On the Optimal Efficiency of A^* with Dominance Pruning

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AAAI'2021

Motivation

- A^*: canonical choice for solving shortest path problems
- A^* is optimally efficient in node expansions (Dechter and Pearl, 1985)

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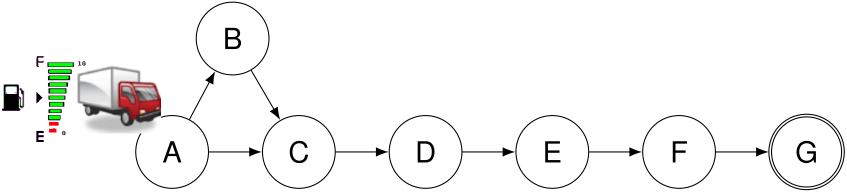
- A^* is optimally efficient in node expansions (Dechter and Pearl, 1985)
- Dominance pruning methods $\rightarrow$ new source of information!
- We use dominance pruning in A^*:
  - Is this a good choice?
  - Could we achieve more pruning with different expansion orders?
  - What tie-breaking strategies are good for dominance pruning?

Outline

Running Example



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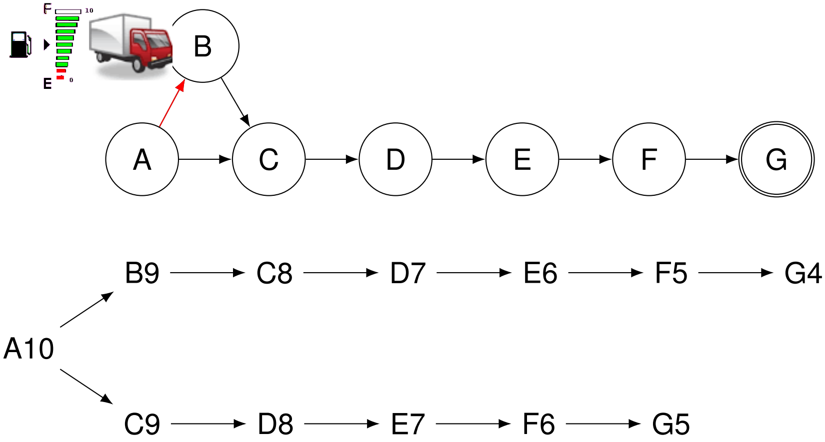

DXBB Algorithms

$UDXBB$: Unidirectional, Deterministic, Expansion-based, Black Box

Access to the state space Θ only via node expansions

Additionally the algorithm is given an admissible heuristic function h
$\rightarrow h(s)$ estimates the distance from s to the goal $h^*(s)$, $h(s) \leq h^*(s)$

h2

A10

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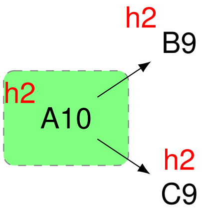

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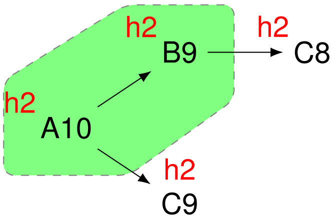

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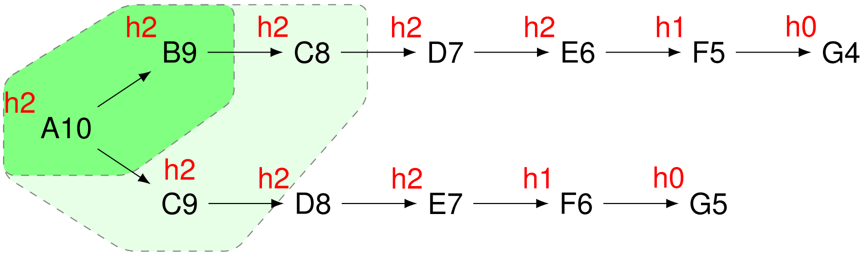

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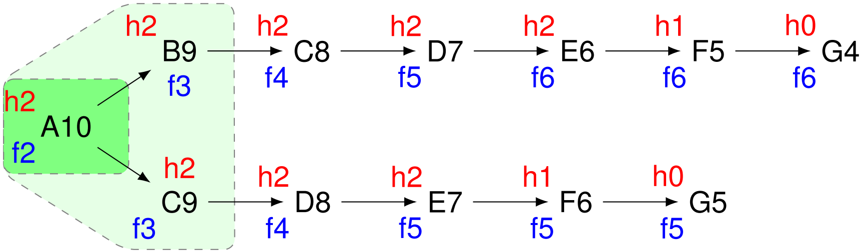
A*: Expand nodes based on f-value: $f(n_s) = g(n_s) + h(s)$

Family of algorithms: tie-breaking strategy may pick any node with minimum f-value



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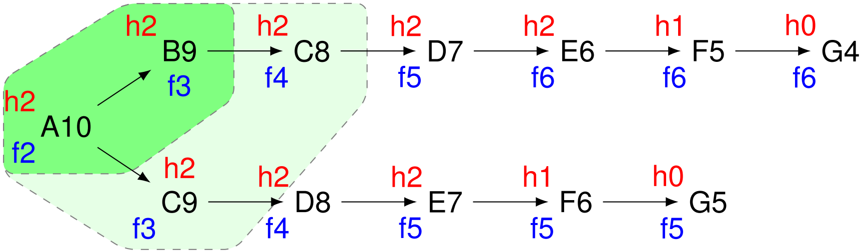

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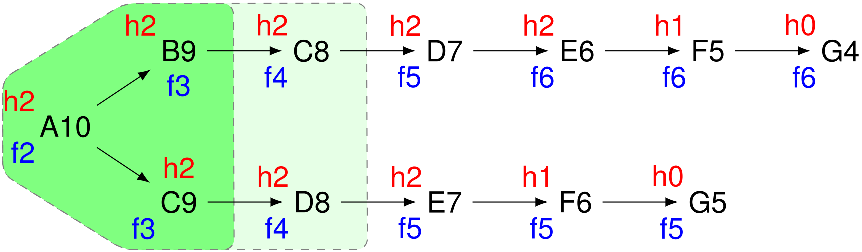

A^* is Optimally Efficient (Dechter and Pearl, 1985)

Class of Algorithms

Domain of Problem Instances	Admissible if $h \leq h^*$ A_{ad}	Globally Compatible with A^* A_{gc}	Best-First A_{bf}
Admissible I_{AD}	A^{**} is 3-optimal No 2-optimal exists	A^* is 1-optimal No 0-optimal exists	A^* is 1-optimal No 0-optimal exists
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A^* **is 1-optimal** on consistent instances

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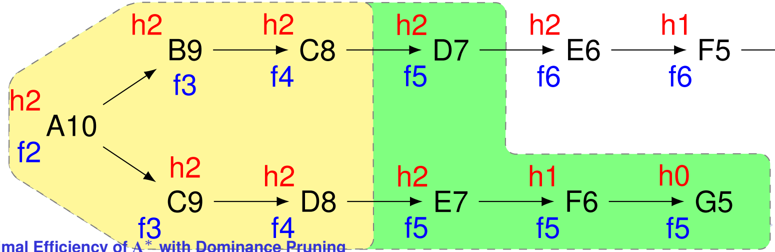

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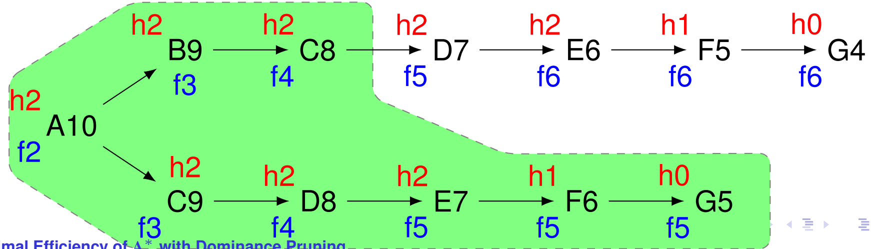

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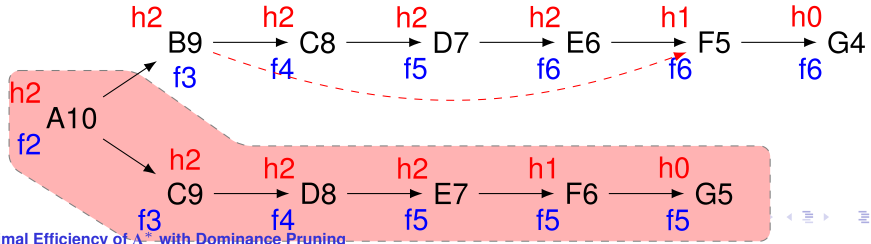

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New source of information: dominance relation directly compares pairs of states

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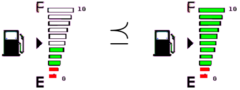

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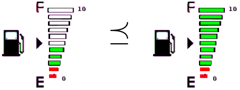

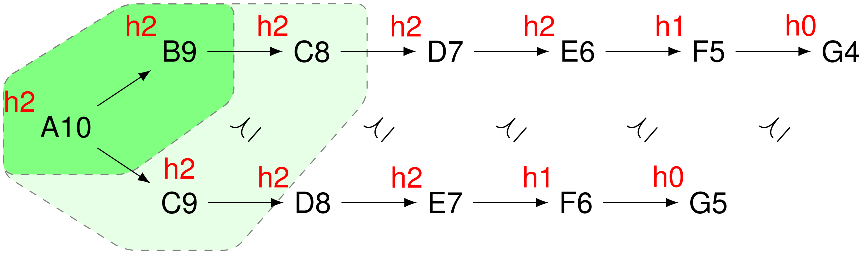

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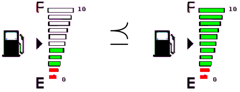

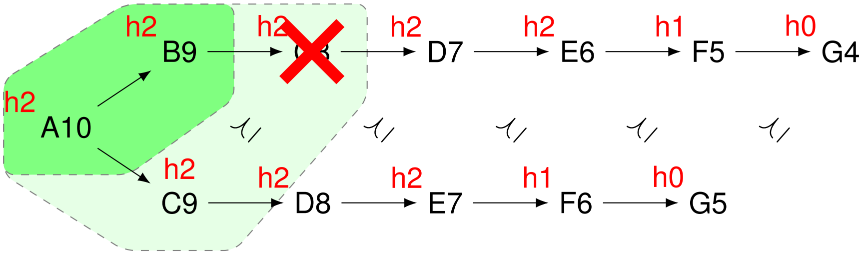

UDXBB with Dominance Pruning

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UDXBB algorithms that can prune a node n_s whenever another n_t has been seen such that $g(n_t) \leq g(n_s)$ and t dominates s.

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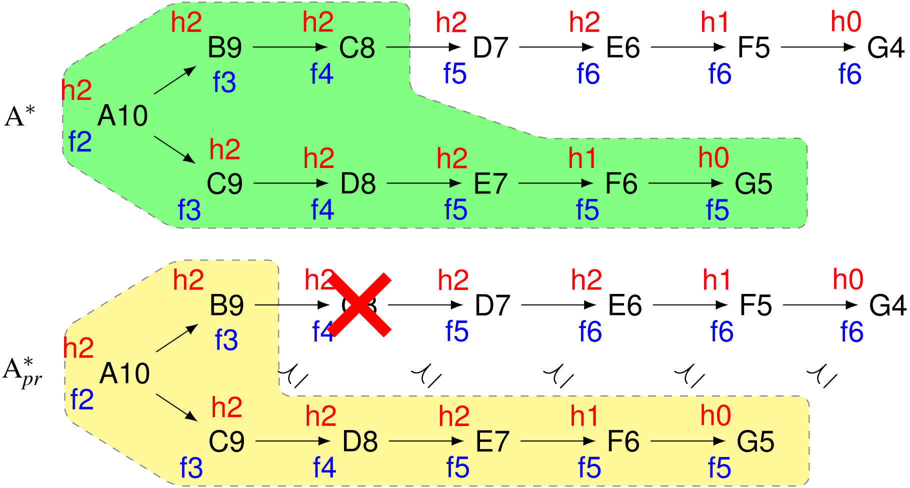
A^* with dominance pruning (A^*_{pr}):

- Expand nodes based on f-value: $f(n_s) = g(n_s) + h(s)$
- Prune any node that can be pruned

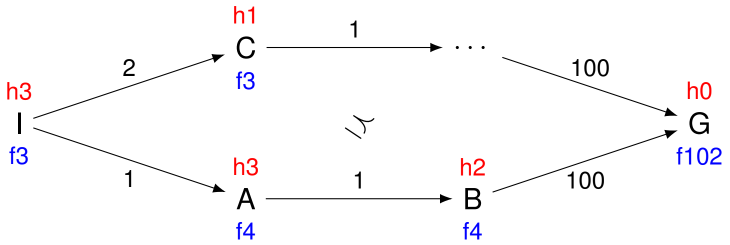
Outline

A^*_{pr} is 1-optimal over A^* (until last f-layer)



A^*_{pr} is not optimally efficient in general

→The expansion order of A^* may not be optimal



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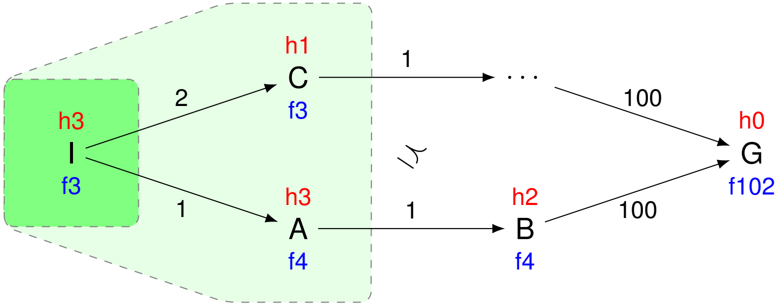

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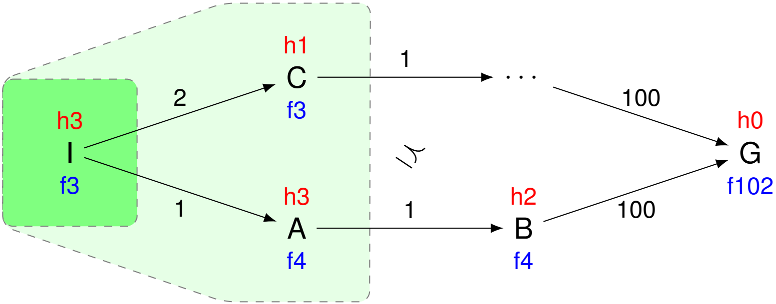

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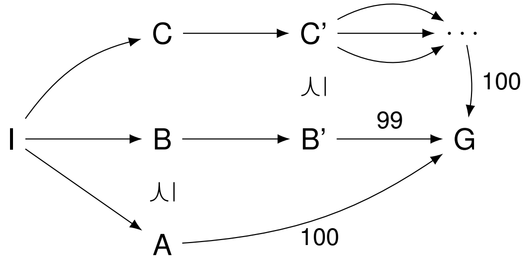
→Sometimes it could be worth it to expand a node even if it can be pruned



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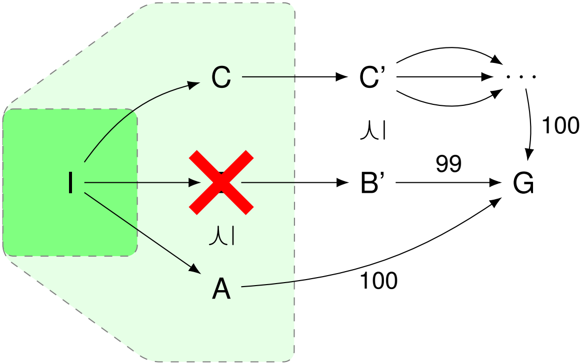

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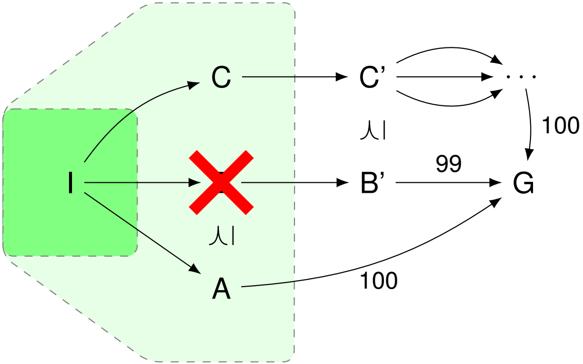

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Consistency

Consistent instances

An instance $I = \langle \Theta, h, \preceq \rangle$ is consistent if:

(i) h is consistent

(ii) $\preceq$ is a transitive cost-simulation relation

(iii) $\preceq$ is consistent with h: $s \preceq t \implies h(t) \leq h(s)$

Consistent Dominance Relations

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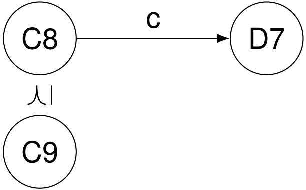

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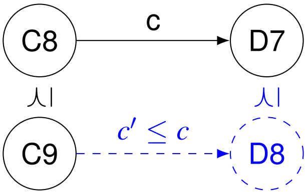

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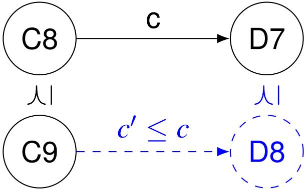

→State-of-the-art methods on dominance pruning derive transitive cost-simulation relations

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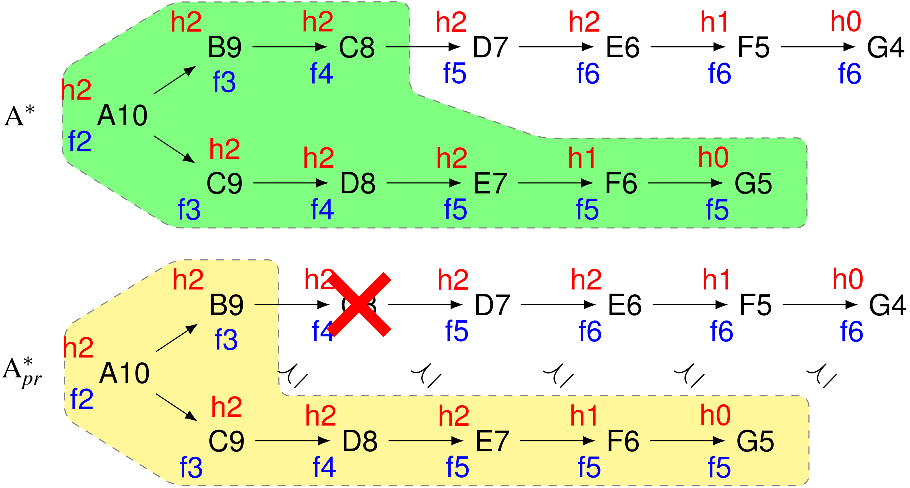
The information of h and $\preceq$ is still complementary!

- When $h(s) < h(t)$, s is more promising but there is no guarantee
- When $t \preceq s$, we are certain that t is as good as s (but if $s \not\preceq t$ we know nothing)

Is A^*_{pr} 1-optimal on consistent instances?



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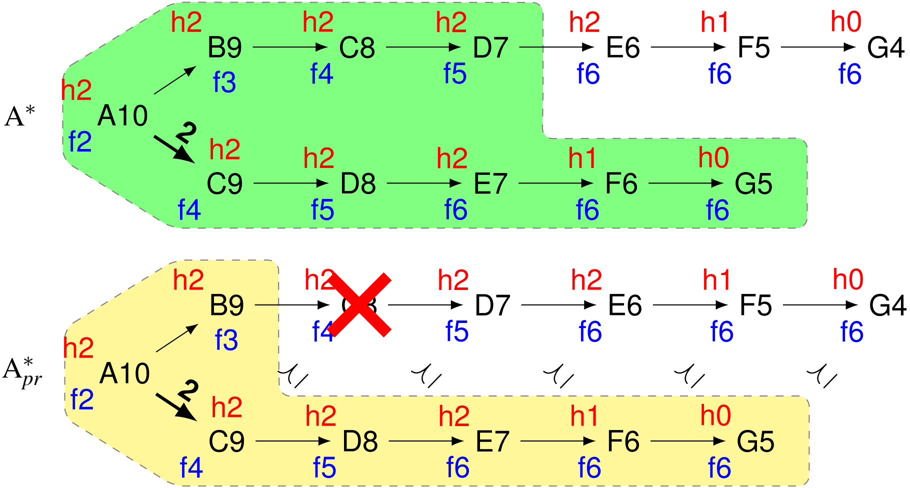

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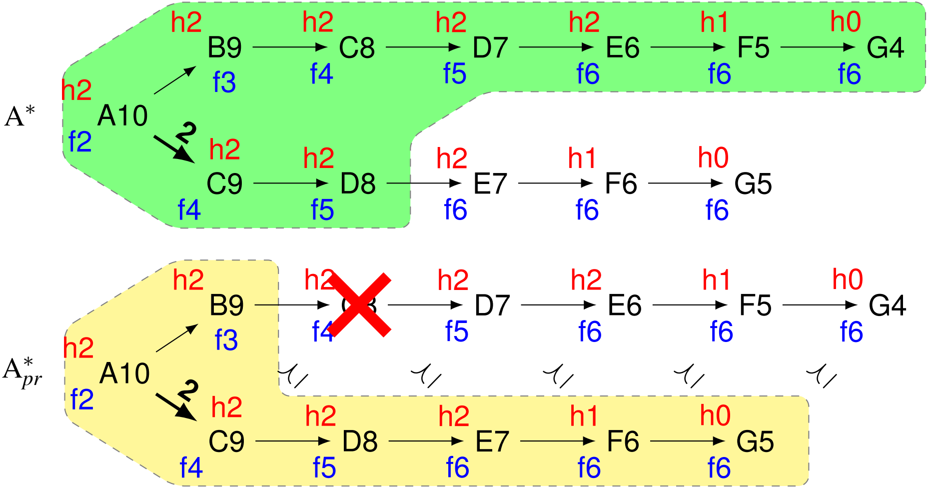

A^*_{pr} is #-optimal

A^*_{pr} is #-optimal on consistent instances wrt. $UDXBB_{pr}$

Let N be the set of states expanded by any admissible $UDXBB_{pr}$ algorithm, then there exists a tie-breaking of A^*_{pr} that expands N' with $|N'| \leq |N|$.

$\rightarrow$It may not expand a subset of nodes but it will expand the least amount of nodes!

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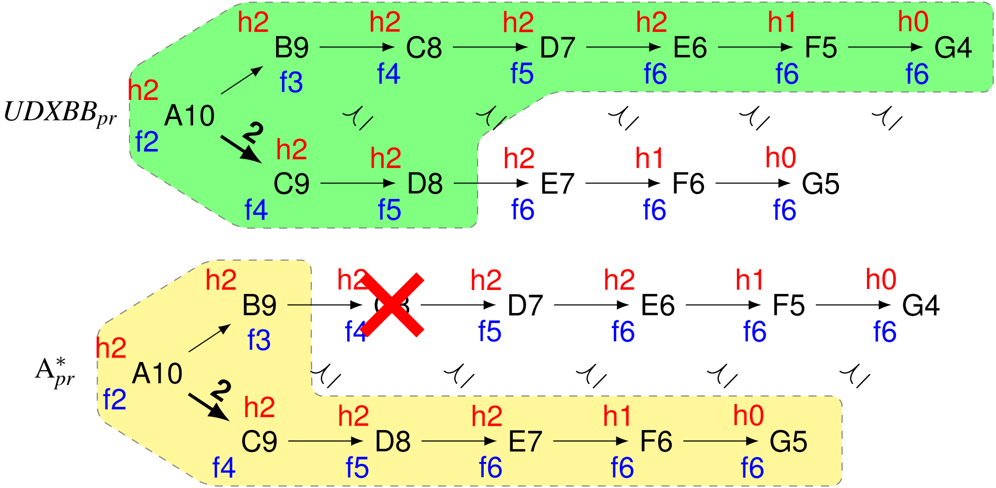

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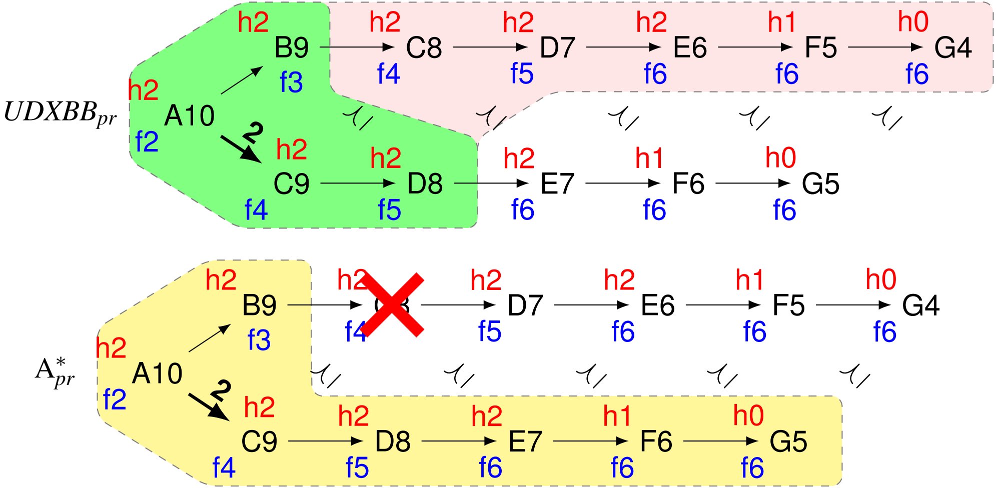

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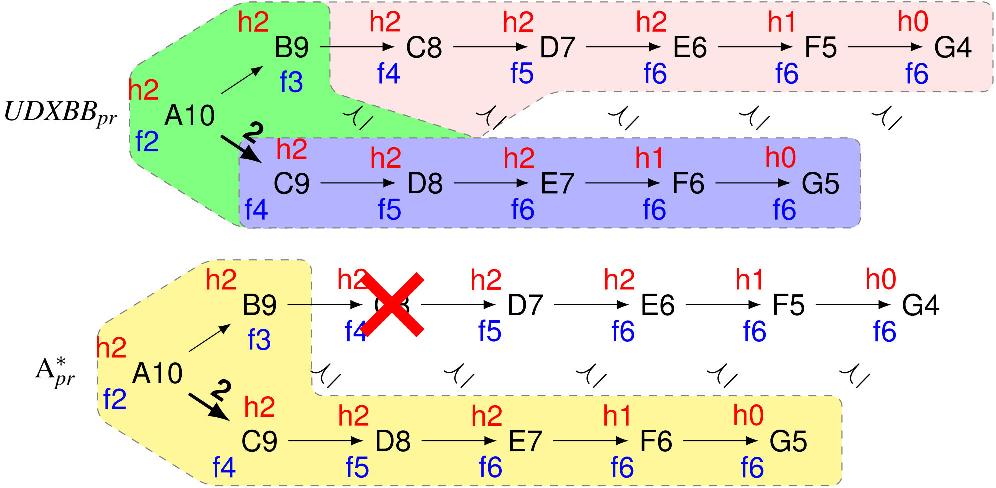

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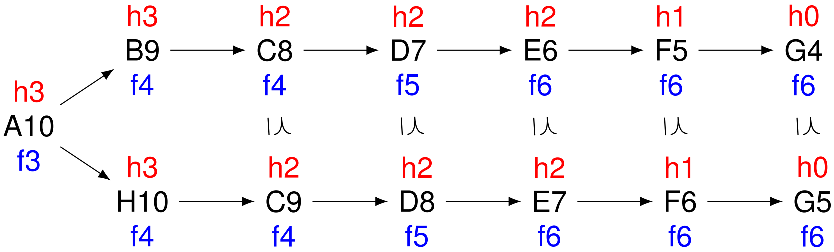
Tie-Breaking

- In A^* tie-breaking is only relevant in the last f-layer
- In A^*_{pr}, tie-breaking is relevant in all layers



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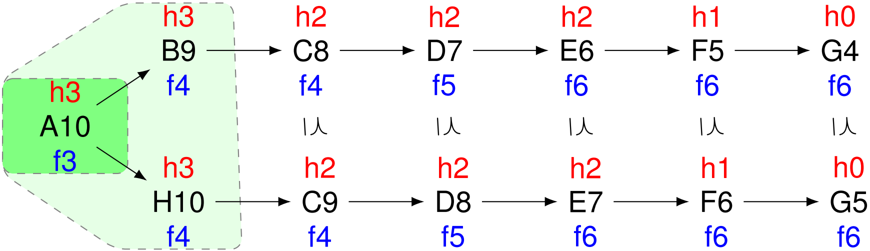

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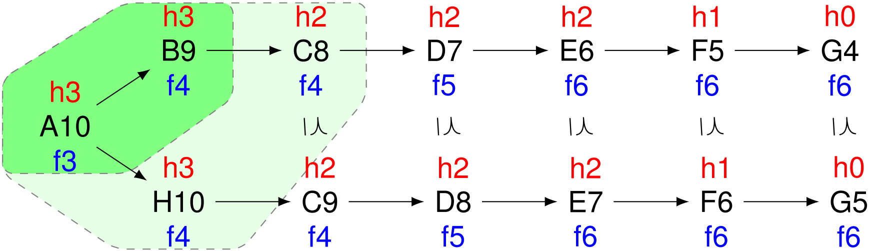

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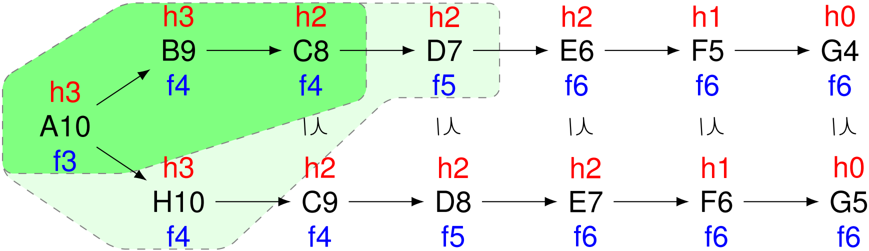

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Prefer nodes with lower h value

- Standard in most implementations of A^*
- Advantage: follow heuristic in the last f-layer

$\rightarrow$We prove that it is not optimally efficient until the last f-layer

$A^*_{g<,pr}$

Prefer nodes with lower g value
$\rightarrow$We prove that it is optimally efficient until the last f-layer

Trade off:

- follow h in the last f-layer
- follow $\preceq$ up to the last f-layer

Conclusion

- Dominance pruning introduces a new source of information for heuristic search algorithms
- Consistent instances:
  1. Consistent heuristic
  2. Dominance relation is a transitive cost-simulation relation
  3. Heuristic and dominance relation are consistent with each other
- A^*_{pr} is #-optimally efficient on consistent instances
- Until last layer is better to break ties in favor of minimum g-value