variables as elements of the algebra. Such a map can be canonically extended to a homomorphism of Ω-algebras*

$$\alpha\colon \hat{\Omega}X \longrightarrow A$$

by defining, for any $f : n \in \Omega$ and any $t_1, \ldots, t_n \in \hat{\Omega}X$,

$$\alpha(f(t_1, \ldots, t_n)) = f^{\mathcal{A}}(\alpha(t_1), \ldots, \alpha(t_n)).$$

We denote by $\Omega[X|\mathcal{A}]$ the set of assignments on $\mathcal{A}$.

Definition 5.2 (Satisfaction). *Let $\mathcal{A} = (A, \Omega^{\mathcal{A}}, d^{\mathcal{A}})$ be a quantitative algebra, X a set of variables and $\{s_i =_{\varepsilon_i} t_i \mid i \in I\} \vdash s =_{\varepsilon} t \in \mathcal{J}(\hat{\Omega}X)$ a quantitative (equation/Horn) judgement on $\hat{\Omega}X$.*

$\mathcal{A}$ satisfies *the judgement under the assignment $\alpha \in \Omega[X|\mathcal{A}]$, if*

$$[\ d^{\mathcal{A}}(\alpha(t_i), \alpha(s_i)) \leq \varepsilon_i \ \text{ for all } \ i \in I\ ] \ \text{ implies } \ d^{\mathcal{A}}(\alpha(s), \alpha(t)) \leq \varepsilon.$$

We denote this by $\{s_i =_{\varepsilon_i} t_i \mid i \in I\} \models_{\mathcal{A},\alpha} s =_{\varepsilon} t$.

We say that $\mathcal{A}$ satisfies the judgement*, or that it is a* model *of the judgement, if $\mathcal{A}$ satifies it under all assignments $\alpha \in \Omega[X|\mathcal{A}]$. We denote this by $\{s_i =_{\varepsilon_i} t_i \mid i \in I\} \models_{\mathcal{A}} s =_{\varepsilon} t$.*

Similarly, for a set of quantitative judgements $\mathcal{U}$, we say that $\mathcal{A}$ is a model of $\mathcal{U}$ if $\mathcal{A}$ satisfies every element of $\mathcal{U}$. For a collection C of quantitative Ω-algebras, we write $\Gamma \models_C s =_{\varepsilon} t$, to mean that $\Gamma \models_{\mathcal{A}} s =_{\varepsilon} t$, for all $\mathcal{A} \in C$.

A set of quantitative judgments is *satisfiable* if it has a non-degenerate model. Otherwise, it is *not satisfiable.*

We denote by $\Omega[S]$ both the class of models of signature Ω of the quantitative equational/Horn theory axiomatized by the set of judgements S, and the full subcategory of Ω-quantitative algebras satisfying S. Note that $\Omega[S]$ is not a variety in the traditional sense, as demonstrated in [57].

[6]We use the term "isometry" to mean a distance-preserving map between metric spaces and not necessarily an isomorphism of metric spaces, i.e., it doesn't need to be a bijection.

Theorem 5.3 (Soundness). *Let $\mathcal{A} = (A, \Omega, d)$ be an Ω quantitative algebra and $\hat{\Omega}X$ the set of Ω-terms over the set X of variables. Define the following sets of $\hat{\Omega}X$ judgements:*

$$Horn_{\hat{\Omega}X}(\mathcal{A}) = \{\Gamma \vdash s =_\varepsilon t \in \mathcal{J}(\hat{\Omega}X) \mid \Gamma \models_{\mathcal{A}} s =_\varepsilon t\}\,,$$
$$Eq_{\hat{\Omega}X}(\mathcal{A}) = \{\Gamma \vdash s =_\varepsilon t \in \mathcal{J}(X) \mid \Gamma \models_{\mathcal{A}} s =_\varepsilon t\}\,.$$

Then, $Horn_{\hat{\Omega}X}(\mathcal{A})$ is a quantitative Horn theory, and $Eq_{\hat{\Omega}X}(\mathcal{A})$ is a quantitative equational theory.

Proof. We need to show that both $Horn_{\hat{\Omega}X}(\mathcal{A})$ and $Eq_{\hat{\Omega}X}(\mathcal{A})$ are closed under the rules in Table 1. The proof is similar for the two cases, and for this reason we only do it for $Horn_{\hat{\Omega}X}(\mathcal{A})$.

(Assumpt): If $s =_\varepsilon t \in \Gamma$, then obviously $\Gamma \models_{\mathcal{A}} s =_\varepsilon t$, since for any assignment $\alpha \in \Omega[X|\mathcal{A}]$ such that all the quantitative equalities $u =_\delta v \in \Gamma$ are such that $d(\alpha(u), \alpha(v)) \leq \delta$, we have in particular $d(\alpha(s), \alpha(t)) \leq \varepsilon$.

(Refl): Since d is a metric, for any assignment α, $d(\alpha(t), \alpha(t)) = 0$, meaning that $\models_{\mathcal{A}} t =_0 t$.

(Symm): Since d is symmetric, we have $\Gamma \models_{\mathcal{A}} s =_\varepsilon t$ iff $\Gamma \models_{\mathcal{A}} t =_\varepsilon s$.

(Triang): If $\Gamma \models_{\mathcal{A}} t =_{\varepsilon_1} u$ and $\Gamma \models_{\mathcal{A}} u =_{\varepsilon_2} s$, then under any assignment α that satisfies all quantitative equalities in Γ, we have $d(\alpha(t), \alpha(u)) \leq \varepsilon_1$ and $d(\alpha(u), \alpha(s)) \leq \varepsilon_2$. Because d is a metric, hence satisfies the triangle inequality, we also get $d(\alpha(t), \alpha(s)) \leq \varepsilon_1 + \varepsilon_2$.

(Max): If $\Gamma \models_{\mathcal{A}} t =_{\varepsilon_1} s$, then for any assignment α that satisfies all the quantitative equalities in Γ we have $d(\alpha(t), \alpha(s)) \leq \varepsilon_1$, implying that we also have $d(\alpha(t), \alpha(s)) \leq \varepsilon_1 + \varepsilon_2$ for any $\varepsilon_2 > 0$.

(Nexp): If for all $i \leq n$, $\Gamma \models_{\mathcal{A}} s_i =_\varepsilon t_i$, then for any assignment α that satisfies all the quantitative equalities in Γ we have $d(\alpha(s_i), \alpha(t_i)) \leq \varepsilon$. Since $\mathcal{A}$ is a quantitative algebra, the interpretation $f^{\mathcal{A}}$ of any operation $f : n \in \Omega$ is nonexpansive. Hence,

$$d(\alpha((t_1, \dots, t_n), \alpha(f(s_1, \dots, s_n))) = d(f^{\mathcal{A}}(\alpha(t_1), \dots, \alpha(t_n))), \alpha(f^{\mathcal{A}}(\alpha(s_1), \dots, \alpha(s_n))) \leq \varepsilon\,.$$

This proves that $\Gamma \models_{\mathcal{A}} f(t_1, \dots, t_n) =_\varepsilon f(s_1, \dots, s_n)$.

(CS): Let $\sigma \in [X|\hat{\Omega}X]$ be a substitution and suppose that $\{u_i =_{\varepsilon_i} v_i \mid i \in I\} \models_{\mathcal{A}} s =_\varepsilon t$ and $\Gamma \models_{\mathcal{A}} \sigma(u_i) =_{\varepsilon_i} \sigma(v_i)$, for all $i \in I$. Note that if α is an assignment on $\mathcal{A}$, then so is $\alpha \circ \sigma$.

Let $\alpha \in \Omega[X|\mathcal{A}]$ be an assignment on $\mathcal{A}$. The first working hypothesis guarantees that

$$[d(\alpha(\sigma(u_i)), \alpha(\sigma(v_i))) \leq \varepsilon_i, \text{ for all } i \in I] \text{ implies } d(\alpha(\sigma(s)), \alpha(\sigma(t))) \leq \varepsilon\,. \tag{1}$$

The second hypothesis gives us that

$$[d(\alpha(u), \alpha(v)) \leq \delta \text{ for any } u =_\delta v \in \Gamma] \text{ implies } [d(\alpha(\sigma(u_i)), \alpha(\sigma(v_i))) \leq \varepsilon_i, \text{ for all } i \in I]\,. \tag{2}$$

By combining (1) and (2) we obtain $\Gamma \models_{\mathcal{A}} \sigma(s) =_\varepsilon \sigma(t)$.

(Cont): Suppose that for a given $\varepsilon \in \mathbf{R}_+$ we have $\Gamma \models_{\mathcal{A}} s =_{\varepsilon'} t$, for all $\varepsilon' > \varepsilon$. This means that for any assignment $\alpha \in \Omega[X|\mathcal{A}]$

$$[d(\alpha(u), \alpha(v)) \leq \delta \text{ for any } u =_\delta v \in \Gamma] \text{ implies } [d(\alpha(s), \alpha(t)) \leq \varepsilon', \text{ for all } \varepsilon' > \varepsilon].$$

As the statement $[d(\alpha(s), \alpha(t)) \leq \varepsilon', \text{ for all } \varepsilon' > \varepsilon]$ is equivalent to $d(\alpha(s), \alpha(t)) \leq \varepsilon$, the above implication actually means $\Gamma \models_{\mathcal{A}} s =_\varepsilon t$.

Note that in this proof, we can freely assume that all the elements in Γ are quantitative equalities among variables only, and in this way we get the soundness proof for $Eq_{\hat{\Omega}X}(\mathcal{A})$. ∎

A direct consequence of this soundness result is that for any set U of judgements that axiomatize the Horn/equational theory $\mathcal{U}$, we have

$$\Omega[U] = \Omega[\mathcal{U}]\,.$$

Moreover, the following corollary also holds:

Corollary 5.4. *Let U be a set of quantitative equations that axiomatize both the equational theory $\mathcal{U}$ and the Horn theory $\mathcal{H}$. Then $\mathcal{U} \subseteq \mathcal{H}$ and moreover, $\Omega[\mathcal{U}] = \Omega[\mathcal{H}]$.*

Proof. Since quantitative equations are, in particular, Horn judgements and since both equational theories and Horn theories are closed under the rules in Table 1, it follows that $\mathcal{U} \subseteq \mathcal{H}$.

For the second part of the statement, let $\mathcal{A} \in \Omega[\mathcal{U}]$. Then $\mathcal{A}$ satisfies all equations in $\mathcal{U}$, and in particular those in U. By soundness, $\mathcal{A}$ must therefore satisfy all the judgements derivable from U, namely all judgements in $\mathcal{H}$. Hence $\mathcal{A} \in \Omega[\mathcal{H}]$. Conversely, suppose that $\mathcal{A} \in \Omega[\mathcal{H}]$. Then $\mathcal{A}$ satisfies all the judgements in $\mathcal{H}$. Since $\mathcal{U} \subseteq \mathcal{H}$, it follows that $\mathcal{A}$ satisfies all the equations in $\mathcal{U}$. Therefore, $\mathcal{A} \in \Omega[\mathcal{U}]$. ∎

The previous corollary essentially shows that, in concrete examples of theories axiomatized by quantitative equations, it makes no difference whether we work with the equational theories themselves or with the larger Horn theories they generate. Indeed, both have the same models, and hence the same expressive power from a semantic perspective.

We conclude this section by demonstrating the role of the infinitary rule (Cont) in quantitative reasoning and how its effect cannot be reproduced by the other finitary rules.

Example 5.5. *Consider the signature $\Omega = \{a : 0, b : 0, f : 1\}$, consisting of two constants, a and b, and a unary operator f and the set of variables $X = \{x, y\}$. Let U be the set of judgements over $\hat{\Omega}[X]$, containing the quantitative equations*

$$\begin{aligned} &\vdash a =_2 b \\ &x =_{1+\varepsilon} y \vdash x =_{1+\frac{\varepsilon}{2}} y \\ &x =_1 y \vdash f(x) =_0 f(y) \end{aligned}$$

for arbitrary $\varepsilon \in \mathbf{R}_+$.

Let $\overline{\mathcal{U}}$ denote the Horn theory axiomatized by $\mathcal{U}$, and $\tilde{\mathcal{U}}$ the smallest set of judgements that contains U and is closed under all rules in Table 1 except for (Cont). Clearly, $\mathcal{U} \subseteq \tilde{\mathcal{U}} \subseteq \overline{\mathcal{U}}$.

Next, we show that the last inclusion is strict. Specifically we prove that $\vdash f(a) =_0 f(b)$ is in $\mathcal{U}$ but not in $\tilde{\mathcal{U}}$. First, note that for every integer $n \geq 1$, we have $\vdash a =_{1+\frac{1}{2^n}} b \in \tilde{\mathcal{U}}$. This is shown by a straightforward induction on n: the base case, $\vdash a =_{1+\frac{1}{2}} b \in \tilde{\mathcal{U}}$ follows from the first two axioms together with (CS); for the inductive step, one simply applies (CS) to the induction hypothesis.

Now, we can apply (Max), and we obtain that for any $\varepsilon > 0$,

$$\vdash a =_{1+\varepsilon} b \in \tilde{\mathcal{U}}.$$

In $\overline{\mathcal{U}}$ we can apply (Cont) to prove $\vdash a =_1 b \in \overline{\mathcal{U}}$; then the third axiom gives us $\vdash f(a) =_0 f(b) \in \overline{\mathcal{U}}$.

However, we cannot prove $\vdash a =_1 b$ in $\tilde{\mathcal{U}}$. Indeed, suppose such a proof is possible. Since in $\tilde{\mathcal{U}}$ no infinitary proof is allowed, we must have a finitary proof using a finite set of hypotheses of type $\vdash a =_{1+\varepsilon} b$ obtained from the axioms. So, let $\delta > 0$ be the smallest value such that $\vdash a =_{1+\delta} b$ is used in the proof of $\vdash a =_1 b$. Consider now the quantitative algebra $\mathcal{A} = (A, \Omega, d)$ such that $d(a^{\mathcal{A}}, b^{\mathcal{A}}) = 1 + \delta$. This should be a model for all the hypotheses that allow us to prove $\vdash a =_1 b$, and due to soundness, it should also be a model for $\vdash a =_1 b$. However, this is obviously not the case since $\delta > 0$. Hence, such a proof is not possible. Since the only way to infer any relationship between $f(a)$ and $f(b)$ would require the use of the third axiom, we conclude that $\vdash f(a) =_0 f(b) \notin \tilde{\mathcal{U}}$.

Note that, although U has only quantitative equations, a similar argument can be also carried out using proper Horn judgments as axioms.

There is one more interesting observation that we can make from this example: it is not the case that $\overline{\mathcal{U}}$ can be computed by only applying (Cont) to elements of $\tilde{\mathcal{U}}$. Because, in the example above, if we apply (Cont) to elements of $\tilde{\mathcal{U}}$ we get $\vdash a =_1 b$, but not $\vdash f(a) =_0 f(b)$, which requires more finitary proofs following an infinitary proof.

We conclude this section by proving that for any quantitative (equational/Horn) theory $\mathcal{U}$, $\Omega[\mathcal{U}]$ is closed under taking subalgebras. This is a result that we will use later, both for proving universality and the completeness of quantitative equational logic. However, the classic variety and quasivariety theorems of Birkhoff do not hold for quantitative algebras; and in [57] novel concepts of variety and quasivariety have been introduced, that generalize to quantitative settings the concepts of Birkhoff, and for these, quantitative variety and quasivariety theorems are proven.

Lemma 5.6. *Let Ω be an algebraic signature and $\mathcal{U}$ a quantitative (equational/Horn) theory over $\hat{\Omega}X$. If $\mathcal{A} \in \Omega[\mathcal{U}]$ and $\mathcal{B} \leq \mathcal{A}$, then $\mathcal{B} \in \Omega[\mathcal{U}]$.*

Proof. Since $\mathcal{B} \leq \mathcal{A}$, the canonical inclusion $\mathcal{B} \hookrightarrow \mathcal{A}$ is a homomorphism of quantitative algebras. Suppose that $\{s_i =_{\varepsilon_i} t_i \mid i \in I\} \vdash s =_\varepsilon t \in \mathcal{U}$. Hence, $\{s_i =_{\varepsilon_i} t_i \mid i \in I\} \models_{\mathcal{A}} s =_\varepsilon t$, meaning that for any asslgnment $\alpha \in \Omega[X|\mathcal{A}]$ on $\mathcal{A}$,

$$[d^{\mathcal{A}}(\alpha(s_i), \alpha(t_i)) \leq \varepsilon_i \text{ for all } i \in I] \text{ implies } d^{\mathcal{A}}(\alpha(s), \alpha(t) \leq \varepsilon.$$

Let $\iota \in \Omega[X|\mathcal{B}]$ be an arbitrary assignment on $\mathcal{B}$ and suppose $d^{\mathcal{B}}(\alpha(s_i), \iota(t_i)) \leq \varepsilon_i$, for all $i \in I$. Clearly, ι is also an assignment on $\mathcal{A}$ (via post composition with $\mathcal{B} \hookrightarrow \mathcal{A}$). Therefore, we also have $d^{\mathcal{A}}(\iota(s), \iota(t)) \leq \varepsilon$. Hence, $d^{\mathcal{B}}(\iota(s), \iota(t)) \leq \varepsilon$. ∎

6 Fundamental examples

We describe some key examples of quantitative algebras which play an important role in the rest of the paper. These examples are not merely illustrations of the theory, but are important in their own right as they play a foundational role in potential applications, as well as being of independent mathematical interest. In this section, we introduce these algebras and state the main claims about them. Proofs of these claims appear in later sections after we have developed the requisite machinery.

In 1949, Stone [78] introduced what he called barycentric algebras, a terminology that we follow here, though they are often called "convex algebras". This is an equational theory in the traditional sense of equation; our quantitative algebra examples will build on this theory. The canonical examples of such algebras are spaces of probability distributions. In this section, we always assume that the underlying metrics are one-bounded, *i.e.*, they take values in $[0, 1]$.

Consider the signature

$$\mathcal{B} = \{+_e : 2 \mid e \in [0, 1]\}$$

containing, for each $e \in [0, 1]$, a binary operator $+_e$. We call it the *barycentric signature.*

Let X be a set of variables. Stone's barycentric equational theory is generated by the following axioms, for $x, y, z \in X$, $\varepsilon \in [0, 1]$ and $e, e' \in [0, 1)$:

(B1) $\vdash x +_1 y =_0 x$
(B2) $\vdash x +_e x =_0 x$
(SC) $\vdash x +_e y =_0 y +_{1-e} x$
(SA) $\vdash (x +_e y) +_{e'} z =_0 x +_{ee'} (y +_{\frac{e'-ee'}{1-ee'}} z)$

Stone's theory is an algebraic theory in the traditional sense. Above we replaced the usual $=$ with $=_0$ to present it as a quantitative equational theory. (SC) stands for *skew commutativity* and (SA) for *skew associativity.* We call the models satisfying these 4 equations *barycentric algebras.*

It is well-known and easy to see that the finitely-supported probability distributions on a set with the usual notion of convex sums satisfy these axioms. The free algebras of this signature are, in fact, isomorphic to the space of finitely-supported probability distributions on a set. We will extend these ideas to quantitative algebras built on top of this signature.

6.1 Left-invariant barycentric algebras.

Note that from Stone's axioms above we can only prove relations between terms that are at distance 0 of each other, *i.e.*, equal in traditional sense. We now consider adding a new equation which is explicitly quantitative, in the sense that it states about $=_\varepsilon$ relations for arbitrary $\varepsilon \in [0,1]$.

Definition 6.1 (Left-invariant barycentric theory). *This is the quantitative theory $\mathcal{L}$ induced by Stone's equational theory presented above ((B1), (B2), (SC), (SA)), to which we add the left-invariance axiom (LI) stated below for arbitrary $x, y, z \in X$ and $\varepsilon \in (0,1)$:*

(LI) $\vdash y +_\varepsilon x =_\varepsilon z +_\varepsilon x$

The quantitative algebras satisfying $\mathcal{L}$ are called *left-invariant barycentric algebras.* Using the notation we defined in the previous sections, these algebras are the elements of the class $\mathcal{B}[\mathcal{L}]$. The crucial new ingredient introduced by (LI) is a notion of distance between convex sums of a special kind. As ε goes to 0 the terms $y +_\varepsilon x$ and $z +_\varepsilon x$ both tend to x and the distance goes to 0.

The key result regarding this algebra is that the free algebras generated by an discrete metric space $\mathcal{M}$ is isomorphic to the space of finitely supported Borel probability distributions on $\mathcal{M}$ with the total variation distance. We will not give the proofs in this paper as they are similar in spirit, though not in detail, to the proofs on interpolative barycentric algebras that appear later on in this paper. What is striking is that nothing is said in the axioms about probability distributions or about anything like the total variation distance; yet, they emerge naturally from these simple equations.

6.2 Interpolative barycentric algebras.

For the next example, we again start with the *barycentric signature* $\mathcal{B} = \{+_e : 2 \mid e \in [0,1]\}$ and from Stone's axioms (B1), (B2), (SC), (SA). To these we add a new quantitative equation called (I_p) where p is a real number, that characterizes a special class of barycentric algebras which we call *interpolative barycentric algebra.* This axiom expresses a kind of "interpolation" property and we get, as the free algebras, spaces of finitely supported probability distributions but with the p-Wasserstein metrics, and with the Kantorovich metric when $p = 1$. These metrics emerged from the work of Kantorovich and his students on optimization in the 1940's [84]. The results hold both for the finitary case, where the free algebras are isomorphic to the space of finitely-supported distributions, and for the case of complete metric spaces, where the free algebra is isomorphic to the space of all Borel probability measures. We give detailed proofs of these facts in a later section.

Definition 6.2 (p-interpolative barycentric theory). *Given a set X of variables and $p \geq 1$, the p-*interpolative barycentric theory *is the quantitative theory of type $\mathcal{B}$ induced by the axioms (B1), (B2), (SC), (SA) of Stone together with the axiom (I_p) stated below for arbitrary $x, x', y, y' \in X$ and $e, \varepsilon_1, \varepsilon_2 \in [0,1]$:*

(I_p) $x =_{\varepsilon_1} y,\ x' =_{\varepsilon_2} y' \vdash x +_e x' =_\delta y +_e y'$, *where* $\delta = (e\varepsilon_1^p + (1-e)\varepsilon_2^p)^{1/p}$.

We denote this theory by $\mathbb{I}_p$. We will devote significant space to studying the free algebras of this theory.

6.3 Quantitative semilattices with zero.

The last example that we consider does not come from probability theory, but from the familiar notion of Hausdorff metric: a metric defined on subsets of a metric space. This metric is used extensively in computer vision.

We start with the ordinary equational theory of semilattices and add a quantitative equation. In later sections we show how this axiomatization induces Hausdorff distances both in the finitary and in the continuous case. For details on Hausdorff distance and Hausdorff duality, see Appendix A.

Consider the algebraic similarity type of (bounded join-)semilattices

$$\mathbb{S} = \{+ : 2, 0 : 0\}$$

containing one binary operator + and one constant 0. We shall call it the *semilattice signature.*

Definition 6.3 (Quantitative Semilattice Theory). *Given a set* X *of variables, the* quantitative semilattice theory *over* $\mathbb{S}X$ *is the theory axiomatized by the following axioms, stated for arbitrary* $x, x', y, y', z \in X$ *and* $\varepsilon, \varepsilon' \in [0, 1]$*:*

(S0) $\vdash x + 0 =_0 x$
(S1) $\vdash x + x =_0 x$
(S2) $\vdash x + y =_0 y + x$
(S3) $\vdash (x + y) + z =_0 x + (y + z)$
(S4) $x =_\varepsilon y,\ x' =_{\varepsilon'} y' \vdash x + x' =_{\max\{\varepsilon,\varepsilon'\}} y + y'$.

We denote this quantitative theory by $\mathcal{S}$ and its models, which are the elements of $\mathbb{S}[\mathcal{S}]$, we call *quantitative semilattices with zero.* We will see later that these axioms, given a metric space, can freely generate the the space of finite sets with the Hausdorff metric, while for the case of complete metric spaces, the free algebra is isomorphic to the space of compact sets with Hausdorff metric.

7 Free algebras

In this section, given a signature Ω and a quantitative equational/Horn theory $\mathcal{U}$ over $\hat{\Omega}X$, we define a quantitative analogue of a *term algebra.* We endow $\hat{\Omega}X$, which is a term algebra in the usual sense, with a metric that turns it into a quantitative algebra. The intention is that the metric should be defined so that $\hat{\Omega}X$ satisfies $\mathcal{U}$. To do this, we first define a pseudometric[7] on $\hat{\Omega}X$ and afterwards we take the quotient w.r.t. $=_0$ to obtain an extended metric space.

We define $|-,-|_{\mathcal{U}} \colon \hat{\Omega}X \times \hat{\Omega}X \longrightarrow \overline{\mathbf{R}}_+$ as

$$|s,t|_{\mathcal{U}} = \inf\{\varepsilon \mid\ \vdash s =_\varepsilon t \in \mathcal{U}\}\,.$$

Observe that the infimum above is always well defined, and $|t,s|_{\mathcal{U}} = \infty$ if there is no ε such that $t =_\varepsilon s \in \mathcal{U}$. By (Refl), (Symm), and (Triang), it follows immediately that $|-,-|_{\mathcal{U}}$ is a well-defined extended pseudometric on $\hat{\Omega}X$.

This pseudometric induces on $\hat{\Omega}X$ the following equivalence relation

$$\sim_{\mathcal{U}} = \{(s,t) \in \hat{\Omega}X^2 \mid |s,t|_{\mathcal{U}} = 0\}\,.$$

As it follows that

$$|s,t|_{\mathcal{U}} = 0 \ \text{ iff } \ \vdash s =_0 t \in \mathcal{U}\,,$$

it is not difficult to observe that $\sim_{\mathcal{U}}$ is a congruence relation w.r.t. Ω. Indeed, let $f : n \in \Omega$ and let $s_1, \dots, s_n, t_1, \dots, t_n \in \hat{\Omega}X$, be such that $s_i \sim_{\mathcal{U}} t_i$ for all $i \leq n$. Since $s_i \sim_{\mathcal{U}} t_i$, we have that $\vdash s_i =_0 t_i \in \mathcal{U}$. By using the rule (NExp) we deduce $\vdash f(s_1, \dots, s_n) =_0 f(t_1, \dots, t_n) \in \mathcal{U}$. Hence, $f(s_1, \dots, s_n) \sim_{\mathcal{U}} f(t_1, \dots, t_n)$.

[7]Recall that pseudometrics may assign a distance of zero to distinct elements, whereas in a metric space, only identical points can have zero distance.

Let $[s]^{\mathcal{U}}$ denote the $\sim_{\mathcal{U}}$-equivalence class of $s \in \hat{\Omega}X$ and $[\hat{\Omega}X]^{\mathcal{U}}$ be the quotient of $\hat{\Omega}X$ w.r.t. $\sim_{\mathcal{U}}$. Then, $[\hat{\Omega}X]^{\mathcal{U}}$ can be organized as an Ω-algebra by interpreting an arbitrary $f : n \in \Omega$ on the quotient as the operator $f^{\sim}$:

$$f^{\sim}([s_1]^{\mathcal{U}}, \dots, [s_n]^{\mathcal{U}}) = [f(s_1, \dots, s_n)]^{\mathcal{U}} .$$

The set of equivalence classes $[\hat{\Omega}X]^{\mathcal{U}}$ becomes an extended metric space with the metric $|-,-|_{\mathcal{U}} \colon [\hat{\Omega}X]^{\mathcal{U}} \times [\hat{\Omega}X]^{\mathcal{U}} \longrightarrow \overline{\mathbf{R}}_+$ defined as $|[s]^{\mathcal{U}}, [t]^{\mathcal{U}}|_{\mathcal{U}} = |s,t|_{\mathcal{U}}$. Moreover, under the interpretations $\Omega^{\mathcal{U}} = \{f^{\sim} \mid f \in \Omega\}$ given above, $[\hat{\Omega}X]^{\mathcal{U}}$ is a quantitative algebra.

To see this, we need to prove that $f^{\sim}$ is non-expansive with respect to $|-,-|_{\mathcal{U}}$, for all the operators $f : n \in \Omega$. Suppose that for each $i \leq n$, $|[s_i]^{\mathcal{U}}, [t_i]^{\mathcal{U}}|_{\mathcal{U}} \leq \varepsilon$, where $s_i, t_i \in \hat{\Omega}X$. Hence, for any $i \leq n$, $\vdash s_i =_{\varepsilon} t_i \in \mathcal{U}$, and by (NExp) we deduce

$$\vdash f(s_1, \dots, x_n) =_{\varepsilon} f(t_1, \dots, t_n) \in \mathcal{U} .$$

This implies $|f^{\sim}([s_1]^{\mathcal{U}}, \dots, [s_n]^{\mathcal{U}}), f^{\sim}([t_1]^{\mathcal{U}}, \dots, [t_n]^{\mathcal{U}})|_{\mathcal{U}} \leq \varepsilon$. Thus, $f^{\sim}$ is indeed non-expansive with respect to $|-,-|_{\mathcal{U}}$.

Definition 7.1. *The quantitative term algebra induced by the theory $\mathcal{U}$ on $\hat{\Omega}X$ is the algebra*

$$\Omega^{\mathcal{U}}X = ([\hat{\Omega}X]^{\mathcal{U}}, \Omega^{\mathcal{U}}, |-,-|_{\mathcal{U}})$$

defined as above.

Although the definition of the quantitative term algebra is given over theories, in what follows we will overload the notation and denote by $\Omega^{U}X$ the quantitative term algebra obtained from the theory axiomatised by U, also when U is a set of equational/Horn judgements that is not deductively closed.

Note that, the same construction can also be done on a subset $Y \subseteq X$ of variables. The restriction of $\sim_{\mathcal{U}}$ to $\hat{\Omega}Y$ remains a congruence, and it is the one induced by the restriction of $|-,-|_{\mathcal{U}}$ on $\hat{\Omega}Y$. In fact, this would coincide with the quantitative term algebra $\Omega^{\mathcal{V}}Y$, over the theory

$$\mathcal{V} = \left\{ \{s_i =_{\varepsilon_i} t_i \mid i \in I\} \vdash s =_{\varepsilon} t \in \mathcal{U} \mid s_i, t_i, s, t \in \hat{\Omega}Y \right\}$$

obtained as the restriction of $\mathcal{U}$ to $\hat{\Omega}Y$. Hereafter, we abuse the notation and simply denote $\Omega^{\mathcal{V}}Y$ by $\Omega^{\mathcal{U}}Y$.

The next is a trivial but interesting consequence of the previous construction.

Lemma 7.2. *Let Ω be an algebraic signature, X a set of variables, $\mathcal{U}$ a quantitative Horn/equational theory on $\hat{\Omega}X$, and $Y \subseteq X$. Then, $\Omega^{\mathcal{U}}Y \leq \Omega^{\mathcal{U}}X$, i.e., $\Omega^{\mathcal{U}}Y$ is a subalgebra of $\Omega^{\mathcal{U}}X$.*

Our interest in the definition of the quantitative term algebra is justified by the following result.

Theorem 7.3. *Let Ω be an algebraic signature, X a set of variables and $\mathcal{U}$ a quantitative Horn/equational theory over $\hat{\Omega}X$. Then, $\Omega^{\mathcal{U}}X$ is a model of $\mathcal{U}$.*

Proof. First, note that for any assignment $\iota \in \Omega[X|\Omega^{\mathcal{U}}X]$ there is a substitution $\sigma_\iota \in [X|\hat{\Omega}X]$ such that for any $x \in X$, $\sigma(x) \in \hat{\Omega}X$ is an element of the $\sim_{\mathcal{U}}$-equivalence class $\iota(x) \in [\hat{\Omega}X]^{\mathcal{U}}$ (e.g., the representative). For such a substitution, the following equality holds for all $u \in \hat{\Omega}X$.

$$\iota(u) = [\sigma_\iota(u)]^{\mathcal{U}} .$$

This equality is proven by induction on the structure of u. The base case, when $u = x \in X$, follows by definition of σ_ι —independently of the choice of the representative, and similarly for the constants in Ω. As for the inductive step, assume $u = f(u_1, \dots, u_n)$ for some $f : n \in \Omega$ and $u_i \in \hat{\Omega}X$. Then

$$
\begin{aligned}
\iota(f(u_1, \dots, u_n)) &= f^{\sim}(\iota(u_1), \dots, \iota(u_n)) \\
&= f^{\sim}([\sigma_\iota(u_1)]^{\mathcal{U}}, \dots, [\sigma_\iota(u_n)]^{\mathcal{U}}) && \text{(ind. hp)} \\
&= [f(\sigma_\iota(u_1), \dots, \sigma_\iota(u_n))]^{\mathcal{U}} && \text{(def. } f^{\sim}) \\
&= [\sigma_\iota(f(u_1, \dots, u_n))]^{\mathcal{U}} .
\end{aligned}
$$

Now we prove the actual statement of the theorem. Assume that $\{s_i =_{\varepsilon_i} t_i \mid i \in I\} \vdash s =_\varepsilon t \in \mathcal{U}$. Take an assignment $\iota \in \Omega[X|\Omega^{\mathcal{U}}X]$ such that $|\iota(s_i), \iota(t_i)|_{\mathcal{U}} \leq \varepsilon_i$, for all $i \in I$.

Hence, we get that, for all $i \in I$, $|[\sigma_\iota(s_i)]^{\mathcal{U}}, [\sigma_\iota(t_i)]^{\mathcal{U}}|_{\mathcal{U}} \leq \varepsilon_i$. By definition of $|-,-|_{\mathcal{U}}$, this is equivalent to the statement that

$$\text{for all } \; i \in I, \vdash \sigma_\iota(s_i) =_{\varepsilon_i} \sigma_\iota(t_i) \in \mathcal{U}.$$

Now, we apply (CS) and obtain $\vdash \sigma_\iota(s) =_\varepsilon \sigma_\iota(t) \in \mathcal{U}$. But from the way σ_ι was constructed, this last fact is equivalent to $|[\iota(s)]^{\mathcal{U}}, [\iota(t)]^{\mathcal{U}}|_{\mathcal{U}} \leq \varepsilon$. ∎

The previous theorem and Lemmas 7.2 and 5.6 give us the following corollary.

Corollary 7.4. *Let Ω be an algebraic signature, X a set of variables and $\mathcal{U}$ a quantitative Horn/equational theory over $\hat{\Omega}X$. For any subset $Y \subseteq X$ of variables, $\Omega^{\mathcal{U}}Y$ is a model of $\mathcal{U}$.*

This corollary provides us a lot of examples of elements of $\Omega[\mathcal{U}]$. But we can do more and describe the initial object in this category.

Theorem 7.5 (Initial object). *Let Ω be an algebraic signature, X a set of variables, and $\mathcal{U}$ a Horn/equational theory over $\hat{\Omega}X$. Then, $\Omega^{\mathcal{U}}\emptyset$ is the initial object in the category $\Omega[\mathcal{U}]$, i.e, the initial algebra over the models of $\mathcal{U}$.*

Proof. Note that, from Theorem 7.3, $\Omega^{\mathcal{U}}\emptyset$ is an object in $\Omega[\mathcal{U}]$. Consider another object $\mathcal{A} = (A, \Omega, d) \in \Omega[\mathcal{U}]$. From the theory of universal algebras, there exists a unique homomorphism of Ω-algebras

$$m \colon \hat{\Omega}\emptyset \longrightarrow A$$

from the algebra of closed terms to the Ω-algebra over A. Note that, as $\mathcal{A} \in \Omega[\mathcal{U}]$, whenever $\vdash s =_\varepsilon t \in \mathcal{U}$, we also have $d(s^{\mathcal{A}}, t^{\mathcal{A}}) \leq \varepsilon$, for all closed terms $s, t \in \hat{\Omega}\emptyset$. In particular, this means that $|s, t|_{\mathcal{U}} \geq d(s^{\mathcal{A}}, t^{\mathcal{A}})$, for all pairs of closed terms $s, t \in \hat{\Omega}\emptyset$ (the distance function $|-,-|_{\mathcal{U}}$ used here is the pseudometric defined at the beginning of the section).

From the map m above, we define

$$m^{\mathcal{U}} \colon [\hat{\Omega}\emptyset]^{\mathcal{U}} \longrightarrow A$$

on the set of $\sim_{\mathcal{U}}$-classes of $\hat{\Omega}\emptyset$, as $m^{\mathcal{U}}([s]^{\mathcal{U}}) = m(s)$. This map is well-defined, as whenever $s \sim_{\mathcal{U}} t$, we have $d(s^{\mathcal{A}}, t^{\mathcal{A}}) = |s, t|_{\mathcal{U}} = 0$, thus $s^{\mathcal{A}} = t^{\mathcal{A}}$. Moreover, it is non-expansive: for all $s, t \in \hat{\Omega}\emptyset$

$$|[s]^{\mathcal{U}}, [t]^{\mathcal{U}}|_{\mathcal{U}} = |s, t|_{\mathcal{U}} \geq d(s^{\mathcal{A}}, t^{\mathcal{A}}) .$$

We show that $m^{\mathcal{U}}$ is a homomorphism. Take $f : n \in \Omega$ and $s_1, \dots, s_n \in \hat{\Omega}\emptyset$. Then we have that

$$
\begin{aligned}
f^{\mathcal{A}}(m^{\mathcal{U}}([s_1]^{\mathcal{U}}), \dots, m^{\mathcal{U}}([s_n]^{\mathcal{U}})) &= f^{\mathcal{A}}(m(s_1), \dots, m(s_n)) && \text{(def. } m^{\mathcal{U}}) \\
&= m(f(s_1, \dots, s_n)) && (m \text{ homo.}) \\
&= m^{\mathcal{U}}([f(s_1, \dots, s_n)]^{\mathcal{U}}) && \text{(def. } m^{\mathcal{U}}) \\
&= m^{\mathcal{U}}(f^{\sim}([s_1]_{\mathcal{U}}, \dots, [s_n]^{\mathcal{U}})) . && \text{(def. } f^{\sim})
\end{aligned}
$$

It remains to show uniqueness. Let $h\colon [\hat{\Omega}\emptyset]^{\mathcal{U}} \longrightarrow A$ be a homomorphism, possibly different from $m^{\sim}$. We show by induction on $u \in \hat{\Omega}\emptyset$ that, $h([u]^{\mathcal{U}}) = m^{\mathcal{U}}([u]^{\mathcal{U}})$.

The base case is when $u = c$, for some constant $c \in \Omega$. By the property of homomorphism, $h([c]^{\mathcal{U}}) = c^{\mathcal{A}} = m^{\sim}([c]^{\mathcal{U}})$. As for the inductive step, assume $u = f(u_1, \dots, u_n)$ for some $f : n \in \Omega$ and $u_i \in \hat{\Omega}\emptyset$. Then,

$$\begin{aligned} h([f(u_1,\dots,u_n)]^{\mathcal{U}}) &= h(f^{\sim}([u_1]^{\mathcal{U}},\dots,[u_n]^{\mathcal{U}})) \\ &= f^{\mathcal{A}}(h([u_1]^{\mathcal{U}}), h([u_1]^{\mathcal{U}})) && (h \text{ homo}) \\ &= f^{\mathcal{A}}(m^{\mathcal{U}}([u_1]^{\mathcal{U}}), m^{\mathcal{U}}([u_1]^{\mathcal{U}})) \\ &= m^{\mathcal{U}}([f(u_1,\dots,u_n)]^{\mathcal{U}}) . \end{aligned}$$

This concludes the proof. ∎

The next result emphasizes the role of unconditional quantitative equations in relations to the models of a theory. For this, we identify the part of a Horn/equational theory $\mathcal{U}$ that contains all the unconditional equations of $\mathcal{U}$ and only them; we denote it by

$$\sqrt{\mathcal{U}} = \{\vdash s =_{\varepsilon} t \mid \ \vdash s =_{\varepsilon} t \in \mathcal{U}\} .$$

The following lemma characterizes the models of $\sqrt{\mathcal{U}}$.

Lemma 7.6. *Let $\mathcal{U}$ be a Horn/equational theory over $\hat{\Omega}X$ and $\mathcal{A} = (A, \Omega, d)$ an Ω-quantitative algebra. Then the following are equivalent:*

- $\mathcal{A} \in \Omega[\sqrt{\mathcal{U}}]$
- *all assignments $\alpha \in \Omega[X|\mathcal{A}]$ are non-expansive, i.e., for all $s, t \in \hat{\Omega}X$, $|s,t|_{\mathcal{U}} \geq d(\alpha(s), \alpha(t))$.*

Proof. In one direction, assume that $\mathcal{A} \in \Omega[\sqrt{\mathcal{U}}]$. Let $\alpha \in \Omega[X|\mathcal{A}]$ be an assignment Then the following implications hold, for arbitrary $s, t \in \hat{\Omega}X$:

$$|s,t|_{\mathcal{U}} \leq \varepsilon \implies \vdash s =_{\varepsilon} t \in \mathcal{U} \implies \vdash s =_{\varepsilon} t \in \sqrt{\mathcal{U}} \implies d(\alpha(s), \alpha(t)) \leq \varepsilon .$$

This proves that α is non-expansive.

For the converse direction, assume that all assignments in $\Omega[X|\mathcal{A}]$ are non-expansive. Let $\vdash s =_{\varepsilon} t \in \sqrt{\mathcal{U}}$ and $\alpha \in \Omega[X|\mathcal{A}]$. Since α is non-expansive, $d(\alpha(s), \alpha(t)) \leq |s,t|_{\mathcal{U}}$. But $\vdash s =_{\varepsilon} t \in \sqrt{\mathcal{U}}$ guarantees that $|s,t|_{\mathcal{U}} \leq \varepsilon$, implying $d(\alpha(s), \alpha(t)) \leq \varepsilon$. Hence, $\mathcal{A}, \alpha \models s =_{\varepsilon} t$. ∎

As shown in Lemma 7.5, the quantitative generalization of the term algebra introduced in this section provides an explicit characterization of the initial quantitative algebra in $\Omega[\mathcal{U}]$. We now demonstrate that a similar construction yields a quantitative generalization of the free algebras.

Let X be a set of variables and $\mathcal{U}$ a quantitative Horn/equational theory over $\hat{\Omega}X$. Given a metric space $\mathcal{M} = (M, d_M)$ (often referred to as *generator*), our goal is to construct a quantitative algebra $\Omega^{\mathcal{U}}\mathcal{M}$ in $\Omega[\mathcal{U}]$ satisfying the following freeness property:

> Any non-expansive map $\alpha\colon M \longrightarrow A$ into a Ω-quantitative algebra $\mathcal{A} = (A, \Omega^{\mathcal{A}}, d_A)$ that satisfies $\mathcal{U}$ uniquely extends to a morphism of quantitative algebras from $\Omega^{\mathcal{U}}\mathcal{M}$ to $\mathcal{A}$.

To this end, define the signature $\Sigma = \Omega \cup \{m : 0 \mid m \in M\}$ as the extension of Ω including the elements of M as constants. Next, we extend the theory $\mathcal{U}$ on $\hat{\Sigma}X$ by taking[8]:

$$\mathcal{U}^{\mathcal{M}} = \overline{\mathcal{U} \cup \{\, \vdash m =_{d(m,n)} n \mid m, n \in M\}} .$$

[8]Recall that for a set U of Horn judgements, $\overline{U}$ denotes the theory axiomatized by U.

The addition of the axioms $\vdash m =_{d(m,n)} n$ encodes the metric of $\mathcal{M}$ as quantitative equations.

Note that the definition of $\mathcal{U}^{\mathcal{M}}$ guarantees that any quantitative algebra in $\Sigma[\mathcal{U}^{\mathcal{M}}]$ can be seen as a quantitative algebra in $\Omega[\mathcal{U}]$ simply by forgetting the interpretations of the constants in M. This operation is functorial and defines a (forgetful) functor

$$\mathbf{U}^{\mathcal{M}}_{\mathcal{U}} \colon \Sigma[\mathcal{U}^{\mathcal{M}}] \longrightarrow \Omega[\mathcal{U}]\,.$$

Conversely, given an algebra $\mathcal{A} = (A, \Omega^{\mathcal{A}}, d_A) \in \Omega[\mathcal{U}]$, a metric space $\mathcal{M} = (M, d_M)$ and a non-expansive map $\alpha \colon M \longrightarrow A$, one can always extend $\mathcal{A}$ to an algebra in $\Sigma[\mathcal{U}^{\mathcal{M}}]$ by interpreting the constants $m \in M$ as $\alpha(m) \in A$.

Consider now the quantitative term algebra $\Sigma^{\mathcal{U}^{\mathcal{M}}}\emptyset$. By Theorem 7.3, this is a model of $\mathcal{U}^{\mathcal{M}}$ and it is, in fact, the initial object in the category $\Sigma[\mathcal{U}^{\mathcal{M}}]$ (Lemma 7.5). Recall that its construction was based on the pseudometric $|-,-|_{\mathcal{U}^{\mathcal{M}}} : \hat{\Sigma}\emptyset \times \hat{\Sigma}\emptyset \longrightarrow \overline{\mathbf{R}}_+$ defined by

$$|s,t|_{\mathcal{U}^{\mathcal{M}}} = \inf\{\varepsilon \mid \ \vdash s =_\varepsilon t \in \mathcal{U}^{\mathcal{M}}\}.$$

Since, by construction, $\hat{\Sigma}\emptyset = \hat{\Omega}M$, the above is a pseudometric on $\hat{\Omega}M$.

By the considerations above, a natural candidate for the free quantitative algebra over $\mathcal{M}$ is

$$\Omega^{\mathcal{U}}\mathcal{M} = ([\hat{\Omega}M]^{\mathcal{U}^{\mathcal{M}}}, \Omega^{\mathcal{U}^{\mathcal{M}}}, |-,-|_{\mathcal{U}^{\mathcal{M}}})\,,$$

where $[\hat{\Omega}M]^{\mathcal{U}^{\mathcal{M}}}$ denotes the quotient of $\hat{\Omega}M$ with respect to the equivalence relation $\sim_{\mathcal{U}^{\mathcal{M}}}$, as defined at the beginning of this section.

Note that, except for the interpretations of the constant symbols from $\mathcal{M}$, $\Omega^{\mathcal{U}}\mathcal{M}$ is (essentially) the quantitative term algebra $\Sigma^{\mathcal{U}^{\mathcal{M}}}\emptyset$. Thus, as $\mathcal{U} \subseteq \mathcal{U}^{\mathcal{M}}$, by Theorem 7.3 we immediately have $\Omega^{\mathcal{U}}\mathcal{M} \models \mathcal{U}$.

Next, we establish the freeness of $\Omega^{\mathcal{U}}\mathcal{M}$ and relate our constructions to the categorical theory of universal algebras.

First, note that the construction above is functorial on the metric spaces used as generators, resulting in the functor

$$\Omega^{\mathcal{U}} \colon \mathbf{EMet} \longrightarrow \Omega[\mathcal{U}]$$

mapping an arbitrary metric space $\mathcal{M}$ to the quantitative algebra $\Omega^{\mathcal{U}}\mathcal{M}$. We will see that this functor is the left adjoint to the obvious forgetful functor

$$\mathbf{U} \colon \Omega[\mathcal{U}] \longrightarrow \mathbf{EMet}$$

mapping quantitative algebras to their carrier.

We establish this adjunction, by showing the existence of an universal morphism. To this end, define for arbitrary metric space $\mathcal{M} = (M, d_M)$, the function

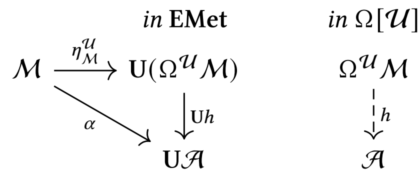
$$\eta^{\mathcal{U}}_{\mathcal{M}} \colon M \longrightarrow [\hat{\Omega}M]^{\mathcal{U}^{\mathcal{M}}}$$

mapping an arbitrary $m \in M$ to its equivalence class $[m]^{\mathcal{U}^{\mathcal{M}}}$ with respect the equivalence relation $\sim_{\mathcal{U}^{\mathcal{M}}}$. Note that for all $\varepsilon \in \mathbf{R}_+$, and $m, n \in M$, the inequality $d_M(m,n) \leq \varepsilon$ implies $\vdash m =_\varepsilon n \in \mathcal{U}^{\mathcal{M}}$, which, in turn gives $|m,n|_{\mathcal{U}^{\mathcal{M}}} \leq \varepsilon$. Therefore, $\eta^{\mathcal{U}}_{\mathcal{M}}$ is a non-expansive map from $\mathcal{M}$ to $\mathbf{U}(\Omega^{\mathcal{U}}\mathcal{M})$, thus a morphism in $\mathbf{EMet}$.

Theorem 7.7. ***(Free quantitative algebra).*** *Given an extended metric space $\mathcal{M} = (M, d_M)$, the pair $(\Omega^{\mathcal{U}}\mathcal{M}, \eta^{\mathcal{U}}_{\mathcal{M}})$ is a universal morphism from $\mathcal{M}$ to the forgetful functor*

$$\mathbf{U} \colon \Omega[\mathcal{U}] \longrightarrow \mathbf{EMet}.$$

Explicitly, this means that, for each $\mathcal{A} \in \Omega[\mathcal{U}]$ *and every non-expansive map* $\alpha\colon \mathcal{M} \longrightarrow \mathbf{U}\mathcal{A}$*, there exists a unique morphism* $h\colon \Omega^{\mathcal{U}}\mathcal{M} \longrightarrow \mathcal{A}$ *of quantitative algebras such that* $\mathbf{U}h \circ \eta^{\mathcal{U}}_{\mathcal{M}} = \alpha$*, i.e., the following diagram commutes*



Proof. Consider an arbitrary quantitative algebra $\mathcal{A} = (A, \Omega^{\mathcal{A}}, d_A) \in \Omega[\mathcal{U}]$ and a non-expansive map $\alpha\colon M \longrightarrow A$. Recall that, by construction of $\Omega^{\mathcal{U}}\mathcal{M}$, to define a Ω-homomorphism from $\Omega^{\mathcal{U}}\mathcal{M}$ is the same as defining a Σ-homomorphism from $\Sigma^{\mathcal{U}^{\mathcal{M}}}\emptyset$, where $\Sigma = \Omega \cup \{m : 0 \mid m \in M\}$ is the extension of the signature Ω with the elements in M acting as constants.

By Lemma 7.5, $\Sigma^{\mathcal{U}^{\mathcal{M}}}\emptyset$ is the initial object in $\Sigma[\mathcal{U}^{\mathcal{M}}]$. Thus, it is natural to use as a candidate map the unique Σ-homomorphism out of the initial quantitative Σ-algebra. We need to see $\mathcal{A}$ as a quantitative Σ-algebra in $\Sigma[\mathcal{U}^{\mathcal{M}}]$.

Define $\mathcal{A}' = (A, \Sigma^{\mathcal{A}}, d_A)$, where $\Sigma^{\mathcal{A}} = \Omega^A \cup \{\alpha(m) \mid m \in M\}$; that is, we interpret the constant symbols from M as $\alpha(m)$ in A. This is clearly a quantitative algebra as all the interpretations of the operators in Σ are non-expansive. It remains to verify that $\mathcal{A}'$ is a model of $\mathcal{U}^{\mathcal{M}}$. This is equivalent to showing that $\mathcal{A}' \models \mathcal{U}$ and $\mathcal{A}'$ satisfies all the equations of the form $\vdash m =_\varepsilon n$, whenever $d_M(m, n) \leq \varepsilon$. The first part, $\mathcal{A}' \models \mathcal{U}$, follows directly from the assumption that $\mathcal{A} \models \mathcal{U}$. For the second, note that $\models_{\mathcal{A}'} m =_\varepsilon n$ is equivalent to $d_A(\alpha(m), \alpha(n)) \leq \varepsilon$. This holds because α is non-expansive, so $d_A(\alpha(m), \alpha(n)) \leq d_M(m, n) \leq \varepsilon$.

Consequently, there exists a unique Σ-homomorphism h of quantitative algebras from $\Sigma^{\mathcal{U}^{\mathcal{M}}}\emptyset$ to $\mathcal{A}'$. As discussed before, this corresponds to an Ω-homomorphism

$$h\colon [\hat{\Omega}M]^{\mathcal{U}^{\mathcal{M}}} \longrightarrow A\,,$$

(recall that $\Sigma^{\mathcal{U}^{\mathcal{M}}}\emptyset$ and $\hat{\Omega}\mathcal{M}$ have same the carrier), that, by construction, is a morphism from $\hat{\Omega}^{\mathcal{U}}\mathcal{M}$ to $\mathcal{A}$ in $\Omega[\mathcal{U}]$. Moreover, by construction, h has indeed the property that $\mathbf{U}h \circ \eta^{\mathcal{U}}_{\mathcal{M}} = \alpha$. The uniqueness of h in $\Omega[\mathcal{U}]$ follows directly from the uniqueness in $\Sigma[\mathcal{U}^{\mathcal{M}}]$, because any $h' \neq h$ in $\Omega[\mathcal{U}]$ satisfying our working hypothesis could be similarly extended to distinct morphisms in $\Sigma[\mathcal{U}^{\mathcal{M}}]$, contradicting the initiality of $\Sigma^{\mathcal{U}^{\mathcal{M}}}\emptyset$ in $\Sigma[\mathcal{U}^{\mathcal{M}}]$. ∎

As discussed earlier, the statement of Theorem 7.7 is equivalent to the existence of the adjunction $\hat{\Omega}^{\mathcal{U}} \dashv \mathbf{U}$, with $\eta^{\mathcal{U}}\colon Id \Rightarrow \mathbf{U}\Omega^{\mathcal{U}}$ as unit.

Adjunctions give rise to monads in a canonical way, and in line with the standard results from the categorical development of universal algebras, we call the monad obtained from the adjunction we just described, *the (metric) term monad.* Next, we give a concrete description of it.

Define the functor

$$\hat{\Omega}^{\mathcal{U}}\colon \mathbf{EMet} \longrightarrow \mathbf{EMet}$$

as the composition $\mathbf{U}\Omega^{\mathcal{U}}$. The overload in notation is intentional, as $\hat{\Omega}^{\mathcal{U}}$ maps an extended metric space $\mathcal{M} \in \mathbf{EMet}$ to the extended metric space of (equivalence classes of) Ω-terms over over $\mathcal{M}$ with metric $|-,-|_{\mathcal{U}^{\mathcal{M}}}$ as defined above.

Maps between extended metric spaces induce maps between the metric term algebras in the evident way.

The functor $\hat{\Omega}^{\mathcal{U}}$ is monadic, with unit the natural transformation $\eta^{\mathcal{U}}\colon Id \Rightarrow \hat{\Omega}^{\mathcal{U}}$ defined above, and multiplication

$$\mu^{\mathcal{U}}\colon \hat{\Omega}^{\mathcal{U}}\hat{\Omega}^{\mathcal{U}} \Rightarrow \hat{\Omega}^{\mathcal{U}},$$

given, for arbitrary extended metric space $\mathcal{M} = (M, d)$, $m \in M$, $t \in \hat{\Omega}M$, $f\colon n \in \Omega$, and $C_1, \dots C_n \in \hat{\Omega}^{\mathcal{U}}(\hat{\Omega}^{\mathcal{U}}\mathcal{M})$, by induction as follows

$$\mu^{\mathcal{U}}_{\mathcal{M}}([t]^{\mathcal{U}^{\mathcal{M}}}) = [t]^{\mathcal{U}^{\mathcal{M}}},$$
$$\mu^{\mathcal{U}}_{\mathcal{M}}(f(C_1, \dots, C_n)) = f\big(\mu_{\mathcal{M}}(C_1), \dots, \mu_{\mathcal{M}}(C_n)\big)\,.$$

Notice that this monad is on **EMet** and the quantitative equational theory $\mathcal{U}$ induces a unique metric on the set of terms.

We conclude this section by stating an important result concerning the term monads just introduced. This result mirrors a well-known theorem from categorical universal algebra. Since the topic is only tangential to the main development of this paper, we omit formal definitions and technical details. Our goal here is simply to complete the broader picture regarding the metric monads defined above; we refer the interested reader to [8] for full details.

A classical result in categorical universal algebra states that, if a functor F admits a free monad F^*, then the Eilenberg–Moore algebras for F^* correspond bijectively to the algebras for F.

A similar correspondence holds in our setting between the Eilenberg–Moore algebras of the term monad $\hat{\Omega}^{\mathcal{U}}$ induced by an equational theory $\mathcal{U}$, and the models of $\mathcal{U}$. Note that this result applies specifically to quantitative equational theories and not to the more general Horn theories.

Theorem 7.8 ([8]). *For any equational theory $\mathcal{U}$ over $\hat{\Omega}X$,*

$$\mathbf{EM}(\hat{\Omega}^{\mathcal{U}}) \cong \Omega[\mathcal{U}]\,,$$

where $\mathbf{EM}(\hat{\Omega}^{\mathcal{U}})$ denotes the category of Eilenberg-Moore algebras for the metric term monad induced by $\mathcal{U}$, and $\cong$ denotes isomorphism of categories.

8 Examples of free algebras

Having developed the general theory of free algebras in Section 7, we can now describe the free algebras for our three main examples in Section 6: left-invariant barycentric algebra, interpolative barycentric algebras and quantitative semilattices.

8.1 Free left-invariant barycentric algebras

Recall the equational axiomatization for the quantitative theory $\mathcal{L}$ of left-invariant barycentric algebras, of signature $\mathcal{B} = \{+_e : 2 \mid e \in [0, 1]\}$, introduced in Section 6:

For $x, y, z \in X$, $\varepsilon \in [0, 1]$ and $e, e' \in [0, 1)$:

(B1) $\vdash x +_1 y =_0 x$
(B2) $\vdash x +_e x =_0 x$
(SC) $\vdash x +_e y =_0 y +_{1-e} x$
(SA) $\vdash (x +_e y) +_{e'} z =_0 x +_{ee'} (y +_{\frac{e'-ee'}{1-ee'}} z)$
(LI) $\vdash x' +_\varepsilon x =_\varepsilon x'' +_\varepsilon x$

Let $\mathcal{M} = (M, d_M)$ be a discrete metric space, by which we mean that when two points are distinct the distance is 1 and when they are the same the distance is zero. Let $\mathcal{D}[\mathcal{M}]$ be the set of Borel probability measures over M. The *total variation distance* between probability measures is defined, for arbitrary $\mu, \nu \in \mathcal{D}[\mathcal{M}]$ by

$$T(\mu, \nu) = \sup_E |\mu(E) - \nu(E)|\,.$$

where E ranges over the Borel sets of $\mathcal{M}$.

The key result regarding left-invariant barycentric algebras is that the free algebras generated by $\mathcal{M}$ is isomorphic to the space of finitely supported Borel probability distributions on $\mathcal{M}$ with the total variation distance. We will not give the proofs here as they are similar in spirit, though not in detail, to the proofs in the next subsection on interpolative barycentric algebras.

Fix a discrete metric space $\mathcal{M} = (M, d_M)$. Let $\mathcal{B}^{\mathcal{L}}\mathcal{M}$ be the quantitative algebra freely generated by $\mathcal{M}$, which is an element in $\mathcal{B}[\mathcal{L}]$.

Denote by $\Pi[\mathcal{M}]$ the set of *finitely-supported* Borel probability measures on $\mathcal{M}$, i.e., those that can be represented as finite convex combinations of Dirac distributions δ_m, for $m \in M$. It can be shown that $\Pi[\mathcal{M}]$ endowed with the total-variation distance can be organized as a quantitative algebra in $\mathcal{B}[\mathcal{L}]$. Moreover, $(\Pi[\mathcal{M}], \delta_{\mathcal{M}})$ is a universal arrow from $\mathcal{M}$ to $\mathbf{U}\colon \mathcal{B}[\mathcal{L}] \longrightarrow \mathbf{EMet}$ for $\delta_{\mathcal{M}}\colon \mathcal{M} \longrightarrow \Pi[\mathcal{M}]$ mapping $m \in M$ to the Dirac distribution δ_m. By the uniqueness (up to isomorphism) of universal arrows of the same type, this proves that the algebras $\Pi[\mathcal{M}]$ and $\mathcal{B}^{\mathcal{L}}\mathcal{M}$ are isomorphic.

Theorem 8.1. *Let $\mathcal{M}$ be a discrete metric space. The algebras $\Pi[\mathcal{M}]$ and $\mathcal{B}^{\mathcal{L}}\mathcal{M}$ are isomorphic via the bijective isometry $h\colon \mathcal{B}^{\mathcal{L}}\mathcal{M} \longrightarrow \Pi[\mathcal{M}]$*

$$h(\eta^{\mathcal{L}}_{\mathcal{M}}(m)) = \delta_m\,, \qquad\qquad h(t +_e s) = eh(t) + (1-e)h(s)\,,$$

where $m \in M$, $t, s \in \mathcal{B}^{\mathcal{L}}\mathcal{M}$ and δ_m is the Dirac measure on m.

Consequently, the metric induced by the quantitative theory $\mathcal{L}$ coincides with the total variation distance on $\Pi[\mathcal{M}]$. The proof is omitted to save space; with the discrete metric, the Kantorovich distance, defined below, is the same as the total variation metric, so this result follows from the results of the next section as a special case. Of course, it is possible to prove it directly.

In this example, the free algebras are actually over a set rather than over metric spaces. This is a special situation that arises because the quantitative equations for this theory are unconditional, and hence the metric information from the generating metric space is not used. In the examples below, the free algebras are indeed generated by metric spaces.

8.2 Interpolative barycentric algebras

Once again we start with the *barycentric signature* $\mathcal{B} = \{+_e : 2 \mid e \in [0,1]\}$ that we considered in the last subsection. Now we consider the extension that we called interpolative barycentric algebra and construct the free algebras over metric spaces. We get, as the free algebras, spaces of finitely-supported probability distributions but with the Kantorovich or p-Wasserstein metrics. These metrics emerged from the work of Kantorovich and his students on optimization in the 1940's [84]. To make the presentation somewhat self-contained, we have included background on the Kantorovich-Rubinstein duality theorem and basic facts from probability measures on metric spaces in Appendix B.

As mentioned before, we introduce a numerical parameter p in the axiom and obtain an axiomatization of the p-Wasserstein metric for $p \geq 1$. For $p = 1$ this reduces to the Kantorovich metric. We call these algebras p-IB algebras for short.

Definition 8.2 (p-interpolative barycentric theory). *Given a set X of variables and $p \geq 1$, the p-*interpolative barycentric theory *is the quantitative theory of type $\mathcal{B}$ induced by the axioms (B1), (B2), (SC), (SA) of Stone together with the axiom (I_p) stated below for arbitrary $x, x', y, y' \in X$ and $e, \varepsilon_1, \varepsilon_2 \in [0,1]$:*

(I_p) $\{x =_{\varepsilon_1} y, x' =_{\varepsilon_2} y'\} \vdash x +_e x' =_\delta y +_e y'$, *where* $\delta = (e\varepsilon_1^p + (1-e)\varepsilon_2^p)^{1/p}$.

We denote this theory by $\mathbb{I}_p$. Note that if we set $\varepsilon_1 = \varepsilon_2$, the axiom (I_p) expresses the non-expansiveness of $+_e$.

If we write (I_1) explicitly, we get the axiom below.

(I_1) $\{x =_{\varepsilon_1} y, x' =_{\varepsilon_2} y'\} \vdash x +_e x' =_\delta y +_e y'$, where $\delta = e\varepsilon_1 + (1-e)\varepsilon_2$.

8.3 Metrics on spaces of probability distributions

In this subsection, we recall the definitions of p-Wasserstein and Kantorovich metrics and state some basic results that will be used in the proofs below.

Definition 8.3. *Let μ and ν be two probability measures defined on the Borel sets of a metric space (M, d). A* coupling *of μ and ν is a joint measure π defined on the product space $M \times M$ such that its marginals are μ and ν respectively.*

We write $C(\mu, \nu)$ for the space of couplings with marginals μ and ν. It is easy to see that the set of couplings is closed under convex sums.

We generally work with Polish spaces in this section. A *Polish space* is a separable topological space, which can be metrized so that it is complete. Note that a space like $(0, 1)$ is Polish even though it is not complete with the *usual* metric. However, it is homeomorphic to $(0, \infty)$, hence can be given a complete metric that gives the same topology.

An important concept is *tightness of a measure* on a space M: this means that for every $\epsilon > 0$ there is a compact set K_ϵ such that the measure of $M \setminus K_\epsilon$ is less than ϵ. If a Borel measure is tight, we call it a *Radon measure.* On complete separable metric spaces, all Borel measures are Radon, so also on a Polish space, all Borel measures are Radon.

Let $\mathcal{M} = (M, d)$ be a one-bounded complete separable metric space and let $p \geq 1$. The *p-Wasserstein metric W_d^p* induced by d on the set $\Delta[\mathcal{M}]$ of Radon probability measures over $\mathcal{M}$, is defined, for arbitrary $\mu, \nu \in \Delta[\mathcal{M}]$ as

$$W_d^p(\mu, \nu) = \inf \left\{ \left(\int d^p \, \mathrm{d}\omega \right)^{\frac{1}{p}} \mid \omega \in C(\mu, \nu) \right\} . \tag{3}$$

The Kantorovich metric induced by d is usually defined by:

$$K_d(\mu, \nu) = \sup \left\{ \left| \int f \, \mathrm{d}\mu - \int f \, \mathrm{d}\nu \right| \mid f \colon M \longrightarrow [0, 1] \text{ non-expansive} \right\} .$$

The relation between the Wasserstein metric for $p = 1$ and the Kantorovich metric is given by the well-known Kantorovich-Rubinstein duality, see e.g., [85]. Kantorovich-Rubinstein duality says that $W_d^1 = K_d$ on $\Delta[\mathcal{M}]$.

Theorem 8.4 (Kantorovich-Rubinstein Duality Theorem). *Let $\mathcal{M}$ be a complete separable metric space with the metric taking real values*[9]*. Then, for arbitrary Radon probability measures $\mu, \nu \in \Delta[\mathcal{M}]$*

$$K_d(\mu, \nu) = \inf \left\{ \int d \, \mathrm{d}\omega \mid \omega \in C(\mu, \nu) \right\} .$$

In the case of $p = 1$, we also know that there exists an optimal coupling for W_d^1, *i.e.*, a coupling that attains the infimum (hence, it is a minimum) in equation (3). It is not known whether this holds for the W_d^p metrics in general, but fortunately, we do not need that in the development below.

[9] So not an extended metric.

8.4 Free algebras for interpolative barycentric algebras

In this subsection, we establish the main result for the finitary case: that the free algebras we get are exactly the spaces of probability distributions equipped with the Kantorovich metric.

Fix a one-bounded metric space $\mathcal{M} = (M, d)$. Let $\mathcal{B}^{\mathbb{I}_p}\mathcal{M}$ be the quantitative algebra in $\mathcal{B}[\mathbb{I}_p]$ freely generated from the metric space $\mathcal{M}$, as constructed in Section 7. By Theorem 7.7, $\mathcal{B}^{\mathbb{I}_p}\mathcal{M}$ has the universal mapping property from $\mathcal{M}$ to $\mathbf{U}\colon \mathcal{B}[\mathbb{I}_p] \longrightarrow \mathbf{EMet}$.

Theorem 8.5. *If $\mathcal{M}$ is a non-degenerate metric space, then $\mathcal{B}^{\mathbb{I}_p}\mathcal{M}$ is a non-degenerate quantitative algebra satisfying $\mathbb{I}_p$. In particular, $\mathbb{I}_p$ is a consistent quantitative theory.*

Recall that $\Pi[\mathcal{M}]$ denotes the set of finitely-supported Borel probability measures on $\mathcal{M}$. Next, we show that $\Pi[\mathcal{M}]$ can be organized as a quantitative algebra in $\mathcal{B}[\mathbb{I}_p]$, with W_d^p as metric. Moreover, we show that this algebra and the morphism $\delta\colon \mathcal{M} \longrightarrow \Pi[\mathcal{M}]$ define a universal arrow from $\mathcal{M}$ to $\mathbf{U}\colon \mathcal{B}[\mathbb{I}_p] \longrightarrow \mathbf{EMet}$; consequently $\Pi[\mathcal{M}]$ and $\mathcal{B}^{\mathbb{I}_p}M$ are isomorphic quantitative algebras.

We regard $\Pi[\mathcal{M}]$ as an algebra of type $\mathcal{B}$ by interpreting $+_e$, for arbitrary $\mu, \nu \in \Pi[\mathcal{M}]$, as

$$\mu +_e \nu = e\mu + (1-e)\nu\,.$$

$\Pi[\mathcal{M}]$ will be viewed as a quantitative algebra by taking the metric W_d^p induced by the metric d on M. Note that finitely supported Borel probability measures are Radon, so W_d^p is a well defined metric on $\Pi[\mathcal{M}]$.

Theorem 8.6. $\Pi[\mathcal{M}] = (\Pi[\mathcal{M}], \mathcal{B}, W_d^p) \in \mathcal{B}[\mathbb{I}_p]$.

Proof. Let $\mathcal{M} = (M, d)$. Note that each assignment $\iota\colon X \longrightarrow \Pi[\mathcal{M}]$ maps variables to Borel measures in $\Pi[\mathcal{M}]$, hence we need to check the axioms on $\mathcal{B}$-terms constructed over $\Pi[\mathcal{M}]$. (Refl), (Symm), (Triang), (Max), and (Cont) follow from the fact that W_d^p is a metric.

(B1), (B2), (SA), and (SC) follow by the definition of the interpretation for $+_e$ and the fact that W_d^p is a metric. As for (I_p), assume $W_d^p(\mu,\mu') \le \varepsilon_1$, $W_d^p(\nu,\nu') \le \varepsilon_2$. Consider an arbitrary $\delta > 0$; and let $\omega \in C(\mu,\mu')$ and $\omega' \in C(\nu,\nu')$ be two couplings such that

$$\int d^p \,\mathrm{d}\omega - \delta \le W_d^p(\mu,\mu') \;\text{ and }\; \int d^p \,\mathrm{d}\omega' - \delta \le W_d^p(\nu,\nu').$$

Then,

$$\begin{aligned}
e\varepsilon_1^p + (1-e)\varepsilon_2^p &\ge eW_d^p(\mu,\mu')^p + (1-e)W_d^p(\nu,\nu')^p && \text{(by hyp.)}\\
&\ge e\int d^p \,\mathrm{d}\omega + (1-e)\int d^p \,\mathrm{d}\omega' - \delta && \text{(hyp. on } \omega,\omega')\\
&= \int d^p \,\mathrm{d}(e\omega + (1-e)\omega') - \delta && \text{(linearity of } \textstyle\int)\\
&\ge W_d^p(e\mu + (1-e)\nu, e\mu' + (1-e)\nu')^p - \delta && \text{(Lemma B.2)}\\
&= W_d^p(\mu +_e \nu, \mu' +_e \nu')^p - \delta\,. && \text{(def. } +_e)
\end{aligned}$$

Since this inequality holds for any $\delta > 0$, we obtain that

$$W_d^p(\mu +_e \nu, \mu' +_e \nu') \le [e\varepsilon_1^p + (1-e)\varepsilon_2^p]^{1/p}\,.$$

Finally, soundness of (NExp) follows as an instance of (I_p), where $\varepsilon_1 = \varepsilon_2$. ∎

Note that the function $\delta_{\mathcal{M}} \colon \mathcal{M} \longrightarrow \Pi[\mathcal{M}]$, mapping $m \in M$ to Dirac measure δ_m is non-expansive, also when $\Pi[\mathcal{M}]$ is endowed with the p-Wasserstein metric —indeed, $W_d^p(\delta_m, \delta_n) = d(m, n)$.

The next theorem shows that the p-interpolative barycentric algebra $\Pi[\mathcal{M}]$ defined above, when paired with $\delta_{\mathcal{M}} \colon \mathcal{M} \longrightarrow \Pi[\mathcal{M}]$, is a universal arrow.

Theorem 8.7. *The pair $(\Pi[\mathcal{M}], \delta_{\mathcal{M}})$ is a universal arrow from $\mathcal{M} \in$ **EMet** to $\mathbf{U} \colon \mathcal{B}[\mathbb{I}_p] \longrightarrow$ **EMet**. Explicitly, for each $\mathcal{A} \in \mathcal{B}[\mathbb{I}_p]$ and every non-expansive map $\alpha \colon \mathcal{M} \longrightarrow \mathbf{U}\mathcal{A}$, there exists a unique morphism $h \colon \Pi[\mathcal{M}] \longrightarrow \mathcal{A}$ of quantitative algebras such the following diagram commutes*

$$
\begin{array}{ccc}
\textit{in } \mathbf{EMet} & & \textit{in } \mathcal{B}[\mathbb{I}_p] \\
\mathcal{M} \xrightarrow{\delta_{\mathcal{M}}} \mathbf{U}(\Pi[\mathcal{M}]) & & \Pi[\mathcal{M}] \\
\quad\searrow^{\alpha} \quad \downarrow \mathbf{U}(h) & & \downarrow h \\
\mathbf{U}(\mathcal{A}) & & \mathcal{A}
\end{array}
$$

Proof. Let $\mathcal{A} = (A, \mathcal{B}^{\mathcal{A}}, d^{\mathcal{A}}) \in \mathcal{B}[\mathbb{I}_p]$ and $\alpha \colon M \longrightarrow A$ be a non-expansive map.

By an obvious induction the convex sum of finitely many elements of A is well defined: given $\{c_i\}_{i=1}^k$ with $c_i \in (0,1]$ and $a_i \in A$ we can define $\sum c_i a_i$. For the set $\Pi[\mathcal{M}]$ of finitely supported Borel probability measures over M, we additionally have that any $\mu \in \Pi[\mathcal{M}]$, can be *uniquely* represented as a *finite* convex combination of the form $\mu = \sum_{i=1}^k c_i \delta_{m_i}$, where $supp(\mu) = \{m_1, \ldots, m_k\}$ and $c_i \in (0,1]$ are such that $\sum_{i=1}^k c_i = 1$.

Using these observations, we define the map $h \colon \Pi[\mathcal{M}] \longrightarrow \mathcal{A}$ as follows:

$$h(\sum_{i=1}^k c_i \delta_{m_i}) = \sum_{i=1}^k c_i \alpha(m_i)\,.$$

Clearly $h \circ \delta_{\mathcal{M}} = \alpha$: for any $m \in M$, $h(\delta_{\mathcal{M}}(m)) = h(\delta_m) = \alpha(m)$. Now we show that h is a homomorphism. Let $\mu = \sum_{i=1}^k c_i \delta_{m_i}$, $\nu = \sum_{j=1}^n d_j \delta_{n_j}$ and $e \in [0,1]$. Then the following holds:

$$
\begin{aligned}
h(\mu +_e \nu) &= h(e\mu + (1-e)\nu) && \text{(def. } +_e\text{)} \\
&= h(e \sum_{i=1}^k c_i \delta_{m_i} + (1-e) \sum_{j=1}^n d_j \delta_{n_j}) && \text{(canonical repr.)} \\
&\overset{(*)}{=} e \sum_{i=1}^k c_i \alpha(m_i) + (1-e) \sum_{j=1}^n d_j \alpha(n_j) && \text{(def. } h, \Pi[\mathcal{M}] \text{ and (B2))} \\
&= h(\mu) +_e^{\mathcal{A}} h(\nu)\,. && \text{(def. } h \text{ \& def. } +_e^{\mathcal{A}}\text{)}
\end{aligned}
$$

Note that in $(*)$, the formal definition of h requires that the measure is canonically represented without repetitions of Dirac measures δ_m. The repetitions can be removed by applying (B2) in $\Pi[\mathcal{M}]$ and, once h is applied, they can be recovered by applying $(B2)$ in $\mathcal{A}$ in the reverse direction.

Now, for the uniqueness, assume that there exists another homomorphism h' such $h' \circ \delta_{\mathcal{M}} = \alpha$. It is immediate that h and h' must agree on point measures, $h(\delta_m) = \alpha(m) = h'(\delta_m)$. Let us assume that for all measures with support of size $n-1$ we have equality of h and h'. Consider $\sum_{i=1}^n c_i \delta_{m_i}$.

Let $e = \sum_{i=1}^{n-1} c_i$, then $1 - e = c_n$. We can write this measure as $e \sum_{i=1}^{n-1}(c_i/e)\delta_{m_i} + (1-e)\delta_{m_n}$ so it is a convex sum of a measure of support size $n-1$ and a point measure. Now the induction argument that $h = h'$ is immediate from the fact that both h and h' are homomorphisms.

It remains to show that h is non-expansive, *i.e.*, that for arbitrary $\mu, \nu \in \Pi[\mathcal{M}]$, $W_d^p(\mu,\nu) \geq d^{\mathcal{A}}(h(\mu), h(\nu))$. We proceed by well-founded induction on pairs or measures (μ,ν) partially ordered by $(\mu,\nu) \sqsubseteq (\mu',\nu')$ iff $|\,supp(\mu)| \leq |\,supp(\mu')|$ and $|\,supp(\nu)| \leq |\,supp(\nu')|$; the $|\cdot|$ notation means cardinality of the set. (Note that its corresponding strict order $\sqsubset$ is indeed a well-founded relation.)

(Base case) Assume $\mu = \delta_m$ and $\nu = \delta_n$, for some $m, n \in M$.

$$\begin{aligned} W_d^p(\delta_m, \delta_n) &= d(m,n) && (\text{def. } W_d)\\ &\geq d^{\mathcal{A}}(\alpha(m), \alpha(n)) && (\alpha \text{ non-exp.})\\ &= d^{\mathcal{A}}(h(\delta_m), h(\delta_n))\,. && (h \circ \delta_{\mathcal{M}} = \alpha) \end{aligned}$$

(Inductive step) Assume $|\,supp(\mu)| > 1$ and $|\,supp(\nu)| > 1$. Then, there exist nontrivial measurable partitions $(N_1; N_2)$ of $supp(\mu)$ and $(M_1; M_2)$ of $supp(\nu)$ such that $\mu(N_1), \mu(N_2), \nu(M_1), \nu(M_2) \neq \{0,1\}$.

Let $\omega \in C(\mu,\nu)$ and $R = N_1 \times M_1$. Note that R is measurable (finite sets are always Borel measurable in the product space) and $\omega(R) \notin \{0,1\}$. Let $e = \omega(R)$. By Lemma B.1, for any $\varepsilon > 0$ such that $\int d^p \mathrm{d}\omega \leq W_d^p(\mu,\nu) + \varepsilon$, there exist $\mu_i, \nu_i \in \Pi[M]$ such that

$$\mu = \mu_1 +_e \mu_2 \quad \text{and} \quad \nu = \nu_1 +_e \nu_2\,,$$
$$W_d^p(\mu,\nu)^p + \varepsilon \geq e W_d^p(\mu_1,\nu_1)^p + (1-e) W_d^p(\mu_2,\nu_2)^p\,.$$

Moreover, by the choice of R, we have that $supp(\mu_1) \subseteq N_1$, $supp(\mu_2) \subseteq N_2$, $supp(\nu_1) \subseteq M_1$, and $supp(\nu_2) = M_2$. Thus,

$$(\mu_1,\nu_1) \sqsubset (\mu,\nu) \qquad \text{and} \qquad (\mu_2,\nu_2) \sqsubset (\mu,\nu)\,.$$

Then the following holds:

$$\begin{aligned} W_d^p(\mu,\nu)^p + \varepsilon &\geq e W_d^p(\mu_1,\nu_1)^p + (1-e) W_d^p(\mu_2,\nu_2)^p && (\text{splitting lemma})\\ &\geq e d^{\mathcal{A}}(h(\mu_1), h(\nu_1))^p + (1-e) d^{\mathcal{A}}(h(\mu_2), h(\nu_2))^p && (\text{ind. hypothesis.})\\ &\geq d^{\mathcal{A}}(h(\mu_1) +_e h(\mu_2), h(\nu_1) +_e h(\nu_2))^p && (I_p)\\ &= d^{\mathcal{A}}(h(\mu_1 +_e \mu_2), h(\nu_1 +_e \nu_2))^p && (h \text{ homo.})\\ &= d^{\mathcal{A}}(h(\mu), h(\nu))^p && (\text{splitting lemma}) \end{aligned}$$

Since in the inequality above $\varepsilon > 0$ is arbitrarily chosen, the proof is done. ∎

The next result follows directly from Theorems 7.7 and 8.7.

Corollary 8.8. *The p-interpolative barycentric algebras $\Pi[\mathcal{M}]$ and $\mathcal{B}^{\mathbb{I}_p}\mathcal{M}$ are isomorphic, with isomorphism characterized by $h\colon \mathcal{B}^{\mathbb{I}_p}\mathcal{M} \longrightarrow \Pi[\mathcal{M}]$*

$$h(\eta^{\mathbb{I}_p}_{\mathcal{M}}(m)) = \delta_m\,, \qquad h(t +_\varepsilon s) = \varepsilon h(t) + (1-\varepsilon) h(s)\,,$$

where $m \in M$, $t, s \in \mathcal{B}^{\mathbb{I}_p}\mathcal{M}$ and δ_m is the Dirac measure on m.

This means that the quantitative equational theory $\mathbb{I}_p$, further extended with the axioms relative to the space (M,d), gives rise to W_d^p; and for $p = 1$ it gives rise to the Kantorovich metric K_d; in both cases this happens even though nothing in the equations mentions optimal transport.

8.5 Quantitative semilattices with zero

Fix an extended metric space $\mathcal{M} = (M, d)$. Let $\mathbb{S}^{\mathcal{S}}\mathcal{M}$ be the quantitative semilattice, of signature $\mathbb{S} = \{+ : 2, 0 : 0\}$ in $\mathbb{S}[\mathcal{S}]$ freely generated from $\mathcal{M}$. By Theorem 7.7, $\mathbb{S}^{\mathcal{S}}\mathcal{M}$ has the universal mapping property for $\mathcal{M}$ to

$$\mathbf{U}\colon \mathbb{S}[\mathcal{S}] \longrightarrow \mathbf{EMet}.$$

Denote by $\mathbb{F}[\mathcal{M}]$ the set of all *finite* subsets of M. In what follows we show that $\mathbb{F}[\mathcal{M}]$ can be organized as a quantitative semilattice and it is an element of $\mathbb{S}[\mathcal{S}]$ that has the universal mapping property for $\mathcal{M}$ to $\mathbf{U}\colon \mathbb{S}[\mathcal{S}] \longrightarrow \mathbf{EMet}$. This will prove that $\mathbb{F}[\mathcal{M}]$ and $\mathbb{S}^{\mathcal{S}}\mathcal{M}$ are isomorphic $\mathbb{S}$-quantitative algebras.

We organize $\mathbb{F}[\mathcal{M}]$ as an algebra of type $\mathbb{S}$ by defining, for arbitrary $A, B \in \mathbb{F}[\mathcal{M}]$, $A + B = A \cup B$, $0 = \emptyset$. And we interpret $\mathbb{F}[\mathcal{M}]$ as a metric space by taking the Hausdorff metric H_d induced by d on the set of all compact subsets of $\mathcal{M}$ is defined, for arbitrary compact sets $A, B \subseteq M$ as

$$H_d(A, B) = \max\left\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\right\},$$

where, $d(m, N) = \inf_{n \in N} d(m, n)$ denotes the distance from an element $m \in M$ to a set $N \subseteq M$ (for more details about this metric see Appendix A). Since the finite sets are compact, the restriction of H_d to $\mathbb{F}[\mathcal{M}]$ is a metric.

We have the important soundness result.

Theorem 8.9. $\mathbb{F}[\mathcal{M}] = (\mathbb{F}[\mathcal{M}], \mathbb{S}, H_d) \in \mathbb{S}[\mathcal{S}]$.

Proof. Note that each assignment $\iota \in \mathbb{S}[X|\mathbb{F}[\mathcal{M}]]$ maps variables in X to sets in $\mathbb{F}[\mathcal{M}]$. Hence we only need to check the axioms on $\mathbb{S}$-terms constructed over M. (Refl), (Symm), (Triang), (Max), and (Cont) follow from the fact that H_d is a metric on $\mathbb{F}[\mathcal{M}]$.

(S0)–(S3) follow by the interpretations for $+ : 2 \in \mathbb{S}$ and $0 : 0 \in \mathbb{S}$ and the fact that H_d is a metric on $\mathbb{F}[M]$. To prove that (S4) holds, it is convenient to recall the dual characterisation of H_d in terms of relational couplings (Therorem A.4). Assume $H_d(A, B) \leq \varepsilon$, $H_d(A', B') \leq \varepsilon'$, and let $R \in C(A, B)$ and $R' \in C(A', B')$ optimal, i.e., such that $H_d(A, B) = \max_{(m,n) \in R} d(m, n)$ and $H_d(A', B') = \max_{(m,n) \in R'} d(m, n)$. Then

$$\begin{aligned}
\max\{\varepsilon, \varepsilon'\} &\geq \max\{H_d(A, B), H_d(A', B')\} && \text{(by hypothesis.)}\\
&= \max\left\{\max_{(m,n) \in R} d(m, n), \max_{(m,n) \in R'} d(m, n)\right\} && (R, R' \text{ optimal})\\
&= \max_{(m,n) \in R \cup R'} d(m, n) && (\max \text{ on } \cup)\\
&\geq H_d(A \cup B, A' \cup B') && (R \cup R' \in C(A \cup B, A' \cup B'))\\
&= H_d(A + B, A' + B'). && \text{(def. +)}
\end{aligned}$$

Note that (NExp) is an instance of (S4), where $\varepsilon = \varepsilon'$. ∎

Define $\chi_{\mathcal{M}}\colon \mathcal{M} \longrightarrow \mathbb{F}[\mathcal{M}]$ as the map that assigns to arbitrary $m \in M$, the singleton set $\{m\}$. Note that, $H_d(\{m\}, \{n\}) = d(m, n)$, hence $\chi_{\mathcal{M}}$ is non-expansive.

The next theorem shows that the quantitative semilattice $\mathbb{F}[\mathcal{M}]$, if paired with $\chi_{\mathcal{M}}\colon \mathcal{M} \longrightarrow \mathbb{F}[\mathcal{M}]$, is a universal arrow.

Theorem 8.10. *The pair $(\mathbb{F}[\mathcal{M}], \chi_{\mathcal{M}})$ is an universal arrow from $\mathcal{M} \in$ **EMet** to $\mathbf{U}\colon \mathbb{S}[\mathcal{S}] \longrightarrow$ **EMet**. Explicitly, for each $\mathcal{A} \in \mathbb{S}[\mathcal{S}]$ and every non-expansive map $\alpha\colon \mathcal{M} \longrightarrow \mathbf{U}\mathcal{A}$, there exists a unique* *morphism* $h\colon \mathbb{F}[\mathcal{M}] \longrightarrow \mathcal{A}$ *of quantitative algebras such the following diagram commutes*

$$\begin{array}{ccccc} & & \textit{in}\ \mathbf{EMet} & & \textit{in}\ \mathbb{S}[\mathcal{S}] \\ \mathcal{M} & \xrightarrow{\chi_{\mathcal{M}}} & \mathbf{U}(\mathbb{F}[\mathcal{M}]) & & \mathbb{F}[\mathcal{M}] \\ & \searrow^{\alpha} & \downarrow{\scriptstyle \mathbf{U}(h)} & & \downarrow{\scriptstyle h} \\ & & \mathbf{U}(\mathcal{A}) & & \mathcal{A} \end{array}$$

Proof. Let $\mathcal{A} = (A, \mathbb{S}^{\mathcal{A}}, d^{\mathcal{A}}) \in \mathbb{S}[\mathcal{S}]$ and $\alpha\colon \mathcal{M} \longrightarrow \mathcal{A}$ a non-expansive map. Define the function $h\colon \mathbb{F}[\mathcal{M}] \longrightarrow \mathcal{A}$ by induction on the size of the sets, as follows:

- $h(\emptyset) = 0^{\mathcal{A}}$;
- for $m \in P \in \mathbb{F}[\mathcal{M}]$, $h(P) = \alpha(m) +^{\mathcal{A}} h(P \setminus \{m\})$.

To show that h is well-defined, we need to prove that its definition is independent of the choice of the element $m \in P$. Let $m, n \in P$, be two distinct elements. The following calculation establishes independence from the choice of n or m:

$$\begin{aligned} h(P) &= \alpha(m) +^{\mathcal{A}} (\alpha(n) +^{\mathcal{A}} h(P \setminus \{m,n\})) && \text{(def. } h\text{)} \\ &= (\alpha(m) +^{\mathcal{A}} \alpha(n)) +^{\mathcal{A}} h(P \setminus \{m,n\}) && (\mathcal{A} \models \text{(S3)}) \\ &= (\alpha(n) +^{\mathcal{A}} \alpha(m)) +^{\mathcal{A}} h(P \setminus \{m,n\}) && (\mathcal{A} \models \text{(S2)}) \\ &= \alpha(n) +^{\mathcal{A}} (\alpha(m) +^{\mathcal{A}} h(P \setminus \{m,n\})) && (\mathcal{A} \models \text{(S3)}) \\ &= \alpha(n) +^{\mathcal{A}} h(P \setminus \{n\}) && \text{(def. } h\text{)} \\ &= h(P)\,. && \text{(def. } h\text{)} \end{aligned}$$

Clearly, by definition of h, we have that $h \circ \chi_{\mathcal{M}} = \alpha$. Now we prove that h is a homomorphism. By definition $h(0) = h(\emptyset) = 0^{\mathcal{A}}$. Let $P, Q \in \mathbb{F}[\mathcal{M}]$, by induction on the size of P we show $h(P + Q) = h(P) + h(Q)$.

(Base case) Let $P = \emptyset$

$$\begin{aligned} h(\emptyset + Q) &= h(Q) && \text{(def. +)} \\ &= 0^{\mathcal{A}} +^{\mathcal{A}} h(Q) && (\mathcal{A} \models \text{(S0),(S2)}) \\ &= h(\emptyset) + h(Q)\,. && (def.\ h) \end{aligned}$$

(Inductive step) Assume $m \in P \cap Q$. We consider two cases: $m \in Q$ and $m \notin Q$. We show only the first case; the other can be derived by avoiding the application of (S1) in what follows:

$$\begin{aligned} h(P + Q) &= h(P \cup Q) && \text{(def. +)} \\ &= \alpha(m) +^{\mathcal{A}} h((P \cup Q)\setminus\{m\}) && \text{(def. } h\text{)} \\ &= (\alpha(m) +^{\mathcal{A}} \alpha(m)) +^{\mathcal{A}} h((P\setminus\{m\}) \cup (Q\setminus\{m\})) && (\mathcal{A} \models \text{(S1)}) \\ &= (\alpha(m) +^{\mathcal{A}} \alpha(m)) +^{\mathcal{A}} (h(P \setminus \{m\}) +^{\mathcal{A}} h(Q\setminus\{m\})) && \text{(hyp. ind)} \\ &= (\alpha(m) +^{\mathcal{A}} h(P\setminus\{m\})) +^{\mathcal{A}} (\alpha(m) +^{\mathcal{A}} h(Q\setminus\{m\})) && (\mathcal{A} \models \text{(S2),(S3)}) \\ &= h(P) + h(Q) && \text{(def. } h\text{)} \end{aligned}$$

As for the uniqueness of the homomorphism. Let $h' \neq h$ be another homomorphism such that $h' \circ \chi_{\mathcal{M}} = \alpha$; and let P a minimal set such that $h'(P) \neq h(P)$. If $P = \emptyset$, the contradiction follows from the fact the h', h are homomorphisms and the interpretation of $\emptyset$ as the constant 0. Assume

$P \neq \emptyset$, and let $m \in P$. Then

$$\begin{aligned} h(P) &= \alpha(m) +^{\mathcal{A}} h(P\backslash\{m\}) && \text{(def. } h) \\ &= \alpha(m) +^{\mathcal{A}} h'(P\backslash\{m\}) && \text{(minimality)} \\ &= h'(\{m\}) +^{\mathcal{A}} h'(P\backslash\{m\}) && (h' \circ \chi_{\mathcal{M}} = \alpha) \\ &= h(\{m\} + P\backslash\{m\}) && (h' \text{ homomorphism.}) \\ &= h(P) && \text{(def. +)} \end{aligned}$$

It remains to show that h is non-expansive, *i.e.*, $H_d(P, Q) \geq d^{\mathcal{A}}(h(P), h(Q))$, for all $P, Q \in \mathbb{F}[\mathcal{M}]$. We proceed by well-founded induction on pairs of sets (P, Q) partially ordered by $(P, Q) \sqsubseteq (P', Q')$ iff $|P| \leq |P'|$ and $|Q| \leq |Q'|$ (note that this is a well-founded partial order, *i.e.*, the corresponding strict order $\sqsubset$ is a well-founded relation).

(Base case) Assume $P, Q = \emptyset$. Then, by definition of H_d, $H_d(\emptyset, \emptyset) = \infty$. Hence $H_d(\emptyset, \emptyset) \geq d^{\mathcal{A}}(h(\emptyset), h(\emptyset))$ is trivially satisfied.

(Inductive step) Without loss of generality, assume $P, Q \neq \emptyset$. Then, there exists $m \in P$ and $n \in Q$. If $P = \{m\}$, then

$$\begin{aligned} H_d(\{m\}, Q) &\geq d(m, n) && \text{(def. } H_d) \\ &\geq d^{\mathcal{A}}(\alpha(m), \alpha(n)) && (\alpha \text{ non-expansive}) \\ &= d^{\mathcal{A}}(h(\{m\}), h(\{n\}))\,. && (\alpha = h \circ \chi_{\mathcal{M}}) \end{aligned}$$

Similarly for the case $Q = \{n\}$. Assume that $P \neq \{m\}$ and $Q \neq \{n\}$. Since P and Q are finite, the Hausdorff distance between them must be realised as the distance between some element in P and some element in Q. Consequently, there exist $m \in P$ and $n \in Q$ such that[10]

$$H_d(P, Q) = \max\{H_d(\{m\}, \{n\}), H_d(P\backslash\{m\}, B\backslash\{n\})\}\,. \tag{4}$$

Then, the following holds

$$\begin{aligned} d^{\mathcal{A}}(h(P), h(Q)) &= d^{\mathcal{A}}(\alpha(m) +^{\mathcal{A}} h(P\backslash\{m\}), \alpha(m) +^{\mathcal{A}} h(P\backslash\{m\})) && \text{(def. h)} \\ &\leq \max\{d^{\mathcal{A}}(\alpha(m), \alpha(n)), d^{\mathcal{A}}(h(P\backslash\{m\}), h(B\backslash\{n\}))\} && (\mathcal{A} \models (S4)) \\ &\leq \max\{d^{\mathcal{A}}(\alpha(m), \alpha(n)), H_d(P\backslash\{m\}, B\backslash\{n\})\} && \text{(hyp. ind.)} \\ &\leq \max\{d(m, n), H_d(P\backslash\{m\}, B\backslash\{n\})\} && (\alpha \text{ non-expansive}) \\ &= \max\{H_d(\{m\}, \{n\}), H_d(P\backslash\{m\}, B\backslash\{n\})\} && \text{(def. } H_d) \\ &= H_d(P, Q) && \text{(equation (4))} \end{aligned}$$

This concludes the proof. ∎

An immediate consequence of Theorem 7.7 and 8.10 is the following.

Corollary 8.11. *The quantitative semilattices $\mathbb{F}[\mathcal{M}]$ and $\mathbb{S}^{\mathcal{S}}\mathcal{M}$ are isomorphic, with isomorphism characterized by $h\colon \mathbb{S}^{\mathcal{S}}\mathcal{M} \longrightarrow \mathbb{F}[\mathcal{M}]$*

$$h(\eta^{\mathcal{S}}_{\mathcal{M}}(m)) = \{m\}\,, \qquad\qquad h(t + s) = h(t) \cup h(s)\,,$$

where $m \in M$ and $t, s \in \mathbb{S}^{\mathcal{S}}\mathcal{M}$.

[10]Notice that this equation is reminiscent of the *splitting lemma* used, for example, with the Kantorovich metric.

Hence, the distance induced by the quantitative equational theory $\mathcal{S}$ extended with the axioms relative to the generator $\mathcal{M} = (M, d)$ is the Hausdorff metric induced by d. Thus we say that $\mathcal{S}$ gives rise to the Hausdorff distance on $\mathbb{F}[\mathcal{M}]$.

9 Complete quantitative algebras

An extended metric space $\mathcal{M} = (M, d)$ is complete if every Cauchy sequence in M converges to a point in M. For any metric space $\mathcal{M}$, it is possible to construct a complete metric space $(\!|\mathcal{M}|\!) = ((\!|M|\!), (\!|d|\!))$ that embeds M as a dense subset. This construction is known as the *Cauchy completion* of $\mathcal{M}$. The embedding $e\colon \mathcal{M} \longrightarrow (\!|\mathcal{M}|\!)$ is an isometry such that every element of $(\!|\mathcal{M}|\!)$ is the limit of a Cauchy sequence[11] in $\mathcal{M}$ and the distance of two elements $m = \lim_i m_i$ and $n = \lim_i n_i$ in $(\!|\mathcal{M}|\!)$ is simply $(\!|d|\!)(m, n) = \lim_i d(m_i, n_i)$. For any non-expansive function $f\colon \mathcal{M} \longrightarrow \mathcal{N}$, where $\mathcal{N}$ is a complete metric space, there is a unique non-expansive extension $f^*\colon (\!|\mathcal{M}|\!) \longrightarrow \mathcal{N}$ defined in $m \in (\!|M|\!)$ by $f^*(m) = \lim_i f(m_i)$, for an arbitrary Cauchy sequence (m_i) converging to m. The map f^* is the unique extension of f to the completion $(\!|\mathcal{M}|\!)$ of $\mathcal{M}$.

Let **CEMet** be the full subcategory of **EMet**, whose objects are complete extended metric spaces. The Cauchy completion of a metric space extends to a functor $(\!|\cdot|\!)\colon$ **EMet** $\longrightarrow$ **CEMet**, mapping a metric space $\mathcal{M}$ to its completion $(\!|\mathcal{M}|\!)$. This functor is left adjoint to the embedding **CEMet** $\hookrightarrow$ **EMet**, making **CEMet** a reflective subcategory of **EMet**; this explains, in abstract categorical terms, the existence and uniqueness of the extension f^* on the completion.

It is easy to observe that for any Ω-quantitative algebra $\mathcal{A} = (A, \Omega, d)$ and any n-ary operation symbol $f \in \Omega$, if each sequence $(a_i^j)_i$ in A is Cauchy for $1 \le j \le n$, then the sequence $(f^{\mathcal{A}}(a_i^1, \dots, a_i^n))_i$ is also Cauchy; this follows directly from the non-expansiveness of $f^{\mathcal{A}}$. Thanks to it, the following definition of completion of a quantitative algebra is well-defined.

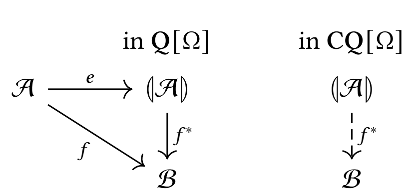
Definition 9.1 (Completions of Quantitative Algebras). *Given a quantitative algebra $\mathcal{A} = (A, \Omega, d)$, its Cauchy completion is the quantitative algebra $(\!|\mathcal{A}|\!) = ((\!|A|\!), \Omega^{(\!|\mathcal{A}|\!)}, (\!|d|\!))$, where $((\!|A|\!), (\!|d|\!))$ is the Cauchy completion of (A, d); and for arbitrary $f : n \in \Omega$, $a_1, \dots, a_n \in A$ and $(b_i^j)_i$ for $j \le n$ Cauchy sequences in A, the following hold*

- $f^{(\!|\mathcal{A}|\!)}(a_1, \dots, a_n) = f^{\mathcal{A}}(a_1, \dots, a_n)$;
- $f^{(\!|\mathcal{A}|\!)}(\lim_i b_i^1, \dots, \lim_i b_i^n) = \lim_i f^{(\!|\mathcal{A}|\!)}(b_i^1, \dots, b_i^n)$

Notice that, indeed, $(\!|\mathcal{A}|\!)$ is a quantitative algebra. What we need to verify for this is that $f^{(\!|\mathcal{A}|\!)}$ are non-expansive with respect to $(\!|d|\!)$.

Consider the category of Ω-quantitative algebras $\mathbf{Q}[\Omega]$ and denote by $\mathbf{CQ}[\Omega]$ the full subcategory formed by all Ω-quantitative algebras having as carrier a complete metric space.

Recall that, the universal property of the adjunction is equivalently characterised as follows: for any $\mathcal{A} \in \mathbf{Q}[\Omega]$ and morphism $f\colon \mathcal{A} \longrightarrow \mathcal{B}$ of Ω-quantitative algebras where $\mathcal{B} \in \mathbf{CQ}[\Omega]$, there exists a unique morphism $f^*\colon (\!|\mathcal{A}|\!) \longrightarrow \mathcal{B}$ of (complete) Ω-quantitative algebras making the following diagram commute



[11]Actually one needs to define an equivalence relation on Cauchy sequences and work with equivalence classes, but these details are well known and we will not belabour the point.

where $e\colon \mathcal{A} \longrightarrow (\!|\mathcal{A}|\!)$ is the embedding of $\mathcal{A}$ into its completion. The homomorphism f^* of quantitative algebra is the unique extension of f on the completion.

There is an interesting relation between the judgements satisfied by an algebra and the judgements satisfied by its completion. To state it, we need to introduce the concept of *continuous equation scheme.*

Definition 9.2 (Continuous equation scheme). *Let Ω be a signature and X a set of variables. A continuous equation scheme (CES) is a syntactic construction over $\hat{\Omega}X$ of the form*

$$x_1 =_{v_1} y_1, \dots, x_n =_{v_n} y_n \vdash s =_{f(v_1,\dots,v_n)} t$$

where $x_1, \dots, x_n \in X$, $s, t \in \hat{\Omega}X$, $v_1, \dots, v_n$ are variables over $\mathbf{R}_+$ and

$$f : \mathbf{R}_+^n \longrightarrow \mathbf{R}_+$$

is a continuous function on $\mathbf{R}_+$.

Note that a CES is not a quantitative judgement, because the quantitative equations of a CES use indices that are not positive reals, but rather variables and functions on positive reals. However, a CES represents a set of quantitative equations on $\hat{\Omega}X$, which are all the equations one gets by instantiating the variables $v_1, \dots, v_n$ with positive numbers. Concretely, the CES in Definition 9.2 represents the following set of quantitative equations on $\hat{\Omega}X$:

$$\left\{x_1 =_{\varepsilon_1} y_1, \dots, x_n =_{\varepsilon_n} y_n \vdash s =_{f(\varepsilon_1,\dots,\varepsilon_n)} t \;\middle|\; \varepsilon_1, \dots, \varepsilon_n \in \mathbf{R}_+\right\}$$

Note that for some choice of $\varepsilon_1, \dots, \varepsilon_n, \varepsilon \in \mathbf{R}_+$ a quantitative equation of the form

$$x_1 =_{\varepsilon_1} y_1, \dots, x_n =_{\varepsilon_n} y_n \vdash s =_{\varepsilon} t$$

is not a particular case of a continuous equation scheme with the continuous function the constant function ε. This is because the quantitative equalities on the left-hand side are using a concrete n-tuple of positive reals $\varepsilon_1, \dots, \varepsilon_n$ and not all the possible such n-tuples, as required by the definition of continuous equation scheme.
On the other hand, any unconditional judgement is a particular case[12] of a CES, the only kind of CES having a singleton as the set of its instances.

Hereafter, we will use a CES to speak collectively of all of its instantiations. For this reason, for an algebra $\mathcal{A}$ and a CES as in Definition 9.2, we write

$$x_1 =_{v_1} y_1, \dots, x_n =_{v_n} y_n \models_{\mathcal{A}} s =_{f(v_1,\dots,v_n)} t$$

when for all n tuples $(\varepsilon_1, \dots, \varepsilon_n) \in \mathbf{R}_+^n$ we have that

$$x_1 =_{\varepsilon_1} y_1, \dots, x_n =_{\varepsilon_n} y_n \models_{\mathcal{A}} s =_{f(\varepsilon_1,\dots,\varepsilon_n)} t$$

Definition 9.3 (Continuous theories). *A quantitative equational/Horn theory $\mathcal{U}$ over $\hat{\Omega}X$ is a continuous equational/Horn theory if it admits an axiomatization containing all the instances of a set of continuous equational schemes and no other axioms.*

The next theorem identifies a couple of situations where the completion of a term algebra induced by a theory remains a model of that theory.

Lemma 9.4. *Let Ω be a signature, X a set of variables, $\mathcal{A}$ a quantitative Ω-algebra and $(\!|\mathcal{A}|\!)$ its completion.*

(1) If $\Gamma \vdash \phi$ is a quantitative judgement on $\hat{\Omega}X$, then

$$\Gamma \models_{(\!|\mathcal{A}|\!)} \phi \;\; \textit{implies} \;\; \Gamma \models_{\mathcal{A}} \phi.$$

[12]We can interpret vacuously the presence of the real variables.

(2) If $\Gamma \vdash \phi$ is a closed judgement on $\hat{\Omega}X$ (i.e., involving no variables), then

$$\Gamma \models_{(\!|\mathcal{A}|\!)} \phi \;\; \text{iff} \;\; \Gamma \models_{\mathcal{A}} \phi.$$

(3) If $\Gamma \vdash \phi$ is a CES on $\hat{\Omega}X$, then

$$\Gamma \models_{(\!|\mathcal{A}|\!)} \phi \;\; \text{iff} \;\; \Gamma \models_{\mathcal{A}} \phi.$$

(4) If $s, t \in \hat{\Omega}X$, then

$$\models_{(\!|\mathcal{A}|\!)} s =_{\varepsilon} t \;\; \text{iff} \;\; \models_{\mathcal{A}} s =_{\varepsilon} t.$$

Proof. Let $\mathcal{A} = (A, \Omega, d)$ and $(\!|\mathcal{A}|\!) = ((\!|A|\!), \Omega, (\!|d|\!))$.

1. Suppose $\Gamma \vdash \phi$ is the judgement $\{s_i =_{\varepsilon_i} t_i \mid i \in I\} \vdash s =_{\varepsilon} t$.

Since $\{s_i =_{\varepsilon_i} t_i \mid i \in I\} \models_{(\!|\mathcal{A}|\!)} s =_{\varepsilon} t$, this means that for any assignment $\alpha \in \Omega[X|(\!|\mathcal{A}|\!)]$,

$$[(\!|d|\!)(\alpha(s_i), \alpha(t_i)) \leq \varepsilon_i \;\text{ for all }\; i \in I] \;\text{ implies }\; (\!|d|\!)(\alpha(s), \alpha(t)) \leq \varepsilon.$$

Since any $\beta \in \Omega[X|\mathcal{A}]$ can also be seen as an assignment in $\Omega[X|(\!|\mathcal{A}|\!)]$, we have that for any $\beta \in \Omega[X|\mathcal{A}]$,

$$[(\!|d|\!)(\beta(s_i), \beta(t_i)) \leq \varepsilon_i \;\text{ for all }\; i \in I] \;\text{ implies }\; (\!|d|\!)(\beta(s), \beta(t)) \leq \varepsilon.$$

2. Suppose now that the judgement is closed, hence in $\Omega\emptyset$. This means that there exists only one assignment, both in $\Omega[\emptyset|\mathcal{A}]$ and in $\Omega[\emptyset|(\!|\mathcal{A}|\!)]$ and it is the one associating to each term $s \in \hat{\Omega}\emptyset$ its interpretation $s^{\mathcal{A}}$ and $s^{(\!|\mathcal{A}|\!)}$ respectively. But from the way the completion of $\mathcal{A}$ was defined, we know that $s^{\mathcal{A}} = s^{(\!|\mathcal{A}|\!)} \in A$, since the closed terms in $\hat{\Omega}\emptyset$ do not include limits of Cauchy sequences that are not definable in $\hat{\Omega}\emptyset$. This establishes the equivalence we need to prove.

3. The left-to-right implication follows as $(\!|d|\!)$ agrees with d on A, and any assignment $\alpha \in \Omega[X|\mathcal{A}]$ can be turned into a assignment $\beta \in \Omega[X|(\!|\mathcal{A}|\!)]$ as $\beta = e \circ \alpha$, where $e\colon A \longrightarrow (\!|A|\!)$ is the isometric embedding of A into its completion.

In the remainder we focus on the converse implication. Assume that $\mathcal{A}$ is a model for the CES

$$x_1 =_{v_1} y_1, \ldots, x_n =_{v_n} y_n \vdash s =_{f(v_1,\ldots,v_n)} t$$

This means that for any $\varepsilon_1, \ldots, \varepsilon_n \in \mathbf{R}_+$ and any assignment $\alpha \in \Omega[X|\mathcal{A}]$,

$$[\forall i.\, d(\alpha(x_i), \alpha(y_i)) \leq \varepsilon_i] \quad \text{implies} \quad d(\alpha(s), \alpha(t)) \leq f(\varepsilon_1, \ldots, \varepsilon_n)\,. \tag{5}$$

Consider an arbitrary sequence $\varepsilon_1, \ldots, \varepsilon_n \in \mathbf{R}_+$ and $\beta \in \Omega[X|(\!|\mathcal{A}|\!)]$ an assignment in $(\!|\mathcal{A}|\!)$ such that $(\!|d|\!)(\beta(x_i), \beta(y_i)) \leq \varepsilon_i$, for all $i = 1, \ldots, n$. We need to prove

$$(\!|d|\!)(\beta(s), \beta(t)) \leq f(\varepsilon_1, \ldots, \varepsilon_n)\,.$$

For each $x \in X$, let $(a^x_j)_j$ be a Cauchy sequence in A converging to $\beta(x)$.

For each j, define the assignment $\iota_j \in \Omega[X|\mathcal{A}]$ such that $\iota_j(x) = a^x_j$. Note that, for any term $r \in \hat{\Omega}X$, this construction guarantees that $(\iota_j(r))_j$ is a Cauchy sequence in A converging to $\beta(r)$ in $(\!|A|\!)$. This easily follows by induction, as the interpretation in $\mathcal{A}$ of all the operations in Ω are non-expansive.

For each $i = 1, \ldots, n$ and j, define $e^i_j = (\!|d|\!)(\beta(x_i), \beta(y_i)) - d(\iota_j(x_i), \iota_j(y_i))$. Clearly,

$$d(\iota_j(x_i), \iota_j(y_i)) \leq \varepsilon_i - e^i_j\,.$$

By applying the implication (5) to the assignment ι_j, we get

$$d(\iota_j(s), \iota_j(t)) \leq f(\varepsilon_1 + e^1_j, \ldots, \varepsilon_n + e^n_j)\,.$$

Since $(\!|d|\!)(\beta(x_i),\beta(y_i)) = \lim_j d(\iota_j(x_i),\iota_j(y_i))$, clearly $\lim_j e^i_j = 0$. By taking the inequality above to the limit after j and using the continuity of f in all variables, we obtain

$$(\!|d|\!)(\beta(s),\beta(t)) = \lim_j d(\iota_j(s),\iota_j(t)) \leq f(\varepsilon_1,\dots,\varepsilon_n)\,.$$

4. This follows from (3), as any unconditional equation is a continuous equation scheme. ∎

We provide below an example of a judgement that is satisfied by an algebra, but not by its completion, proving in this way that the right-to-left implication in Theorem 9.4(1) is not valid.

Example 9.5. *Consider a signature containing only one function symbol f of arity 1 and consider two variables x and y. Let $\mathcal{U}$ be the theory axiomatized by the axiom*

$$x =_2 y \vdash f(x) =_1 f(y).$$

Consider now the quantitative algebra $\mathcal{A}$ supported by the set $[0,1) \cup (3,4]$ with the usual metric on reals and where f is interpreted as the identity function. Note that

$$x =_2 y \models_{\mathcal{A}} f(x) =_1 f(y),$$

because for any assignment α, if $|\alpha(x) - \alpha(y)| \leq 2$, then $\alpha(x)$ and $\alpha(y)$ are either both in $[0,1)$ or both in $[3,4)$, meaning that $|\alpha(x) - \alpha(y)| \leq 1$, and also satisfying $|\alpha(f(x)) - \alpha(f(y))| \leq 1$, since f is the identity.

Consider now the completion $(\!|\mathcal{A}|\!)$, which is supported by the set $[0,1] \cup [3,4]$ with the usual metric and with f still interpreted as identity. In this case, we have that $x =_2 y \vdash f(x) =_1 f(y)$ is not satisfied by $(\!|\mathcal{A}|\!)$ because the assignment α such that $\alpha(x) = 1$ and $\alpha(y) = 3$, satisfies $|\alpha(x) - \alpha(y)| \leq 2$ but not $|\alpha(x) - \alpha(y)| \leq 1$.

The previous lemma shows us a few cases where satisfiability is preserved by completing models. This suggests the following definition.

Definition 9.6 (Model-completable theories). *A theory $\mathcal{U}$ over $\hat{\Omega}X$ is model-completable if for any Ω-quantitative algebra $\mathcal{A}$ we have that $\mathcal{A} \in \Omega[\mathcal{U}]$ implies $(\!|\mathcal{A}|\!) \in \Omega[\mathcal{U}]$.*

Lemma 9.7. *The following statements hold:*

(1) The continuous theories are model-completable.
(2) If U contains only closed judgements, then $\overline{U}$ (the theory axiomatised by U) is model-completable.
(3) If $\mathcal{U}_1, \mathcal{U}_2$ are model-completable, then so is $\overline{\mathcal{U}_1 \cup \mathcal{U}_2}$.

Proof. (1–2) are direct consequences of Lemma 9.4. As for (3), let $\mathcal{A} \in \Omega[\overline{\mathcal{U}_1 \cup \mathcal{U}_2}]$. Then for each $i \in \{1,2\}$, $\mathcal{A} \in \Omega[\mathcal{U}_i]$. Since $\mathcal{U}_i$ is model-completable, we get that $(\!|\mathcal{A}|\!) \in \Omega[\mathcal{U}_i]$. Hence, $(\!|\mathcal{A}|\!) \in \Omega[\mathcal{U}_1 \cup \mathcal{U}_2]$. ∎

If $\mathcal{U}$ is a quantitative theory over $\hat{\Omega}X$, we denote by $\mathbf{C}\Omega[\mathcal{U}]$ the full subcategory of $\mathbf{CQ}[\Omega]$ whose objects are models of $\mathcal{U}$; hence, the carriers of the quantitative algebras in $\mathbf{C}\Omega[\mathcal{U}]$ are complete metric spaces.

Corollary 9.8. *Let Ω be a signature, X a set of variables, $\mathcal{U}$ a quantitative equational/Horn theory on $\hat{\Omega}X$ and $\mathcal{M}$ an extended metric space. If $\mathcal{U}$ is a model-completable theory, then the completion $(\!|\Omega^{\mathcal{U}}\mathcal{M}|\!)$ of the quantitative term algebra is a model of $\mathcal{U}$, i.e., $(\!|\Omega^{\mathcal{U}}\mathcal{M}|\!) \in \mathbf{C}\Omega[\mathcal{U}]$.*

Proof. A direct consequence of Theorem 7.7, which states that $\Omega^{\mathcal{U}}\mathcal{M} \in \Omega[\mathcal{U}]$. ∎

An immediate consequence of Corollary 9.8 is that the completion functor restricts to a functor

$$(\!|\cdot|\!) : \Omega[\mathcal{U}] \longrightarrow \mathrm{C}\Omega[\mathcal{U}]$$

whenever $\mathcal{U}$ is a continuous equational theory. Moreover, this functor is left adjoint to the embedding $\mathrm{C}\Omega[\mathcal{U}] \hookrightarrow \Omega[\mathcal{U}]$.

Corollary 9.9. *Let $\mathcal{U}$ be a continuous equational/Horn theory. Then, $(\!|\cdot|\!) : \Omega[\mathcal{U}] \longrightarrow \mathrm{C}\Omega[\mathcal{U}]$ is left adjoint to the embedding $\mathrm{C}\Omega[\mathcal{U}] \hookrightarrow \Omega[\mathcal{U}]$.*

Proof. By Lemma 9.8, $(\!|\cdot|\!) : \Omega[\mathcal{U}] \longrightarrow \mathrm{C}\Omega[\mathcal{U}]$ is a well-defined functor. Note that $(\!|\cdot|\!)$ acts on the carriers as the Cauchy completion. Since this is a left adjoint to the embedding **CEMet** $\hookrightarrow$ **EMet**, the thesis follows by direct application of the same universal property that holds on this adjunction. ∎

By combining Theorem 7.7 and Corollary 9.9, we obtain the following characterisation of free complete quantitative algebra of a theory $\mathcal{U}$. Note that the result is stated only for theories $\mathcal{U}$ that are continuous.

Theorem 9.10 (Free complete quantitative algebra). *Let $\mathcal{U}$ be a continuous quantitative theory. For any complete extended metric space $\mathcal{M} \in$ **CEMet**, the pair $((\!|\Omega^{\mathcal{U}}\mathcal{M}|\!), \eta^{\mathcal{U}}_{\mathcal{M}})$ is a universal morphism from $\mathcal{M}$ to $\mathbf{U} : \mathrm{C}\Omega[\mathcal{U}] \longrightarrow$ **CEMet**.*
Explicitly, this means that, for each $\mathcal{A} \in \mathrm{C}\Omega[\mathcal{U}]$ and every non-expansive map $\alpha : \mathcal{M} \longrightarrow \mathbf{U}\mathcal{A}$, there exists a unique morphism $h : (\!|\Omega^{\mathcal{U}}\mathcal{M}|\!) \longrightarrow \mathcal{A}$ of complete quantitative algebras such that $\mathbf{U}h \circ \eta^{\mathcal{U}}_{\mathcal{M}} = \alpha$, i.e., the following diagram commutes

$$\begin{array}{ccc} & \textit{in } \mathbf{CEMet} & \\ \mathcal{M} & \xrightarrow{\eta^{\mathcal{U}}_{\mathcal{M}}} & \mathbf{U}((\!|\Omega^{\mathcal{U}}\mathcal{M}|\!)) \\ & \searrow^{\alpha} & \downarrow \mathbf{U}h \\ & & \mathbf{U}\mathcal{A} \end{array} \qquad \begin{array}{c} \textit{in } \mathrm{C}\Omega[\mathcal{U}] \\ (\!|\Omega^{\mathcal{U}}\mathcal{M}|\!) \\ \downarrow h \\ \mathcal{A} \end{array}$$

10 Examples of complete quantitative algebras

In Section 8 we have constructed free algebras over metric spaces for our main examples. These constructions have a purely algebraic flavour; in this section, we will construct the corresponding free algebras over *complete* metric spaces. In each case, we will need to do a suitable completion procedure, and this will typically have a more analytic character. We will only treat the cases of interpolative barycentric algebras and quantitative semilattices, as these will illustrate the different ideas that come into play.

10.1 Interpolative barycentric algebras

We focus on the class of the general Borel probability measures over a one-bounded, complete, separable metric space and prove that it forms a $\mathcal{B}$-quantitative algebra satisfying the axioms $\mathbb{I}_p$. It turns out that this is the free algebra in the category of the models of $\mathbb{I}_p$, defined over one-bounded, complete, separable metric spaces. Note that, unlike in Section 8, we are no longer restricting finitely-supported distributions.

It is important to observe that, in this section, we will be working within a different underlying category than in previous parts of the paper. Requiring separability is a standard assumption when one considers probabilistic situations, ensuring well-behaved measure-theoretic constructions and avoiding technical complications. While our results are formulated for separable spaces, they can be extended to the non-separable case using the techniques developed in [59]. However, even in that work, the development begins with separable spaces and only later extends the results to general complete metric spaces. Here we just stick to the separable case for simplicity.

Consider a complete separable metric space $\mathcal{M} = (M, d)$ with the metric taking values in $[0, 1]$. Let $\Delta[\mathcal{M}]$ be the set of all Borel probability measures on $\mathcal{M}$. Note that since $\mathcal{M}$ is complete and separable, all the measures in $\Delta[\mathcal{M}]$ are Radon. We endow $\Delta[\mathcal{M}]$ with the signature $\mathcal{B}$, where we define for arbitrary $\mu, \nu \in \Delta[\mathcal{M}]$ and $e \in [0, 1]$,

$$\mu +_e \nu = e\mu + (1 - e)\nu.$$

As shown previously, $\mathcal{B}^{\mathbb{I}_p}\mathcal{M}$ is a barycentric algebra isomorphic to $\Pi[\mathcal{M}]$. However, we prove below that the carrier of $\mathcal{B}^{\mathbb{I}_p}\mathcal{M}$ is separable, but it is not a complete metric space.

Consider the metric space $((\!|\mathcal{B}^{\mathbb{I}_p}\mathcal{M}|\!), (\!||-, -|_{\mathbb{I}_p}|\!))$ obtained by the completion of $(\mathcal{B}^{\mathbb{I}_p}\mathcal{M}, |-, -|_{\mathbb{I}_p})$. To tackle this, we need to understand how convergence works in this setting. The p-Wasserstein distance W_d^p is related to a standard notion in probability theory: weak convergence. We recall the explicit definition of weak convergence below (for more details see [85, Definition 6.8]).

Definition 10.1. *Given a sequence of probability measures ν_i defined on a metric space $\mathcal{M}$ we say that ν_i* ***converges weakly*** *to ν if for any bounded continuous real-valued function f defined on $\mathcal{M}$ we have $\int_M f \mathrm{d}\nu_i \longrightarrow \int_M f \mathrm{d}\nu$.*

The topology of weak convergence is called the *weak topology.*

More generally, we can define a so-called p-weak topology. We recall the explicit definition of the p-weak topology and the fact that this topology is metrized by the p-Wasserstein distance.

The following key lemma is well known (see, for example, Chapter 8 of [13]).

Proposition 10.2. *Let $\mathcal{M}$ be a Polish space and $c_1 \ldots, c_k$ be positive real numbers such that $\sum_{i=1}^k c_i = 1$, and $m_1, \ldots, m_k$ be points in $\mathcal{M}$. Then, measures of the form $\sum_{i=1}^k c_i \delta_{m_i}$ are p-weakly dense in $\Delta[\mathcal{M}]$.*

It immediately shows that $\Pi[\mathcal{M}]$ is dense in $\Delta[\mathcal{M}]$. We give a simplified proof in the Appendix B, as most of the references in the literature establish it after a long chain of more general results.

Hence, if $\mathcal{M} = (M, d)$ is a complete separable metric space, then p-Wasserstein metric W_d^p metrizes the p-weak topology on $\Delta[\mathcal{M}]$ (see Theorem 6.9 and Corollary 6.13 in [85]).

We are now ready to establish the result that the completion of free barycentric algebra on complete separable metric spaces is isomorphic to the space of Radon probability measures with the p-Wasserstein metric.

Let $\overline{\mathbb{K}}_p$ be the class of quantitative algebras in $\mathcal{B}[\mathbb{I}_p]$ supported by a complete separable metric space. Note that $(\!|\mathcal{B}^{\mathbb{I}_p}\mathcal{M}|\!)$ is the free quantitative algebra in $\overline{\mathbb{K}}_p$ generated over the separable metric space $\mathcal{M}$. This follows as in Theorem 9.10, by simply observing that $\mathbb{I}_p$ is a continuous axiom scheme and that the completion functor $(\!|\cdot|\!)$ preserves the separability of the spaces.

Next we prove that $\Delta[\mathcal{M}] = (\Delta[\mathcal{M}], \mathcal{B}, W_d^p)$ endowed with p-Wasserstein metric is isomorphic, as a quantitative algebra, to $(\!|\mathcal{B}^{\mathbb{I}_p}\mathcal{M}|\!)$.

Theorem 10.3. *If $\mathcal{M}$ is a complete separable metric space, then $\Delta[\mathcal{M}] \in \overline{\mathbb{K}}_p$. Moreover, $\Delta[\mathcal{M}]$ is isomorphic to $(\!|\mathcal{B}^{\mathbb{I}_p}\mathcal{M}|\!)$.*

Proof. Verifying the axioms of the barycentric algebras for $\Delta[\mathcal{M}]$ is routine and follows closely the proof of Theorem 8.6. What we need to prove further is that $\Delta[M]$ is a complete separable metric space.

Let $D \subseteq M$ be a countable dense subset of M (its existence is guaranteed by the fact that $\mathcal{M}$ is a separable space). Now $\Pi[D]$ is of course not countable but we can take all distributions that assign only rational measures to points and get a countable set. We call this $\mathcal{P}[D]$ for short. We now show that it is dense in $\Delta[M]$.

Let $\rho \in \Delta[\mathcal{M}]$. Since $\mathcal{M}$ is Polish, W_d^p metrizes the p-weak-topology on $\Delta[\mathcal{M}]$, which is also a Polish space (Corollary 6.13 in [85]). Moreover, $\Pi[\mathcal{M}]$ is dense in $\Delta[\mathcal{M}]$ with respect to this topology by Proposition 10.2. Hence, there exists a sequence $(\rho_i)_{i\in\mathbb{N}} \subseteq \Pi[\mathcal{M}]$ of measures with finite support on M that converges to ρ. Since D is dense in M and the rationals are dense in $[0,1]$, for any sequence $(\varepsilon_i)_{i\in\mathbb{N}} \in [0,1]$ that converges to 0, we can find a sequence $(\rho'_i)_{i\in\mathbb{N}} \subseteq \mathcal{P}[D]$ such that $W_d^p(\rho_i, \rho'_i) < \varepsilon_i$. Thus, $(\rho'_i)_{i\in\mathbb{N}}$ is a Cauchy sequence in $\Pi[\mathcal{M}]$ and since $(\rho_i)_{i\in\mathbb{N}}$ converges to ρ and $\Pi[\mathcal{M}]$ is complete, also $(\rho'_i)_{i\in\mathbb{N}}$ converges to ρ. And this proves that $\mathcal{P}[D]$ is dense in $\Delta[\mathcal{M}]$.

In Corollary 8.8 we have shown that $\Pi[\mathcal{M}]$ and $\mathcal{B}^{\mathbb{I}_p}\mathcal{M}$ are isomorphic. Since the completion of $\Pi[D]$ is unique, we obtain the isomorphism between $(\!|\mathcal{B}^{\mathbb{I}_p}\mathcal{M}|\!)$ and $\Delta[M]$. ∎

We conclude by remarking that all the results presented in this section can be readily extended to the case of subprobability measures by introducing a new constant in the signature, where the "missing mass" can reside. These are called *pointed* interpolative barycentric algebras; we have worked out this case, but it is too similar to the present case to merit an explicit presentation.

10.2 Quantitative semilattices

We now focus on the collection of compact subsets of a complete metric space and prove that it too can be organized as a quantitative semilattice. Note that in this case we do not restrict to separable metric spaces as it makes no difference to the proofs. We interpret, as before, + by $\cup$, 0 by $\emptyset$ and the constants as singletons. It turns out that this is the freely generated algebra in the category of quantitative semilattices over complete metric spaces. As might be expected, the proofs here are more analytic rather than the combinatorial proofs of the previous subsection.

Consider a complete metric space $\mathcal{M} = (M, d)$. Let $\mathbb{G}[\mathcal{M}]$ be the set of compact subsets of $\mathcal{M}$ in the open-ball topology of d. Note that a subset of M is compact if and only if it is closed and bounded, so the union of two compact sets is clearly closed and bounded, hence compact. We need to show that by endowing $\mathbb{G}[\mathcal{M}]$ with the Hausdorff metric H_d, we obtain a quantitative semilattice that satisfies $\mathcal{S}$.

As shown in the previous section, we can construct the freely generated quantitative semilattice $\mathbb{S}^{\mathcal{S}}\mathcal{M}$, which is isomorphic to $\mathbb{F}[\mathcal{M}]$. However, the carrier of $\mathbb{S}^{\mathcal{S}}\mathcal{M}$ is not a complete metric space; we need to consider the free quantitative semilattice in the category of complete metric spaces. The main claim of this section is that it is precisely $\mathbb{G}[\mathcal{M}]$ with the Hausdorff metric.

Let $\overline{\mathbb{K}}$ be the subcategory of quantitative semilattices with zero defined on complete separable metric spaces. Consider $(\!|\mathbb{S}^{\mathcal{S}}\mathcal{M}|\!)$, the completion of $\mathbb{S}^{\mathcal{S}}\mathcal{M}$. Since the axioms for quantitative semilattices form a continuous equation scheme, by Theorem 9.10, $(\!|\mathbb{S}^{\mathcal{S}}\mathcal{M}|\!)$ is the free algebra in $\overline{\mathbb{K}}$ over $\mathcal{M}$. Since $\mathbb{S}^{\mathcal{S}}\mathcal{M}$ is isomorphic to $\mathbb{F}[\mathcal{M}]$, their completions must be isomorphic. We prove that $\mathbb{G}[\mathcal{M}] = (\mathbb{G}[\mathcal{M}], \mathbb{S}, H_d)$ and $(\!|\mathbb{S}^{\mathcal{S}}\mathcal{M}|\!)$ are isomorphic quantitative semilattices with zero.

Theorem 10.4. *If $\mathcal{M}$ is a complete metric space, then $\mathbb{G}[\mathcal{M}] \in \overline{\mathbb{K}}$. Moreover, $\mathbb{G}[\mathcal{M}]$ is isomorphic to $(\!|\mathbb{S}^{\mathcal{S}}\mathcal{M}|\!)$.*

Proof. Verifying the axioms of quantitative semilattices with zero for $\mathbb{G}[\mathcal{M}]$ is straightforward and follows closely the proof of Theorem 8.9.

In addition, we need to prove that $(\mathbb{G}[\mathcal{M}], H_d)$ is a complete metric space. In other words, we need to show that the completion of the collection of finite subsets in the Hausdorff metric is precisely the collection of compact sets. However, it is a standard result that the collection of compact subsets of a complete metric space equipped with the Hausdorff metric is itself a complete metric space (see, for example, [77, pp.67–69] for an explicit proof). The finite subsets are, of course, compact, so any Cauchy sequence of finite subsets converges in the Hausdorff metric to a compact set.

We need to show that the finite subsets are dense in the space of compact subsets. We use the following notation and definition of the Hausdorff metric. Given a subset $A \subset M$ and $\varepsilon > 0$, we write:

$$(A)_\varepsilon := \bigcup_{x \in A} N_\varepsilon(x) \text{ where } N_\varepsilon(x) = \{y | d(x,y) < \varepsilon\}.$$

Here N_ε is the basic open neighbourhood centred at x and with radius ε. With this notation, the Hausdorff metric can be defined as follows:

$$H_d(X,Y) = \inf\{\varepsilon | Y \subseteq X_\varepsilon \text{ and } X \subseteq Y_\varepsilon\}.$$

Now we need to show that every compact set is the limit of a Cauchy sequence of finite sets, thus establishing that the finite sets are dense in the collection of compact sets and hence the closure of the collection of finite sets is the collection of compact sets.

Let C be any compact subset of M. For any $n \geq 1$ the collection $\{N_{1/n}(x) | x \in C\}$ is an open cover, hence has a finite subcover. We may write this finite subcover as $\{N_{1/n}(x) | x \in A_n\}$ where A_n is some finite subset of C. We claim that this sequence A_n is a Cauchy sequence and converges to C. Now $C \subseteq (A_n)_{1/n}$ by definition of the A_n. Thus, for any $m \geq n$ we have $A_m \subseteq C \subseteq (A_n)_{1/n}$ and $A_n \subseteq C \subseteq (A_m)_{1/m} \subseteq (A_m)_{1/m}$ which means $H_d(A_n, A_m) \leq 1/n$. Thus we have a Cauchy sequence. For any $m \geq n$ we have $C \subseteq (A_m)_{1/m} \subseteq (A_m)_{1/n}$ and $A_m \subseteq C \subseteq (C)_{1/n}$. In other words $H_d(C, A_m) < 1/n$.

Finally, we invoke Theorem 9.10 and the fact that the axioms for a quantitative semilattices form a continuous equation scheme to note that the completion of the free quantitative semilattice over metric spaces is the free quantitative semilattice over complete metric spaces. ∎

11 Completeness of Quantitative Equational Logic

In this section, we present a series of completeness results for quantitative equational logic.

11.1 General completeness for quantitative algebras

In this subsection, we prove the most general form of completeness result for quantitative equational logic against the class of quantitative algebras. But before doing this, we need to prove some lemmas that will be important ingredients of the proof of the main theorem.

Consider a signature Ω and a set X of variables. Let $\mathbb{C}$ be a set of fresh constants in one-to-one correspondence with the variables in X, i.e., there is a bijection $\iota : X \longrightarrow \mathbb{C}$; we also have its inverse $\iota^{-1} : \mathbb{C} \longrightarrow X$.

Let Σ be the signature we obtain by adding all the constants in $\mathbb{C}$ to Ω. In what follows, we will extend ι and it's inverse ι^{-1} to terms, obtaining the maps

$$\iota \colon \hat{\Omega}X \longrightarrow \hat{\Sigma}\emptyset \qquad\qquad \iota^{-1} \colon \hat{\Sigma}\emptyset \longrightarrow \hat{\Omega}X$$

inductively defined as follows:

- if $x \in X$, then $\iota(x) \in \mathbb{C} \subseteq \hat{\Sigma}\emptyset$;
- if $f : n \in \Omega$ and $t_i \in \hat{\Omega}X$, let $\iota(f(t_1, \ldots, t_n)) = f(\iota(t_1), \ldots, \iota(t_n))$.

and

- if $c \in \mathbb{C}$, then $\iota^{-1}(c) \in X \subseteq \hat{\Omega}X$;
- if $f : n \in \Omega$ and $t_i \in \hat{\Sigma}\emptyset$, let $\iota^{-1}(f(t_1, \ldots, t_n)) = f(\iota^{-1}(t_1), \ldots, \iota^{-1}(t_n))$.

These maps are extended to quantitative equations and sets of equations as expected.

When moving from Ω to Σ, any theory $\mathcal{U}$ on $\hat{\Omega}X$ (taken as a set of quantitative judgments) generates the theory $\mathcal{U}_{\mathbb{C}}$ on $\hat{\Sigma}\emptyset$ by systematically applying the map ι to the judgments in $\mathcal{U}$ and then closing the set that is obtained this way by the rules in Table 1.

In what follows we fix a quantitative theory $\mathcal{U}$ on $\hat{\Omega}X$.

If a judgement $\Gamma \vdash \phi$ over $\hat{\Omega}X$ is such that all the variables in X that appear in this judgement are in the list $\overline{x}$ of pairwise-distinct variables, we make this explicit by writing $\Gamma(\overline{x}) \vdash \phi(\overline{x})$.

Lemma 11.1. *In the context provided above,*

(1) If $\Delta \vdash \psi \in \mathcal{U}$, then $\iota(\Delta) \vdash \iota(\psi) \in \mathcal{U}_{\mathbb{C}}$;
(2) If $\Delta \vdash \psi \in \mathcal{U}_{\mathbb{C}}$, then $\iota^{-1}(\Delta) \vdash \iota^{-1}(\psi) \in \mathcal{U}$.

Proof. 1. Follows directly by the definition of $\mathcal{U}_{\mathbb{C}}$.

2. Any judgement in $\mathcal{U}_{\mathbb{C}}$ has a proof from the axioms in $\mathcal{U}$. If we apply uniformly i^{-1} to a proof of $\Delta \vdash \phi$ in $\mathcal{U}_{\mathbb{C}}$ from the axioms in $\mathcal{U}$, we will get a legal proof in $\mathcal{U}$ for $i^{-1}(\Delta) \vdash i^{-1}(\psi)$. ∎

For a collection C of quantitative Ω-algebras, we write $\Gamma \models_C s =_\varepsilon t$, to mean that $\Gamma \models_{\mathcal{A}} s =_\varepsilon t$, for all $\mathcal{A} \in C$.

If $\mathcal{A}$ is an Ω-quantitative algebra, let $\mathcal{A}_\Sigma$ represent the collection of all quantitative Σ-algebras extending $\mathcal{A}$ with an interpretation for the constants in $\mathbb{C}$ (*i.e.*, all the Σ-quantitative algebras having the same carrier as that of $\mathcal{A}$).

Lemma 11.2. *In the context provided above, for any judgement $\Gamma \vdash \phi$ in $\hat{\Omega}X$ and any $\mathcal{A} \in \mathbf{Q}[\Omega]$,*

$$\Gamma \models_{\mathcal{A}} \phi \;\; \textit{iff} \;\; \Gamma \models_{\mathcal{A}_\Sigma} \phi \;\; \textit{iff} \;\; \iota(\Gamma) \models_{\mathcal{A}_\Sigma} \iota(\phi).$$

Proof. Assume, without loss of generality, that the judgement $\Gamma \vdash \phi$ is of type $\Gamma(\overline{x}) \vdash \phi(\overline{x})$ where $\overline{x}$ is the list of all the variables involved in this judgement.

To simplify the presentation of this proof, for a list $\overline{a}$ of elements in $\mathcal{A}$ (since $\overline{a}$ can be seen as an assignment of the variables in the list $\overline{x}$) we use $\Gamma(\overline{a})$ as the Boolean predicate stating that under the assignment that maps $\overline{x}$ to $\overline{a}$, all the quantitative equalities in Γ are true. Similarly, $\phi(\overline{a})$ states that the quantitative equality ϕ is true when the variables in $\overline{x}$ are given the assignment $\overline{a}$.

1. $\Gamma(\overline{x}) \models_{\mathcal{A}} \phi(\overline{x})$ iff $\Gamma(\overline{x}) \models_{\mathcal{A}_\Sigma} \phi(\overline{x})$:

We have $\Gamma(\overline{x}) \models_{\mathcal{A}} \phi(\overline{x})$ iff, for any assignment $\alpha \in \Omega[X|\mathcal{A}]$, $\Gamma(\overline{x}) \models_{\mathcal{A},\alpha} \phi(\overline{x})$.

α choses a list $\alpha(\overline{x})$ of elements in $\mathcal{A}$ and for those it states that $\Gamma(\alpha(\overline{x}))$ implies $\phi(\alpha(\overline{x}))$. Since this is stated for all possible assignments, it means that it is stated for any possible list $\overline{a}$ of elements in $\mathcal{A}$. Hence, $\Gamma(\overline{x}) \models_{\mathcal{A}} \phi(\overline{x})$ is equivalent to, for any list $\overline{a}$ of elements in $\mathcal{A}$, $\Gamma(\overline{a})$ implies $\phi(\overline{a})$.

Let $\mathcal{B} \in \mathcal{A}_\Sigma$. Since the support of $\mathcal{B}$ is the support of $\mathcal{A}$, any list $\overline{a}$ of elements in $\mathcal{A}$ can be seen as the list $\beta(\overline{x})$ for some assignment $\beta \in \Sigma[X|\mathcal{B}]$. This means that the previous statement is further equivalent to the statement saying that, for any $\mathcal{B} \in \mathcal{A}_\Sigma$ and any assignment $\beta \in \Sigma[X|\mathcal{B}]$, $\Gamma(\overline{x}) \models_{\mathcal{B},\beta} \phi(\overline{x})$. But this is just the definition of $\Gamma(\overline{x}) \models_{\mathcal{A}_\Sigma} \phi(\overline{x})$.

2. $\Gamma(\overline{x}) \models_{\mathcal{A}_\Sigma} \phi(\overline{x})$ implies $\iota(\Gamma(\overline{x})) \models_{\mathcal{A}_\Sigma} \iota(\phi(\overline{x}))$:

Consider an arbitrary $\mathcal{B} \in \mathcal{A}_\Sigma$ and let $\theta : \mathbb{C} \longrightarrow \mathcal{A}$ be its interpretation of the constants in $\mathbb{C}$. Since from the hypothesis we know that for any list $\overline{a}$ of elements in $\mathcal{A}$ we have $[\Gamma(\overline{a})$ implies $\phi(\overline{a})]$, we also have in particular, that $[\Gamma(\theta(\iota(\overline{x})))$ implies $\phi(\theta(\iota(\overline{x})))]$. Hence, $\Gamma(\iota(\overline{x})) \models_{\mathcal{B}} \phi(\iota(\overline{x}))$, which is $\iota(\Gamma(\overline{x})) \models_{\mathcal{B}} \iota(\phi(\overline{x}))$. Since this happens for arbitrary $\mathcal{B} \in \mathcal{A}_\Sigma$, we get $\iota(\Gamma(\overline{x})) \models_{\mathcal{A}_\Sigma} \iota(\phi(\overline{x}))$.

3. $\iota(\Gamma(\overline{x})) \models_{\mathcal{A}_\Sigma} \iota(\phi(\overline{x}))$ implies $\Gamma(\overline{x}) \models_{\mathcal{A}} \phi(\overline{x})$:

Let $\alpha \in \Omega[X|\mathcal{A}]$ be an assignment; and let $\mathcal{B} \in \mathcal{A}_\Sigma$ be such that its interpretation $\theta : \mathbb{C} \longrightarrow \mathcal{A}$ of the constants in $\mathbb{C}$ is $\theta(\iota(\overline{x})) = \alpha(\overline{x})$ — the existence of this algebra is guaranteed by the fact that the constants in $\mathbb{C}$ are fresh and pairwise-distinct, and the elements in the list $\overline{x}$ are pairwise distinct.

$\iota(\Gamma(\overline{x})) \models_{\mathcal{A}_\Sigma} \iota(\phi(\overline{x}))$ guarantees that $[\Gamma(\theta(\iota(\overline{x})))$ implies $\phi(\theta(\iota(\overline{x})))]$. Hence, for any assignment $\alpha \in \Omega[X|\mathcal{A}]$ we have $[\Gamma(\alpha(\overline{x}))$ implies $\phi(\alpha(\overline{x}))]$, which means that $\Gamma(\overline{x}) \models_{\mathcal{A}} \phi(\overline{x})$. ∎

In the lemma below, $\Omega[\mathcal{U}]$, $\mathcal{A}_\Sigma$ and $\Sigma[\mathcal{U}_\mathbb{C}]$ are seen as sets.

Lemma 11.3. *In the context provided above, if $\mathcal{U}$ is a theory on $\hat{\Omega}X$, then*

$$\mathcal{A} \in \Omega[\mathcal{U}] \;\; \textit{iff} \;\; \mathcal{A}_\Sigma \subseteq \Sigma[\mathcal{U}_\mathbb{C}] \;\; \textit{iff} \;\; \mathcal{A}_\Sigma \bigcap \Sigma[\mathcal{U}_\mathbb{C}] \neq \emptyset.$$

Proof. 1. $\mathcal{A} \in \Omega[\mathcal{U}]$ implies $\mathcal{A}_\Sigma \subseteq \Sigma[\mathcal{U}_\mathbb{C}]$:

Let $\Delta \vdash \psi \in \mathcal{U}_\mathbb{C}$. From Lemma 11.1(1), we get that $\iota^{-1}(\Delta) \vdash \iota^{-1}(\psi) \in \mathcal{U}$. From the hypothesis this means that $\iota^{-1}(\Delta) \models_{\mathcal{A}} \iota^{-1}(\psi)$. Applying Lemma 11.2 we get $\Delta \models_{\mathcal{A}_\Sigma} \psi$, that implies $\mathcal{A}_\Sigma \subseteq \Sigma[\mathcal{U}_\mathbb{C}]$.

2. $\mathcal{A}_\Sigma \subseteq \Sigma[\mathcal{U}_\mathbb{C}]$ implies $\mathcal{A}_\Sigma \bigcap \Sigma[\mathcal{U}_\mathbb{C}] \neq \emptyset$:

The hypothesis gives us $\mathcal{A}_\Sigma \bigcap \Sigma[\mathcal{U}_\mathbb{C}] = \mathcal{A}_\Sigma$ and obviously $\mathcal{A}_\Sigma \neq \emptyset$.

3. $\mathcal{A}_\Sigma \bigcap \Sigma[\mathcal{U}_\mathbb{C}] \neq \emptyset$ implies $\mathcal{A} \in \Omega[\mathcal{U}]$:

The hypothesis ensures that there exists $\mathcal{B} \in \mathcal{A}_\Sigma$ so that $\mathcal{B}$ is a model of $\mathcal{U}_\mathbb{C}$. Since $\mathcal{U} \subseteq \mathcal{U}_\mathbb{C}$, this means that $\mathcal{B}$ is a model of $\mathcal{U}$. Hence, for any assignment $\beta \in \Sigma[X|\mathcal{A}]$, $\mathcal{B}$ under the assignment β satisfies all the judgements in $\mathcal{U}$. Recall that all these judgements in $\mathcal{U}$ are, in fact, judgements on $\hat{\Omega}X$, meaning they involve no constants from $\mathbb{C}$. Since the support of $\mathcal{B}$ coincides with the support of $\mathcal{A}$, the assignments in $\mathcal{B}$ coincide with the assignments in $\mathcal{A}$, as they are uniquely defined by maps $X \longrightarrow \mathcal{A}$. Hence, for any assignment $\alpha \in \Omega[X|\mathcal{A}]$ we have that $\mathcal{A}$ under the assignment α satisfies all the judgements in $\mathcal{U}$, meaning that $\mathcal{A} \in \Omega[\mathcal{U}]$. ∎

The previous results allow us to prove the following lemma, which is the key to the completeness proof.

Lemma 11.4. *In the context provided above, for any theory $\mathcal{U}$ and any judgement $\Gamma \vdash \phi$ in $\hat{\Omega}X$,*

$$\Gamma \models_{\Omega[\mathcal{U}]} \phi \;\; \textit{iff} \;\; \iota(\Gamma) \models_{\Sigma[\mathcal{U}_\mathbb{C}]} \iota(\phi).$$

Proof. Assume, without losing generality, that the judgement $\Gamma \vdash \phi$ is of the form $\Gamma(\overline{x}) \vdash \phi(\overline{x})$ where $\overline{x}$ is the list of all the variables involved in this judgement.

$(\Longrightarrow)$: Let $\mathcal{B} \in \Sigma[\mathcal{U}_\mathbb{C}]$. $\mathcal{B}$ must be the extension of some algebra $\mathcal{A} \in \mathbf{Q}[\Omega]$, i.e., $\mathcal{B} \in \mathcal{A}_\Sigma$. Hence, $\mathcal{B} \in \Sigma[\mathcal{U}_\mathbb{C}] \bigcap \mathcal{A}_\Sigma$. Applying Lemma 11.3, we get that $\mathcal{A} \in \Omega[\mathcal{U}]$. The hypothesis gives us now $\Gamma(\overline{x}) \models_{\mathcal{A}} \phi(\overline{x})$. Next, Lemma 11.2 gives us that $\iota(\Gamma(\overline{x})) \models_{\mathcal{A}_\Sigma} \iota(\phi(\overline{x}))$. Because $\mathcal{B} \in \mathcal{A}_\Sigma$, this means that $\iota(\Gamma(\overline{x})) \models_{\mathcal{B}} \iota(\phi(\overline{x}))$. And since $\mathcal{B}$ was chosen arbitrary from $\Sigma[\mathcal{U}_\mathbb{C}]$, we get $\iota(\Gamma(\overline{x})) \models_{\Sigma[\mathcal{U}_\mathbb{C}]} \iota(\phi(\overline{x}))$.

$(\Longleftarrow)$: Let $\mathcal{A} \in \Omega[\mathcal{U}]$. from Lemma 11.3 we get that $\mathcal{A}_\Sigma \subseteq \Sigma[\mathcal{U}_\mathbb{C}]$. Then, the hypothesis gives us $\iota(\Gamma(\overline{x})) \models_{\mathcal{A}_\Sigma} \iota(\phi(\overline{x}))$. Next, Lemma 11.2 guarantees that $\Gamma(\overline{x}) \models_{\mathcal{A}} \phi(\overline{x})$. Since this happens for an arbitrary $\mathcal{A} \in \Omega[\mathcal{U}]$, ge get that $\Gamma(\overline{x}) \models_{\Omega[\mathcal{U}]} \phi(\overline{x})$. ∎

Now we can state and prove the most general completeness theorem for quantitative equational logic against the the class of quantitative algebras.

Theorem 11.5 (Completeness). *Let Ω be an algebraic signature, X a set of variables and $\mathcal{U}$ a quantitative equational/Horn theory on $\hat{\Omega}X$. Then,*

$$\Gamma \models_{\Omega[\mathcal{U}]} \phi \;\; \textit{implies} \;\; \Gamma \vdash \phi \in \mathcal{U}.$$

Proof. We prove the theorem for the most general case of quantitative Horn theories, the case of equational theories is similar, only restricted to equations.

For the beginning, we prove the completeness for the case when the judgement $\Gamma \vdash \phi$ involves no variables from X, i.e., it is a judgement in $\hat{\Omega}\emptyset$.

In order to simplify the notation, for any set $S \subseteq \mathcal{E}(\hat{\Omega}X)$, we write

$$S^{\vdash} = \{ \vdash \psi \mid \psi \in S\}.$$

Let $\mathcal{U}^+$ be the quantitative Horn theory generated by $\mathcal{U} \cup \{ \vdash \psi \mid \psi \in \Gamma\}$, i.e., $\mathcal{U}^+ = \overline{\mathcal{U} \cup \Gamma^{\vdash}}$. Obviously, $\mathcal{U}^+$ is a theory over $\hat{\Omega}X$. From Theorem 7.3, we know that $\Omega^{\mathcal{U}^+}X$ is a model for $\mathcal{U}^+$, hence both for $\mathcal{U}$ and for $\Gamma^{\vdash}$.

Because $\Omega^{\mathcal{U}^+}X \in \Omega[\mathcal{U}^+] \subseteq \Omega[\mathcal{U}]$, $\Gamma \models_{\Omega^{\mathcal{U}^+}X} \phi$. And because $\Omega^{\mathcal{U}^+}X$ is a model of $\Gamma^{\vdash}$, the soundness of (CS) guarantees that $\Omega^{\mathcal{U}^+}X$ is also a model for $\vdash \phi$. Assuming ϕ is the quantitative equality $s =_e t$, this means that

$$\inf\{\varepsilon \mid \ \vdash s =_\varepsilon t \in \mathcal{U}^+\} \leq e, \ \ \text{i.e.,} \ \ |s,t|_{\mathcal{U}^+} \leq e.$$

Suppose now that $\Gamma \vdash s =_e t \notin \mathcal{U}$.
If $\vdash s =_e t \in \mathcal{U}$, applying (CS) we get that $\Gamma \vdash s =_e t \in \mathcal{U}$ - contradiction.
Also, $\vdash s =_e t \notin \overline{\Gamma^{\vdash}}$. Because otherwise, $\vdash s =_e t$ is derived from the hypothesis in Γ and the use of some of the closure conditions in Definition 3.3. However, since all the quantitative equalities in Γ contain no variables, none of the elements in $\overline{\Gamma^{\vdash}}$ are derived from $\Gamma^{\vdash}$ applying (Subst). Hence, $\Gamma \vdash s =_e t$ must be an element of any quantitative equational theory, and in particular, $\Gamma \vdash s =_e t \in \mathcal{U}$; which is a contradiction. Consequently, $\vdash s =_e t \notin \mathcal{U} \cup \overline{\Gamma^{\vdash}}$.

If $\vdash s =_e t \in \mathcal{U}^+$, then there exists $\Gamma_0 \subseteq \Gamma$ and $\Delta \subseteq \mathcal{E}(\hat{\Omega}X)$ with $\Delta^{\vdash} \in \mathcal{U}$, such that $\Gamma_0 \cup \Delta \vdash s =_e t \in \mathcal{U}$. Then, using (Assumpt), $\Gamma \cup \Delta \vdash s =_e t \in \mathcal{U}$. Because $\vdash \rho \in \mathcal{U}$ for all $\rho \in \Delta$, (Assumpt) gives us $\Gamma \vdash \rho \in \mathcal{U}$ for all $\rho \in \Delta$. Hence, we get further $\Gamma \vdash \rho \in \mathcal{U}$ for all $\rho \in \Gamma \cup \Delta$. Since $\Gamma \cup \Delta \vdash s =_e t \in \mathcal{U}$, by (CS) we get $\Gamma \vdash s =_e t \in \mathcal{U}$, which is a contradiction. Hence, $\vdash s =_e t \notin \mathcal{U}^+$.

Let $i = \inf\{\varepsilon \mid \ \vdash s =_\varepsilon t \in \mathcal{U}^+\} = |s,t|_{\mathcal{U}^+}$. Then, from the definition of $|\ |_{\mathcal{U}^+}$, $\vdash s =_i t \in \mathcal{U}^+$. Since $\vdash s =_e t \notin \mathcal{U}^+$, (Max) and (Cont) guarantee that $i > e$. Hence, $|s,t|_{\mathcal{U}^+} > e$. But this contradicts $|s,t|_{\mathcal{U}^+} \leq e$ proven above. This concludes the proof for the case where $\Gamma \vdash \phi$ contains no variables.

Next, we prove the completeness theorem for the general case of an arbitrary judgement $\Gamma \vdash \phi$. As before in this section, we consider a set $\mathbb{C}$ of fresh constants in one-to-one correspondence with the set X of variables and let $\iota : X \longrightarrow \mathbb{C}$ a bijection. Also, as before, let Σ be the signature we get by adding all the constants in $\mathbb{C}$ to Ω. In what follows, we use the notation defined above.

By Lemma 11.4, we know that $\Gamma \models_{\Omega[\mathcal{U}]} \phi$ is equivalent to $\iota(\Gamma) \models_{\Sigma[\mathcal{U}_\mathbb{C}]} \iota(\phi)$. Observe now that $\iota(\Gamma) \vdash \iota(\phi)$ contains no variables, so we can apply the completeness result for judgements without variables proven at the beginning of this proof. Thus, from $\iota(\Gamma) \models_{\Sigma[\mathcal{U}_\mathbb{C}]} \iota(\phi)$ we get $\iota(\Gamma) \vdash \iota(\phi) \in \mathcal{U}_\mathbb{C}$. Finally, we apply Lemma 11.1, and from $\iota(\Gamma) \vdash \iota(\phi) \in \mathcal{U}_\mathbb{C}$, we get $\Gamma \vdash \phi \in \mathcal{U}$. ■

11.2 Completeness for rational quantitative equational logic

By rational quantitative equational logic, we mean the fragment of quantitative equational logic involving only quantitative equalities $s =_\varepsilon t$ with $\varepsilon \in \mathbf{Q}_+$. For this, we prove a completeness result stating that one can only use quantitative equalities with rational indices and still and characterize exactly the same models.

Let $\mathcal{E}_\mathbb{Q}(\hat{\Omega}X)$ be the class of *rational quantitative equalities* on $\hat{\Omega}X$, which are the quantitative equalities involving only positive rational indexes, i.e., constructions of the form

$$s =_\varepsilon t \ \text{ with } \ \varepsilon \in \mathbf{Q}_+.$$

And let $\mathcal{J}_\mathbb{Q}(\hat{\Omega}X)$ be the class of *rational quantitative Horn judgements* on $\hat{\Omega}X$, which are the Horn judgements involving only rational quantitative equalities, i.e., constructions of the form

$$\{s_i =_{\varepsilon_i} t_i \mid i \in I\} \vdash s =_\varepsilon t,$$

where $\varepsilon_i, \varepsilon \in \mathbf{Q}_+$ for all $i \in I$.

Let $\mathcal{U}$ be an arbitrary quantitative Horn/equational theory on $\hat{\Omega}X$ and let

$$\mathcal{U}_{\mathbb{Q}} = \mathcal{U} \cap \mathcal{J}_{\mathbb{Q}}(\hat{\Omega}X).$$

In what follows we prove that the rational judgements are "dense" in the set of quantitative judgements, in the sense that whatever one can prove with real judgements, one can also prove with rational judgements only.

Theorem 11.6 (Density of rational judgements). *Let $\mathcal{U}$ be an arbitrary quantitative Horn/equational theory on $\hat{\Omega}X$. Then, $\mathcal{U}_{\mathbb{Q}}$ axiomatizes $\mathcal{U}$, i.e.,*

$$\overline{\mathcal{U}_{\mathbb{Q}}} = \mathcal{U}.$$

Proof. Let $rat : 2^{\mathcal{E}(\hat{\Omega}X)} \longrightarrow 2^{\mathcal{E}_{\mathbb{Q}}(\hat{\Omega}X)}$ defined as follows:

$$\begin{aligned} rat(\emptyset) &= \emptyset\,, \\ rat(\{s =_\varepsilon t\}) &= \begin{cases} \{s =_\varepsilon t\} & \text{if } \varepsilon \in \mathbf{Q}_+\,, \\ \{s =_e t \mid e \in \mathbf{Q}_+, \varepsilon < e\} & \text{if } \varepsilon \in \mathbf{R}_+ \setminus \mathbf{Q}_+\,, \end{cases} \\ rat(\Gamma) &= \bigcup_{\phi \in \Gamma} rat(\{\phi\})\,. \end{aligned}$$

We prove now that

$$\Gamma \vdash \phi \in \mathcal{U} \;\text{ iff }\; rat(\Gamma) \vdash \phi \in \mathcal{U}.$$

To do this, it is sufficient to observe that for any $s =_\varepsilon t \in \Gamma$, (Cont), (Max) and (CS) guarantees that

$$rat(\{s =_\varepsilon t\}) \vdash s =_\varepsilon t \in \mathcal{U},$$

and also, (Max) guarantees that for any $s =_e t \in rat(\{s =_\varepsilon t\})$,

$$s =_\varepsilon t \vdash s =_e t \in \mathcal{U}.$$

Hence, for any $\phi \in \Gamma$, $rat(\Gamma) \vdash \phi \in \mathcal{U}$, and for any $\psi \in rat(\Gamma)$, $\Gamma \vdash \psi \in \mathcal{U}$. Using these and applying (CS), we get $\Gamma \vdash \phi \in \mathcal{U}$ iff $rat(\Gamma) \vdash \phi \in \mathcal{U}$.

Now we prove that

$$\Gamma \vdash \phi \in \mathcal{U} \;\text{ iff }\; [\text{for all } \psi \in rat(\phi),\; \Gamma \vdash \psi \in \mathcal{U}].$$

(Max) guarantees that $\phi \vdash \psi \in \mathcal{U}$, and involving (CS) we get the left-to-right implication. For right-to-left, we use (Max) and (CS).

Combining the previous two results we get that

$$\Gamma \vdash \phi \in \mathcal{U} \;\text{ iff }\; [\text{for all } \psi \in rat(\phi),\; rat(\Gamma) \vdash \psi \in \mathcal{U}].$$

The previous equivalence concludes our proof, since it argues that by taking $\overline{\mathcal{U}_{\mathbb{Q}}}$ as an axiomatization we can prove all the judgements in $\mathcal{U}$. ∎

A corollary of this result is the following theorem.

Corollary 11.7. *Let $\mathcal{U}$ be a quantitative Horn/equational theory on $\hat{\Omega}X$. Then,*

$$\mathcal{A} \in \Omega[\mathcal{U}] \;\; \textit{iff} \;\; \mathcal{A} \in \Omega[\mathcal{U}_{\mathbb{Q}}].$$

Combining the above, we can prove the completeness of rational equational logic.

Theorem 11.8 (Rational completeness). *Let Ω be an algebraic signature, X a set of variables and $\mathcal{U}$ a quantitative equational/Horn theory on $\hat{\Omega}X$. Let $\Gamma \vdash \phi \in \mathcal{J}_{\mathbb{Q}}(\hat{\Omega}X)$ be a rational judgement on $\hat{\Omega}X$. Then,*

$$\Gamma \models_{\Omega[\mathcal{U}_{\mathbb{Q}}]} \phi \;\; \textit{implies} \;\; \Gamma \vdash \phi \in \mathcal{U}_{\mathbb{Q}}.$$

Proof. From Corollary 11.7, $\Gamma \models_{\Omega[\mathcal{U}_{\mathbb{Q}}]} \phi$ implies $\Gamma \models_{\Omega[\mathcal{U}]} \phi$. Next, Theorem 11.5 guarantees that $\Gamma \models_{\Omega[\mathcal{U}]} \phi$ implies $\Gamma \vdash \phi \in \mathcal{U}$, and applying Theorem 11.6, we get $\Gamma \vdash \phi \in \mathcal{U}_{\mathbb{Q}}$. ∎

Note, in all the development of this subsection, that all that matters is that the set of positive rationals is dense in the set of positive reals. In fact, any dense subset of the positive reals would have sufficed to establish these results. So it is not a special property of the rationals that is important, just the fact that they are dense in the reals.

11.3 Completeness for models on complete metric spaces

Now we prove a completeness result for quantitative equational logic against the class of quantitative algebras on complete metric spaces.

Because we have seen that given a theory $\mathcal{U}$ over $\hat{\Omega}X$, it is not guaranteed that $(\!|\Omega^{\mathcal{U}}\emptyset|\!)$ or $(\!|\Omega^{\mathcal{U}}X|\!)$ are models of $\mathcal{U}$, we cannot reproduce the proof in subsection 11.1 to get a similar result for the category of complete models of $\mathcal{U}$. However, Lemma 9.7 allows us to prove such a completeness for the model-completable theories.

Theorem 11.9 (Completeness for complete metric spaces). *Let Ω be an algebraic signature, X a set of variables and $\mathcal{U}$ a model-completable theory on $\hat{\Omega}X$. Then,*

$$\Gamma \models_{\mathrm{C}\Omega[\mathcal{U}]} \phi \;\; \textit{implies that} \;\; \Gamma \vdash \phi \in \mathcal{U}.$$

Proof. Consider an arbitrary $\mathcal{A} \in \Omega[\mathcal{U}]$. This means that $\mathcal{A}$ is a model of $\mathcal{U}$. Since $\mathcal{U}$ is model-completable, $(\!|\mathcal{A}|\!)$ is also a model of $\mathcal{U}$. Hence, $(\!|\mathcal{A}|\!) \in \mathrm{C}\Omega[\mathcal{U}]$. Because $\Gamma \models_{\mathrm{C}\Omega[\mathcal{U}]} \phi$, we get now that $\Gamma \models_{(\!|\mathcal{A}|\!)} \phi$. Applying Theorem 9.4 case 1., we get $\Gamma \models_{\mathcal{A}} \phi$. Since this happens for an arbitrary $\mathcal{A} \in \Omega[\mathcal{U}]$, we get that $\Gamma \models_{\Omega[\mathcal{U}]} \phi$. Now we apply Theorem 11.5 and get $\Gamma \vdash \phi \in \mathcal{U}$. ∎

Recall that, in the light of Lemma 9.7, the continuous theories, or the theories that admit an axiomatization containing only closed judgements, are all examples of model-completable theories. The same lemma also guarantees that any union of such theories (so, admitting an axiomatization that combines these three types of axioms) is still model-completable. Hence, all these types of theories are instances of Theorem 11.9, and so we have completeness results against the category of models on complete metric spaces.

12 Related work and subsequent developments

Since we first announced these results at the ACM-IEEE Symposium on Logic in Computer Science in 2016, there has been substantial new work in the area. We have developed the theory ourselves [7–9, 11, 57–59] in a number of directions. There have also been very substantial developments by Matteo Mio, Ralph Sarkis, and Valeria Vignudelli [62, 63] and by Adamek [2] and by Milius and Urbat [60] and Jurka et al. [41]. There has been so much work that we are forced to focus on those developments that are closest to our work.

A closely related prior work is by van Breugel et al. [80], which was an important precursor to our work. This paper really shows why the Hausdorff and Kantorovich metrics are canonical. The second one shows the finitary natures of these monads. In the paper by van Breugel et al. [80], it was shown that the Kantorovich functor is left adjoint to a forgetful functor from a suitable algebraic category, mean-value algebras over one-bounded complete metric spaces, and complete metric spaces. Similarly, they show that a suitable Hausdorff functor can be treated similarly. Their results are intended to exhibit the power of an approach to solving recursive equations using the theory of accessible categories. The work in [80] uses ordinary equations and uses what are called midpoint algebras that are quite special; the theory of midpoint algebras is equivalent to the theory of barycentric algebras restricted to dyadic rationals. The midpoint-algebra approach works well for the $p = 1$ case of the IB_p algebras, but it does not extend to the other values of p where the flexibility of using arbitrary rationals allows the extension easily.

The difference between the mean-value axiomatization and the barycentric axiomatization may seem unimportant, but we feel that barycentric algebras are more fundamental. They allow *all* binary choices to be directly available; they are, of course, all definable from the mean-value *if you allow infinite terms, but certainly not if you want everything to be finitary.* The barycentric algebras axioms are those for abstract convex spaces and arise widely in mathematics; see, for example, the historical remarks in [45] and the original paper by Stone [78]. Barycentric algebras work very well in other settings too. For example, if one takes the free pointed barycentric algebras in other categories like sets or cpos, one gets the structures one expects: finite probability distributions for the case of sets and the valuation powerdomain for the case of continuous dcpos.

Another important precursor is a paper by Adamek et al. [3], which studied the finitary versions of the same functors and gave equational presentations. More precisely, it is shown that every finitary endofunctor of a locally finitely presentable category has an equational presentation using operations with finitary signatures in the sense of Kelly and Power [46]. This general result was used to prove that the Hausdorff functor on complete metric spaces is finitary (i.e., $\aleph_0$-accessible), thus, sharpening the classification given in [80], where the same functor was only deemed accessible *i.e.*, λ-accessible for some unspecified regular cardinal λ.

Since the appearance of [56], the theory of quantitative algebras has advanced significantly. A substantial body of research has focused on investigating the equational classes of quantitative algebras. The first work in this direction is [57], which proved a Birkhoff-type variety theorem for equational and Horn classes of quantitative algebras. In [60], Milius and Urbat clarified the Birkhoff-type theorems stated in [57] from a more general categorical perspective. Later Adámek [2] showed that ω_1-quantitative equational theories correspond to $\aleph_1$-ary enriched monads on metric spaces preserving surjections. Then, Rosický [71] provided a Birkhoff-type theorem for the classes of quantitative algebras of discrete theories.

Other work focused on extensions to the theory proposed in [56]. In [51], Dal Lago et al. investigated higher-order extensions; in [63], Mio et al. extended the framework by allowing non-expansive operations as well as distances that are not metrics. This is a very important extension and greatly broadens the scope of the theory. It is not a mere tweak of the present work; it is a fundamental advance. They were even able to include distance functions that do not satisfy $d(x, x) = 0$ and thus cover an important example that came up in machine learning [16]. Related to [63], there is a recent paper [64] which extends the results of [63] significantly. Dagnino and Pasquali [18] expressed quantitative equational theories in the more general categorical language of Lawvere's doctrines.

Another research direction explored the (monadic) computational effects representable by quantitative theories. Inspired by the seminal work of Hyland et al. [36] on combining of algebraic effects via sum and tensor, we developed an analogous theory for the combination of quantitative effects in [8, 9, 11]. This allowed us to develop some examples of interest to computer scientists by combining existing theories. One of the important developments in equational reasoning was showing how the monads used in defining effectful extensions of functional programming languages could be described by equations and operations [68, 69]. This has become a vast subarea explored under the name of *effects*. In [7], we developed some examples of quantitative effects. Clearly, much more could be done, but it was pleasing to see how smoothly this extension worked. An example that cannot be expressed via sum and tensor was investigated in [62], namely the combination of nondeterminism and probabilistic choice.

Another important extension is the work in [58], where we developed an extension to fixed points. In earlier work [76], the general axiomatics of fixed-point operators was developed based on the notion of order structures and Kleene-style fixed-point theorems. In [58], we did the analogous axiomatization of fixed-point operators on metric spaces based on Banach-style fixed-point theorems. This involved introducing some new machinery to capture contractivity in multi-argument functions. In essence, we introduce something like a type system, which we called Banach patterns. This extension allowed us to express quantities like optimal value functions in Markov decision processes and to prove results [22] relating optimal value functions and bisimulation metrics; these results were originally presented in machine learning conferences using a bare-bones approach exploiting Kantorovich duality.

A very significant recent advance is the work of Matteo Mio [61] where he shows that one can often eliminate the infinitary rule; he calls such theories compact and shows that our theory of interpolative barycentric algebras is compact.

With a notion of quantitative equation in place, it is natural to think about quantitative rewriting. This is especially important as a prelude to include quantitative equational logic in automated reasoning tools. A recent paper by Gavazzo and Di Florio [26] initiated such a theory. Many topics in rewriting theory, for example, narrowing, unification, generalization, and matching, are ripe for quantitative extensions, and indeed activity in these areas is underway. A recent paper[55] has initiated a quantitative analogue of the theory of string diagrams, which is a graphical way to carry out equational reasoning; see also [75]. A good general reference for rewriting is [6] or [48].

Another important development is "Free complete Wasserstein algebras" [59] where we generalise from complete *separable* metric spaces to complete ones. The theory developed in this paper says we have free algebras over complete metric spaces, so the examples are best if they also work in that category. We already have this in the paper for the Hausdorff distance, though we have not considered the case where there are extended metrics.

Lastly, we mention the large amount of work appearing on categorical approaches to probability [17, 24, 25]. This work is clearly an important approach to quantitative reasoning, though a tight connection with the work of the present paper is not clear.

13 Conclusions and future work

The theory of quantitative algebras is a surprisingly rich source of interesting examples. We have laid out the basic theory in this paper and have developed two significant examples in detail: the Hausdorff metric and the W_p metrics, as well as sketching the case of the total variation metric. The surprise is how these metrics emerge from very simple axioms: the free algebras come equipped with these metrics. It is particularly surprising that the Kantorovich metric comes out this way because the axiom just expresses a simple interpolative property whereas the usual way it appears is in optimal transport theory which involves a duality between minimal cost of transport and the maximum value of a certain difference of integrals.

Given the richness of the theory of quantitative equational logic it is natural to consider richer logics. We have begun investigating propositional logics [10] with various extensions being actively pursued. Investigations into induction and recursion principles of higher-order quantitative logics has already been initiated in [12]; this is a topic which we are actively planning to pursue.

It is important to pursue applications of these ideas. One very promising direction that we would like to explore is the application to neurosymbolic AI, there are numerous papers on this topic, see, for example [21]. Here one combines logic with quantitative reasoning, precisely the topic of our research. Another potential application is the development of equational reasoning principles for privacy and security applications [20]; the idea of approximate equality of probability distributions appears as a central idea in the field of differential privacy.

Acknowledgments

Prakash Panangaden is supported by a grant from the Natural Sciences and Engineering Research Council of Canada. Radu Mardare is supported by the EPSRC grant EP/Y000455/1, *A correct-by-construction approach to approximate computation* and by ARIA TA1.1 project *Quantitative Predicate Logic as a Foundation for Verified ML*. We want to thank the anonymous referees, particularly one who pointed out an incorrect theorem and provided the correct version with proof.

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A Hausdorff metric and duality

Let (M, d) be an extended metric space. The *Hausdorff metric induced by d* on the set of all compact subsets of M is defined, for arbitrary compact sets[13] $A, B \subseteq M$ as

$$H_d(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right\},$$

where, $d(m, N) = \inf_{n \in N} d(m, n)$ denotes the distance from an element $m \in M$ to a set $N \subseteq M$.

As usual, we assume that $\sup \emptyset = 0$ and $\inf \emptyset = \infty$; so, for any compact nonempty set A,

$$H_d(\emptyset, A) = H_d(A, \emptyset) = \infty \text{ and } H_d(\emptyset, \emptyset) = 0.$$

Before proving that H_d is indeed a metric, we provide a dual characterization for H_d. For an arbitrary set $A \subseteq M$ and arbitrary $\varepsilon > 0$, let

$$A_\varepsilon = \{x \in M \mid \exists a \in A, d(x, a) \leq \varepsilon\}.$$

Lemma A.1. *If A and B are compact subsets of M, then*

$$H_d(A, B) = \inf\{\varepsilon \mid A \subseteq B_\varepsilon \text{ and } B \subseteq A_\varepsilon\}.$$

Proof. To start, note that $x \in A_\varepsilon$ means that $d(x, A) \leq \varepsilon$. Suppose that A and B are such that $A \subseteq B_\varepsilon$ and $B \subseteq A_\varepsilon$.

Since $A \subseteq B_\varepsilon$, for any $a \in A$, $d(a, B) \leq \varepsilon$. Hence, $\sup_{a \in A} d(a, B) \leq \varepsilon$. Similarly, from $B \subseteq A_\varepsilon$ we get $\sup_{b \in B} d(b, A) \leq \varepsilon$. From these it follows that $H_d(A, B) \leq \varepsilon$ and hence,

$$H_d(A, B) \leq \inf\{\varepsilon \mid A \subseteq B_\varepsilon \text{ and } A \subseteq B_\varepsilon\}.$$

Suppose that $H_d(A, B) < \inf\{\varepsilon \mid A \subseteq B_\varepsilon \text{ and } A \subseteq B_\varepsilon\}$. Let δ be such that $H_d(A, B) < \delta < \inf\{\varepsilon \mid A \subseteq B_\varepsilon \text{ and } A \subseteq B_\varepsilon\}$. Then, either $A \not\subseteq B_\delta$ or $B \not\subseteq A_\delta$.

If $A \not\subseteq B_\delta$, then there exists $a \in A$ such that $a \notin B_\delta$. Hence, for any $b \in B$, $d(a, b) > \delta$, implying further that there exists $a \in A$ such that $d(a, B) \geq \delta$. But then,

$$\sup_{a \in A} d(a, B) \geq \delta,$$

which guarantees that

$$H_d(A, B) \geq \delta$$

[13] Our definition is not quite standard and generalizes for the case of extended metric spaces. The standard definition is restricted to non-empty sets. For our purposes, the non-emptyness assumption on the sets is not needed, as our distances can be infinite.

contradicting the assumption that $H_d(A,B) < \delta$.

Hence, $H_d(A,B) = \inf\{\varepsilon \mid A \subseteq B_\varepsilon \text{ and } A \subseteq B_\varepsilon\}$. ■

Lemma A.2. *Let A, B be nonempty compact sets in the open-ball topology and $a \in A$ an arbitrary point. Then, for any $\varepsilon > 0$, there exists $b \in B$ such that*

$$d(a,b) \leq H_d(A,B) + \varepsilon.$$

Proof. We can assume, without losing generality, that

$$H_d(A,B) = \sup_{x \in A} d(x,B).$$

Hence,

$$H_d(A,B) \geq d(a,B) = \inf_{y \in B} d(a,y).$$

This implies $H_d(A,B) + \varepsilon > \inf_{y \in B} d(a,y)$. Then, there exists $b \in B$ such that $d(a,b) \leq H_d(A,B) + \varepsilon$. ■

We show now that H_d is indeed an extended metric, over the set of compact sets of M.

Theorem A.3. *H_d is an extended metric on the set of compact subsets of M.*

Proof. 1. Assume that $H_d(A,B) = 0$. We prove that $A = B$.
If at least one of the two sets is empty, the other one must be empty too, since otherwise $H_d(A,B) = \infty$.
Assume that $A \neq \emptyset \neq B$. Because $H_d(A,B) = 0$, for any $a \in A$, $d(a,B) = 0$, i.e., $\inf_{b \in B} d(a,b) = 0$. Hence, there exists a sequence (b_i) of elements in B such that

$$\lim_{i \longrightarrow \infty} d(a,b_i) = 0.$$

But then, (b_i) converges to a and since B is compact, $a \in B$. Hence, $A \subseteq B$.
Similarly one can prove that $B \subseteq A$, hence $A = B$.

2. The fact that $H_d(A,B) = H_d(B,A)$ follows from the symmetry of max.

3. We prove now that for arbitrary compact sets A, B, C,

$$H_d(A,C) \leq H_d(A,B) + H_d(B,C).$$

Observe that if at least one of them is empty, the inequality is trivially true since $r + \infty = \infty + \infty = \infty > r$ for any $r \in \mathbf{R}_+$.

Assume they are not empty. Let $a \in A$. For any $\varepsilon > 0$, we can apply Lemma A.2 and conclude that there exists $b \in B$ such that

$$d(a,b) \leq H_d(A,B) + \varepsilon.$$

We apply again Lemma A.2 for $b \in B$ and C and we obtain that there exists $c \in C$ such that

$$d(b,c) \leq H_d(B,C) + \varepsilon.$$

Consequently, after applying the triangle inequality for d, we get

$$d(a,c) \leq d(a,b) + d(b,c) \leq H_d(A,B) + H_d(B,C) + 2\varepsilon.$$

It follows that

$$d(a,C) \leq H_d(A,B) + H_d(B,C) + 2\varepsilon$$

and further that

$$\sup_{a \in A} d(a,C) \leq H_d(A,B) + H_d(B,C) + 2\varepsilon.$$

Symmetrically (using also the symmetry of H_d),

$$\sup_{c \in C} d(c, A) \leq H_d(A, B) + H_d(B, C) + 2\varepsilon.$$

Hence for any $\varepsilon > 0$,

$$H_d(A, C) \leq H_d(A, B) + H_d(B, C) + 2\varepsilon$$

which further entails

$$H_d(A, C) \leq H_d(A, B) + H_d(B, C).$$

∎

Next we prove a dual characterization for H_d in terms of what we call *relational coupling*. For M a set and $A, B \subseteq M$, a relational coupling for the pair (A, B) is a relation $R \subseteq M \times M$ such that

$$\pi_1(R) = A \qquad \text{and} \qquad \pi_2(R) = B\,,$$

where π_1 and π_2 are the canonical projections of R. We denote by $C(A, B)$ the set of couplings for (A, B).

Theorem A.4 (Hausdorff Duality). *Let (M, d) be a metric space and $A, B \subseteq M$ compact sets in the open-ball topology, then*

$$H_d(A, B) = \inf \left\{ \sup_{(m,n) \in R} d(m, n) \mid R \in C(A, B) \right\},$$

Proof. If either A or B is empty while the other one is not empty, the equality is trivially true since $H_d(A, B) = \infty$ and $C(A, B) = \emptyset$ which means that

$$\left\{ \sup_{(m,n) \in R} d(m, n) \mid R \in C(A, B) \right\} = \emptyset.$$

On the other hand, if $A = B = \emptyset$, then $H_d(A, B) = 0$ and moreover

$$\left\{ \sup_{(m,n) \in R} d(m, n) \mid R \in C(A, B) \right\} = \{0\}.$$

Assume now that $A \neq \emptyset \neq B$.

($\leq$): It suffices to show that for any relational coupling $R \in C(A, B)$,

$$H_d(A, B) \leq \sup_{(m,n) \in R} d(m, n).$$

Let $R \in C(A, B)$. Note that, for any $a \in A$ and $b \in B$, since $\pi_1(R) = A$ and $\pi_2(R) = B$, the following sets are nonempty:

$$R_a = \{n \in M \mid (a, n) \in R\} \subseteq \pi_2(R)\,, \qquad R^b = \{m \in M \mid (m, b) \in R\} \subseteq \pi_1(R)\,.$$

Then, the following holds

$$\begin{aligned}
\sup_{(m,n) \in R} d(m, n) &\geq \max \left\{ \sup_{a \in A} d(a, R_a), \sup_{b \in B} d(b, R^b) \right\} && (\text{def. } R_a, R^b) \\
&\geq \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right\} && (R \in C(A, B)) \\
&= H_d(A, B)\,. && (\text{def. } H_d)
\end{aligned}$$

($\geq$): Recall the equivalent characterization of H_d, given as

$$H_d(A, B) = \inf\{\varepsilon \geq 0 \mid A \subseteq B_\varepsilon \text{ and } B \subseteq A_\varepsilon\}\,.$$

If $\{\varepsilon \geq 0 \mid A \subseteq B_\varepsilon \text{ and } B \subseteq A_\varepsilon\} = \emptyset$, since we have assumed that $A \neq \emptyset \neq B$, then $H_d(A,B) = \infty$ and the inequality is trivially satisfied.

Suppose now that $\{\varepsilon \geq 0 \mid A \subseteq B_\varepsilon \text{ and } B \subseteq A_\varepsilon\} \neq \emptyset$. To prove the inequality it is sufficient to show that for any $\varepsilon \geq 0$ such that $A \subseteq B_\varepsilon$ and $B \subseteq A_\varepsilon$, the following inequality holds:

$$\varepsilon \geq \inf \left\{ \sup_{(m,n)\in R} d(m,n) \mid R \in C(A,B) \right\} .$$

Let $\varepsilon \geq 0$ be such that $A \subseteq B_\varepsilon$ and $B \subseteq A_\varepsilon$. We define $R_\varepsilon \subseteq M \times M$ as $R_\varepsilon = \{(a,b) \in A \times B \mid d(a,b) \leq \varepsilon\}$ and we show that $\mathcal{R}_\varepsilon \in C(A,B)$.

$$\begin{aligned}
\pi_1(R_\varepsilon) &= \pi_1(\{(a,b) \in A \times B \mid d(a,b) \leq \varepsilon\}) && \text{(def. } R_\varepsilon\text{)}\\
&= \{a \in A \mid \exists b \in B, d(a,b) \leq \varepsilon\} && \text{(def. } \pi_1\text{)}\\
&= B_\varepsilon \cap A && \text{(def. } B_\varepsilon\text{)}\\
&= A && (A \subseteq B_\varepsilon)
\end{aligned}$$

Similarly, $\pi_2(R_\varepsilon) = B$. Hence:

$$\begin{aligned}
\varepsilon &\geq \sup_{(m,n)\in R_\varepsilon} d(m,n) && \text{(by def. } R_\varepsilon\text{)}\\
&\geq \inf \left\{ \sup_{(m,n)\in R} d(m,n) \mid R \in C(A,B) \right\} . && (R_\varepsilon \in C(A,B))
\end{aligned}$$

The proof is complete. ∎

B Proofs for barycentric algebras

Lemma B.1 (Splitting Lemma). *Let $\mu, \nu \in \Delta[M]$ and $\omega \in C(\mu,\nu)$. If R is a measurable set such that $\omega(R) \notin \{0,1\}$, then*

$$\mu = \omega(R)\mu_1 + (1-\omega(R))\mu_2 \qquad \text{and} \qquad \nu = \omega(R)\nu_1 + (1-\omega(R))\nu_2\,,$$

for some $\mu_i, \nu_i \in \Delta[M]$, such that $supp(\mu_1) \subseteq \pi_1(R)$, $supp(\mu_2) \subseteq \pi_1(R^c)$, $supp(\nu_1) \subseteq \pi_2(R)$, and $supp(\nu_2) \subseteq \pi_2(R^c)$, where $R^c = (M \times M) \setminus R$.

Moreover, for any $\varepsilon > 0$, if $\int d^p \mathrm{d}\omega \leq W_d^p(\mu,\nu) + \varepsilon$ then

$$\omega(R) W_d^p(\mu_1,\nu_1)^p + (1-\omega(R)) W_d^p(\mu_2,\nu_2)^p \leq W_d^p(\mu,\nu)^p + \varepsilon.$$

Proof. They key step is to use the conditional probabilities given R and R^c to construct the splitting. Define $\mu_i, \nu_i \in \Delta[M]$, for an arbitrary Borel set E, as follows:

$$\mu_1(E) = \frac{\omega((E\times M)\cap R)}{\omega(R)}\,, \qquad \mu_2(E) = \frac{\omega((E\times M)\cap R^c)}{1-\omega(R)}\,,$$
$$\nu_1(E) = \frac{\omega((M\times E)\cap R)}{\omega(R)}\,, \qquad \nu_2(E) = \frac{\omega((M\times E)\cap R^c)}{1-\omega(R)}\,.$$

We show that $\mu = \omega(R)\mu_1 + (1-\omega(R))\mu_2$. Let E be any Borel set, then

$$\begin{aligned}
&\omega(R)\mu_1(E) + (1-\omega(R))\mu_2 = \\
&\quad = \omega((E\times M)\cap R) + \omega((E\times M)\cap R^c) && \text{(def. } \mu_1,\mu_2\text{)}\\
&\quad = \omega(E\times M) && \text{(additivity of } \omega\text{)}\\
&\quad = \mu(E)\,. && (\omega \in C(\mu,\nu))
\end{aligned}$$

Similarly, $\nu = \omega(R)\nu_1 + (1-\omega(R))\nu_2$.

Now we prove that $(\omega|R) \in C(\mu_1, \nu_1)$ and $(\omega|R^c) \in C(\mu_2, \nu_2)$, where $(\omega|R)$ $(\omega|R^c)$ are the conditional probability measures of ω given R and R^c, respectively. We show only one membership, the other is similar. Let E be a Borel measurable set. Then, by definitions of μ_1, ν_1 and conditional probability

$$(\omega|R)(E \times M) = \frac{\omega((E \times M) \cap R)}{\omega(R)} = \mu_1(E)\,, \qquad \text{(left marginal)}$$

$$(\omega|R)(M \times E) = \frac{\omega((M \times E) \cap R)}{\omega(R)} = \nu_1(E)\,. \qquad \text{(right marginal)}$$

The conditions on the supports follow immediately by the definitions of μ_i, ν_i, $(i = \{1, 2\})$.

For the last assertion in the lemma we proceed as follows. Fix $\varepsilon > 0$ and suppose that $\int d^p \, \mathrm{d}\omega \le W_d^p(\mu, \nu)^p + \varepsilon$. We compute as follows:

$$\begin{aligned}
&\omega(R) W_d^p(\mu_1, \nu_1)^p + (1 - \omega(R)) W_d^p(\mu_2, \nu_2)^p \\
&\le \omega(R) \int d^p \, \mathrm{d}(\omega|R) + (1 - \omega(R)) \int d^p \, \mathrm{d}(\omega|R^c) && \text{(since } \omega|R \text{ and } \omega|R^c \text{ are couplings)} \\
&= \int_R d^p \, \mathrm{d}\omega + \int_{R^c} d^p \, \mathrm{d}\omega && \text{(def. } \mu|R \text{ \& linearity of } \textstyle\int) \\
&= \int d^p \, \mathrm{d}\omega && \text{(additivity of } \textstyle\int) \\
&\le W_d^p(\mu, \nu)^p + \varepsilon\,. && \text{(hyp. on } \omega)
\end{aligned}$$

∎

In order to proceed we need a basic convexity property of the space of couplings.

Lemma B.2 (Convex Combination of Couplings). *Let $\mu_i, \nu_i \in \Delta[M, \Sigma]$ and $\omega_i \in C(\mu_i, \nu_i)$, for $i \in \{1, 2\}$. Then, for all $e \in [0, 1]$*

$$(e\omega_1 + (1 - e)\omega_2) \in C\Big((e\mu_1 + (1 - e)\mu_2), (e\nu_1 + (1 - e)\nu_2)\Big)\,.$$

Proof. We show only the left marginal, the other is similar. Let $E \in \Sigma$, then

$$\begin{aligned}
(e\omega_1 + (1 - e)\omega_2)(E \times M) &= e\omega_1(E \times M) + (1 - e)\omega_2(E \times M) && \text{(by def.)} \\
&= e\mu_1(E) + (1 - e)\mu_2(E) && \text{(by } \omega_i \in C(\mu_i, \nu_i)) \\
&= (e\mu_1 + (1 - e)\mu_2)(E)\,. && \text{(by def.)}
\end{aligned}$$

∎

The result above states that the set of all couplings between arbitrary measures in $\Delta[M, \Sigma]$ is a convex set (note that $\Delta[M, \Sigma]$ is a convex set too).

The following lemma is well known, see, for example, Chapter 8 of [13]. However, we have simplified the presentation of the proof.

Proposition B.3. *Let M be a Polish space and let $\{c_i\}_{i=1}^k$ be positive real numbers such that $\sum_{i=1}^k c_i = 1$. Let $\{m_i\}_{i=1}^k$ be points in M. Then measures of the form $\sum_{i=1}^k c_i \delta_{m_i}$ are p-weakly dense in $\Delta[M]$.*

Proof. Suppose that we are given a basic open neighbourhood of $\Delta[M]$

$$U = \left\{\mu : \left| \int_M f_i \, \mathrm{d}\nu - \int_M f_i \, \mathrm{d}\mu \right| < \varepsilon, i = 1, \ldots, k \right\},$$

where the f_i are bounded continuous functions, ν is a probability measure and $\varepsilon > 0$.

Fix an $\varepsilon > 0$. Now the functions f_i are measurable so there are simple functions g_i such that $\sup_x |f_i(x) - g_i(x)| < \varepsilon/2$ for each $i = 1, \ldots, k$. We partition M into disjoint Borel sets A_j, $j = 1, \ldots, l$ such that all the g_i are constant over each A_j.

Now choose a point m_j in each A_j and set $c_j = \nu(A_j)$. The measure $\mu := \sum_{j=1}^{l} c_j \delta_{m_j}$ is a convex combination of Dirac measures and has the property that for each A_j, $\mu(A_j) = \nu(A_j)$. Thus for each of the g_i

$$\int g_i \, \mathrm{d}\nu = \int g_i \, \mathrm{d}\mu.$$

Thus we have

$$\left| \int f_i \, \mathrm{d}\nu - \int f_i \, \mathrm{d}\mu \right| = \left| \int f_i \, \mathrm{d}\nu - \int g_i \, \mathrm{d}\nu + \int g_i \, \mathrm{d}\mu - \int f_i \, \mathrm{d}\mu \right| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

This proves that μ is in U. Since U is an arbitrary basic open it shows that measures of the form μ are dense in the weak topology. ∎

Received ??; revised 29 August 2025; accepted 11 May 2026