Revenue Maximization by Viral Marketing: A Social Network Host's Perspective

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***Abstract*—We study the novel problem of revenue maximization of a social network host that sells viral marketing campaigns to multiple competing campaigners. Each client campaigner informs the social network host about her target users in the network, as well as how much money she is willing to pay to the host if one of her target users buys her product. The social network host, in turn, assigns a set of seed users to each of her client campaigners. The seed set for a campaigner is a limited number of users to whom the campaigner provides free samples, discounted price etc. with the expectation that these seed users will buy her product, and would also be able to influence many of her target users in the network towards buying her product. Because of various product-adoption costs, it is very unlikely that an average user will purchase more than one of the competing products. Therefore, from the host's perspective, it is important to assign seed users to client campaigners in such a way that the seed assignment guarantees the maximum aggregated revenue for the host considering all her client campaigners.**

We formulate our problem by following two well-established influence cascading models: the independent cascade model and the linear threshold model. While our problem using both these models is NP-hard, and neither monotonic, nor sub-modular; we develop approximated algorithms with theoretical performance guarantees. However, as our approximated algorithms often incur higher running times, we also design efficient heuristic methods that empirically perform as good as our approximated algorithms. Our detailed experimental evaluation attests that the proposed techniques are effective and scalable over real-world datasets.

I. Introduction

In viral marketing, whenever a social network user buys a product, she is viewed as being influenced or activated. The classical viral marketing problem [9], [13] identifies the top-k seed users in a social network such that the expected number of influenced users in the network, starting from those seed users and following some influence cascading model, is maximized. The budget k on the seed-set size usually depends on the campaigner — in other words, it depends on how many initial users the campaigner can directly influence to buy her product by advertisements, giving free samples, and discounted prices.

The bulk of the research in the domain of viral marketing assumes that the social network structure is available to the campaigners. However, in real-world scenarios, the social network platforms are owned by third-party hosts [19], such as Facebook, Twitter, and LinkedIn; and the hosts keep their social graphs secret for their own benefits and for privacy reasons. Therefore, marketing companies themselves are not able to select their best seed sets due to lack of access to the social network graph.

In this study, we assume that the seed set selections are done by the social network host on behalf of her clients, who are the marketing campaigners. The campaigners, on the other hand, spend their overall budget for viral marketing into two parts. Particularly, each campaigner informs the host about: **(a)** her budget on the seed-set size (i.e., the number of seed users, k), and also **(b)** how much money she is willing to pay to the host for each of her target users if that user adopts her product. While the campaigner might not know the exact social network structure, it is usually easier for her to define her target users, either explicitly, or via some constraints, e.g., people in the age group 20-30, all banking professionals, etc. We note that the number of such target users for a campaigner can be very large, and it is often not possible (or not economical) to give each of them a free sample or discounted price. Therefore, the campaigner still allocates a small k as the number of her seed nodes. She uses rest of her budget to pay the social network host according to the agreement, which can be a small amount of money for each of her target users who adopts her product.

In real-world, multiple companies compete and they launch comparable products around the same time [1] (e.g., Nintendo's Wii vs. Sony's Playstation vs. Microsoft's X-Box; Microsoft's Surface vs. Apple's iPad vs. Samsung Note 3) [18], [19]. Thus, the host often needs to run multiple competing viral marketing campaigns together over the network. However, due to various product-adoption costs, it is very unlikely that an average user will purchase more than one of the competing products. Since most of the users adopt only one of the competing products, it implies that the seed sets of the competing campaigners require to be mutually non-overlapping [18], [19]. Therefore, from the host's perspective, the challenge lies in how to select the seed set for each of her client campaigners so that the host's overall expected revenue is maximized.

Running multiple viral marketing campaigns by a social network host was studied earlier in [19] by Lu et. al. However, [19] studied a different problem. The problem our paper addresses is the problem of maximizing the revenue of the host of the social network. In contrast, [19] studied how to balance the expected spread of each campaign over the network, which we believe is less relevant in practice. When a social network host is selecting the seed sets on behalf of her client campaigners, maximizing the host's overall expected revenue is the problem of interest for most practical scenarios. Furthermore, [19] makes a number of additional assumptions. For instance, [19] did not apply the notion of target users for each client campaigner; they assume that all users in the network are equally important to all campaigners. In reality, a campaigner often promotes her product with a group of target customers in mind [16], [17], and is willing to pay more money to the host if her target users adopt her product. Indeed, as

[1]At the Consumer Electronic Show in January 2011, over 80 new tablets were announced by Motorola, Samsung, and Toshiba. (http://mashable.com/2011/01/12/ces-2011-tablet-videos/)

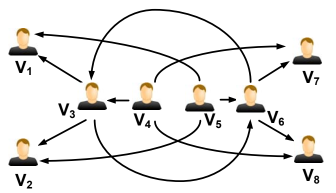


Fig. 1: Revenue maximization: limitations of naïve methods

shown experimentally in [1], a product adopted by customers (or, a campaign reached to users) outside the target group could generate negative impression towards the campaigner, which will affect her image in the long run. In summary, for different users, each campaigner would be willing to pay a different amount of money to the host if those users buy her product.

We next demonstrate with an example why the method in [19] and other naïve approaches are not suitable for our problem.

Example 1: Assume that there are two campaigners — C_1 and C_2, who are viral marketing clients to the social network host as depicted in Figure 1. We also assume that each directed edge has an influence probability 1, and we apply the independent cascade (IC) model [13], described later in Section III-B. The model assumes that the cascade of influence happens in discrete time steps. Each node can be activated only once and by only one of the campaigns; also the node stays activated with that campaign until the end. Let A_{ij} denote the money that campaigner C_i is willing to pay to the host if node V_j adopts her product. We set $A_{11} = A_{12} = A_{13} = A_{14}$=US\$ 10, $A_{15} = A_{16} = A_{17} = A_{18}$=US\$ 1; while $A_{21} = A_{22} = A_{23} = A_{24}$=US\$ 1, and $A_{25} = A_{26} = A_{27} = A_{28}$=US\$ 10. Assume that the budget on the seed set size for each campaigner is 1.

What is the optimal solution? The optimal solution is as follows. If V_3 and V_6 are selected as the seed nodes for C_1 and C_2, respectively, then V_1, V_2, V_3 will be influenced by C_1, while V_6, V_7, and V_8 will be influenced by C_2. This will ensure an aggregated revenue of US\$ 60 to the host.

Why a naïve method will not work? A naïve approach to solve the host's revenue maximization problem would be to first define some order among the campaigners, and then identify the top-k seed nodes for each campaigner such that the host's revenue is maximized considering one campaigner at a time. If some of the top-k seed nodes for the current campaigner have already been assigned to previous campaigners, we identify the next-best seed nodes for the current campaigner until we exhaust the budget k of the current campaigner's seed-set size. If we apply this naïve approach, we get V_4 as the best seed node for C_1. This is because V_4 could eventually influence V_1, V_2, V_3, V_6, V_7, and V_8, and the host will get a revenue of US\$ 43, assuming C_1 is the only campaigner in the network. Similarly, we find that V_5 is the best seed node for C_2, assuming there is no other campaigner. Now, in reality, when the host runs the two viral marketing campaigns simultaneously with campaign of C_1 starting from V_4 and that of C_2 starting from V_5, the host's aggregate revenue will be only US\$ 44. This is because after simultaneous campaigning, V_3, V_4, V_7, and V_8 will be influenced by C_1, while the remaining nodes will be influenced by C_2.

Why the method in [19] will not work? Lu et. al.'s problem formulation [19] identifies the seed sets for the campaigners in a way such that the expected spread of each campaign is almost equal (i.e., maintaining fairness), while also maximizing the overall spread of all campaigns in the network. As an example, selection of V_4 as the seed node of C_1 and V_5 as the seed node of C_2 would be an optimal solution according to [19], since this will result in V_3, V_4, V_7, and V_8 to be influenced by C_1, while the remaining four nodes will be influenced by C_2. Note that the host's revenue in this solution is only US\$ 44.

The above example clearly illustrates that the aforementioned naïve approach as well as [19] may result in a sub-optimal aggregated revenue for the host. In fact, as [19] did not consider the notion of target users for each client campaigner, it is non-trivial to extend their proposed algorithm to solve our current problem. Specifically, their Needy-Greedy algorithm heuristically partitions the initial seed set to balance the expected influence spread of each campaigner, assuming that all users in the network are equally important to all campaigners. It is difficult to adapt such an algorithm to maximize the host's overall revenue from all campaigners in a scenario when each user has a different importance to every campaigner. Indeed, our dynamic programming based seed partitioning strategy introduced in Section 5.2 is different from their Needy-Greedy heuristics.

Our Contributions and Roadmap. Our contributions can be summarized as follows:

- We define the fundamental problem of host's revenue maximization by viral marketing in the presence of $m \geq 2$ competitive campaigners (Sec. III).
- We formulate the problem using two widely-used influence cascading models — the independent cascade (IC) model (Section IV) and the linear threshold (LT) model (Sec. V).
- We show that our problem using both these models is **NP**-hard, and neither monotonic, nor sub-modular. We therefore develop approximated algorithms to solve our problem. In addition, we also design more efficient and scalable heuristic techniques that empirically perform as good as our approximated algorithms.
- We conduct a thorough experimental evaluation using several real-world datasets and with various kinds of revenue distributions (Sec. VII). Our empirical results attest that the proposed methods efficiently generate high-quality results.

II. Related Work

In viral marketing, a social network user is considered influenced or activated by a campaign if she buys a product corresponding to the campaign. The classical viral marketing problem aims at finding a small number of seed nodes that generates the largest expected influence cascade in a social network. Domingos and Richardson [9] formulated viral marketing as an optimization problem. Kempe et. al. [13] proposed the linear threshold model and the independent cascade model, and designed approximation algorithms with provable performance guarantees. However, the computation of influence cascade is still #**P**-hard [7]. Several heuristics have been proposed to improve the efficiency of viral marketing [8], [11]. Very recently, [21] developed almost linear-time viral marketing algorithms, yet providing the same approximation guarantee as Kempe et. al.'s original method. In [16], Lappas et. al. introduced the concept of target marketing and k-effectors — by identifying k seed nodes such that a given activation pattern can be established. The notion of target marketing was also considered in [17] that maximizes influence over a region of the network.

Viral marketing in the presence of a negative campaign was investigated in [2], [5]. These works assume that the later campaign has prior knowledge of rival side's initial seed nodes. Bordin et. al. [3] analyzed the similar problem under the LT model; while [4], [6] attempt at preventing the spread of an existing negative campaign in the network. Recently, [22] studied the viral marketing problem between non-cooperative campaigns who select seeds alternatively. However, as discussed in Section 1, competitive new products from rival companies are often launched around the same time. Thus, [12], [18], [19] considered viral marketing in the presence of multiple competing campaigners, who promote their products in a social network around the same time. We also consider a similar scenario, i.e., multiple rival companies launch and promote competing products at the same time.

While the bulk of the research on viral marketing assumes that the social network structure is available to the campaigners; in reality, the social network platforms are owned by third-party hosts. Lu et al. [19] were the first to consider the viral marketing problem from the social network host's perspective. In their framework, the host selects the seed nodes on behalf of her client campaigners so that the expected influence spread for each client campaigner becomes nearly the same. However, when a social network host is selecting the seed sets on behalf of her client campaigners, maximizing the host's overall expected revenue would be the problem of interest for most practical scenarios. To the best of our knowledge, ours is the first work that studies the host's revenue maximization problem, while also considering the notion of target users for each campaigner. As [19] did not consider the notion of target users for each client campaigner, it is non-trivial to extend their proposed algorithm to solve our current problem.

III. Preliminaries

A. Problem Statement

A social network $\mathcal{G}$ is modeled as a triple (V, E, P), where V is a set of n nodes, $E \subseteq V \times V$ is a set of e directed edges, and $P : E \rightarrow (0, 1)$ is a probability function that assigns a probability to each edge in E. The probability p_{uv} on a directed edge $(u, v) \in E$ represents the probability that node v adopts a product due to the influence of node u, because u adopted that product before. When v adopts that product, it automatically becomes eligible to influence its neighbors who has not adopted that product already. We shall discuss the details of various influence cascading models in Section III-B.

We consider $m \geq 2$ competing campaigners, denoted by $C_1, C_2, \ldots, C_m$, for whom the social network host runs simultaneous viral marketing campaigns. We denote by S_i the seed set for campaigner C_i, and A_{iu} the money that campaigner C_i is willing to pay to the host if node u adopts C_i's product. We refer to $A = (A_{iu})_{m\times n}$ the *revenue matrix.* We denote by $Pr(u, i, S)$ the probability that node u will adopt C_i's product due to the influence of her campaign by following some influence cascading model, where $S = \{S_1, S_2, \ldots, S_m\}$

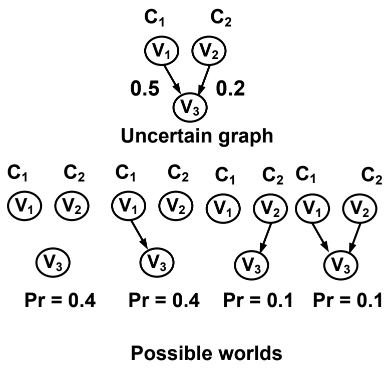


Fig. 2: Example of MCIC model

represents the seed sets for the m campaigners. We are now ready to define our problem.

Problem 1 (Revenue Maximization): Given a network $\mathcal{G} = (V, E, P)$, $m \geq 2$ client campaigners, the revenue matrix A, and a budget k_i on the seed set size for each campaigner C_i, i.e., $|S_i| = k_i$ for $1 \leq i \leq m$, find the seed set for each campaigner such that the expected revenue of the host is maximized. Formally,

$$\underset{S_1,S_2,\ldots,S_m}{\arg\max} \sum_{i=1}^{m} \sum_{u\in V} [A_{iu} \cdot Pr(u, i, S)]$$

$$\text{such that} \quad |S_i| = k_i \quad \forall i \in (1, m)$$
$$\text{and} \quad S_i \bigcap S_j = \phi \quad \forall i \neq j; \quad i, j \in (1, m) \tag{1}$$

B. Influence Cascading Models

We apply two widely-used influence cascading models: independent cascade (IC) and linear threshold (LT) [13].

1) Independent Cascade Model: In the single-campaigner IC model, the campaign starts with an initially active (i.e., adopted her product) set of seed nodes, and then unfolds in discrete steps. When some node u first becomes active at step t, it gets a single chance to activate each of its currently inactive out-neighbors v; it succeeds with probability $p_{u,v}$. If u succeeds, then v will become active at step $t+1$. Whether or not u succeeds at step t, it cannot make any further attempts in the subsequent rounds. If a node v has incoming edges from multiple newly activated nodes, their attempts are sequenced in an arbitrary order. Also, each node can be activated only once and it stays active until the end. The campaigning process runs until no more activations are possible.

Multi-Campaigner Independent Cascade Model. We shall now introduce the multi-campaigner Independent Cascade (MCIC) model [4], which models multiple campaigns that are being run simultaneously in a network. MCIC follows the same process as IC, except two major differences. First, if some node u is activated with campaign of C_i, it attempts to activate its out-neighbors v with the campaign of C_i. Second, an activated node v adopts one campaign uniform at random from all its in-neighbors which were successfully activated in the last round. Each node can be activated only once and by only one of the campaigns; also the node stays activated with that campaign until the end. Therefore, the MCIC model assumes the following influence cascading scenario: people adopt a product when they come in direct contact with their friends who very recently adopted that product.

*Example 2 (*MCIC *Model):* In Figure 2, we show a social network along with edge probabilities. We also assume that V_1 and V_2 are seed nodes for campaigners C_1 and C_2, respectively, and we want to calculate the probability that node V_3 will be influenced by each of these campaigners using the MCIC model. The computation is carried out following the *possible world* semantics [13]. We first identify all possible worlds of the uncertain input graph, where each possible world is a certain instance of the uncertain graph, and obtained by independent sampling of the edges. Each possible world is associated with a probability of existence. For example, the second possible world in Figure 2 has probability of existence 0.4, which is due to the presence of the edge V_1V_3 with probability 0.5 and the absence of the edge V_2V_3 with probability $(1-0.2)$. Hence, the probability of existence of the second possible world is $0.5\times(1-0.2) = 0.4$. In our example, there can be total 4 possible worlds. In each possible world, a node is activated by its closest seed nodes. Thus, V_3 is activated by C_1 in the second possible world, by C_2 in the third possible world, and by either of C_1 and C_2 with equal probability in the fourth possible world. Therefore, the probability that V_3 is activated by C_1 is $0.4 + 0.1 \times \frac{1}{2} = 0.45$, and V_3 is activated by C_2 with probability $0.1 + 0.1 \times \frac{1}{2} = 0.15$.

2) Linear Threshold Model: In the single-campaigner LT model, each node v has an activation threshold $\theta_v \leq 1$. In addition, there is a constraint that the sum of the probabilities of all incoming edges for every node must be at most 1. The campaign starts with an initially active set of seed nodes, and then unfolds in discrete steps. If the sum of the probabilities of the incoming edges from all active nodes is greater than or equal to the activation threshold of an inactive node, then the node gets activated in the next round. Each node can only be activated once and stays active until the end.

Multi-Campaigner Linear Threshold Model. The multi-campaign LT model, also termed as the K-LT model in [19], follows two steps in each round. The first step decides whether a new node will get activated and it works in exactly the same way as the LT model (i.e., without distinguishing among multiple campaigns). However, in the second step, it decides which campaign each of those newly activated nodes will adopt. Let us consider all nodes u that were activated in the last round and contributed to the activation of a node v in the current round. Then, v will adopt the same campaign as that of u with probability $\frac{p_{uv}}{\sum_u p_{uv}}$. With the K-LT model, each node can be activated only once and by only one of the campaigns; also the node stays activated with that campaign until the end. One may note that the K-LT model simulates the following scenario: a user adopts a technology only when more than a threshold number of her neighbors adopted a similar technology. However, once the user decides to adopt the technology, she decides on the specific product only based on her neighbors who most recently adopted that technology.

*Example 3 (*K-LT *Model):* We show an example of the K-LT model in Figure 3. Here, V_1 and V_4 are seeds for campaigners C_1 and C_2, respectively. At time step 1, V_2 becomes active with campaign of C_1, since $p_{v_1,v_2} = 0.8 > \theta_{v_2} = 0.6$. However, V_3 remains inactive as $p_{v_4,v_3} = 0.5 < \theta_{v_3} = 0.6$. At time step 2, V_3 first gets activated as the total incoming influence from its activated neighbors is: $p_{v_2,v_3} + p_{v_4,v_3} = 0.7$, which is higher than its activation threshold $\theta_{v_3} = 0.6$. Finally,

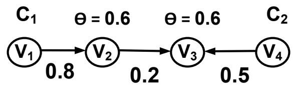


Fig. 3: Example of K-LT model

V_3 selects campaign of C_1 with probability 1, because V_2 is the only neighbor of V_3 which was activated in the last round, and V_2 was activated with campaign of C_1.

C. Hardness Results

We first prove that Problem 1 is **NP**-hard both with the MCIC and K-LT influence cascading models. We consider the decision version of our problem: Given a social network $\mathcal{G} = (V, E, P)$, $m \geq 2$ client campaigners, the revenue matrix A, a budget k_i on the seed set size for each campaigner C_i, that is, $|S_i| = k_i$ for $1 \leq i \leq m$, and a positive integer R, can we find a seed set for each campaigner such that the expected revenue of the host is at least R?

Theorem 1: Following the MCIC model of influence cascading, the decision version of Problem 1 is **NP**-hard.

Proof: We shall prove the **NP**-hardness by performing a reduction from the **NP**-complete set-cover problem. Let us consider an instance of the set-cover problem, defined by a collection of subsets $S = \{S_1, S_2, \ldots, S_r\}$ of a ground set $U = \{u_1, u_2, \ldots u_n\}$; we wish to know whether there exist k of the subsets whose union is equal to U. Now, we consider another identical instance of the previous set cover problem, given by a collection of subsets $S' = \{S'_1, S'_2, \ldots, S'_r\}$ of the ground set $U' = \{u'_1, u'_2, \ldots u'_n\}$. We construct our revenue maximization problem for $m = 2$ competing campaigners: C_1 and C_2, as follows. For each $u_i \in U$, $u'_i \in U'$, $S_i \in S$, and $S'_i \in S'$, we include a node in the network. The two nodes corresponding to element pairs u_i, u'_i are connected by a bi-directed edge of probability 1. If a subset S_i covers an element u_j, we add a directed edge of probability 1 from node S_i to node u_j in the network. Analogously, if some subset S'_i covers an element u'_j, we also add a directed edge of probability 1 from node S'_i to node u'_j in the network. The revenue matrix has the following form: $A_{1,u_i} = 1$ for all $u_i \in U$, $A_{2,u'_j} = 1$ for all $u'_j \in U'$, all other entries in the revenue matrix are 0. Finally, we also assume that there is a budget k on seed-set size for each campaigner. In this setting, there is a solution to our revenue maximization problem with the host's expected revenue at least $2n$, if and only if there is a solution to the set cover problem. Hence, the theorem. ■

Theorem 2: Following the K-LT model of influence cascading, the decision version of Problem 1 is **NP**-hard.

Proof: We prove the **NP**-hardness by performing a reduction from the **NP**-complete set cover problem, defined by a collection of subsets $S = \{S_1, S_2, \ldots, S_r\}$ of a ground set $U = \{u_1, u_2, \ldots u_n\}$; and we want to know whether there exist k of the subsets whose union is equal to U. Now, we construct our revenue maximization problem with the K-LT model and for $m = 2$ competing campaigners: C_1 and C_2, as follows. For each element $u_i \in U$, there is a node in the network with activation threshold $\frac{1}{r+1}$. For each subset $S_i \in S$, we add two nodes v_i and v'_i in the network, each having an activation threshold 1. Now, if a subset S_i covers an element u_j, we add a directed edge of probability $\frac{1}{2r}$ from node v_i to node u_j, and another directed edge of probability $\frac{1}{2r}$ from

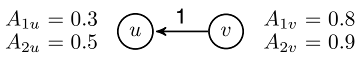


Fig. 4: Counter-example of monotonicity

node v_i' to node u_j. Note that the sum of the probabilities of all incoming edges to any node u_j is at most 1. The revenue matrix A has the following form: $A_{1,u_i} = 1$ for all $u_i \in U$, $A_{2,u_i} = 1$ for all $u_i \in U$, and all other entries in A are 0. Finally, we also assume that there is a budget k on seed-set size for each campaigner. In this setting, there is a solution to our revenue maximization problem with the host's expected revenue at least n, if and only if there is a solution to the set cover problem. Hence, the theorem. ■

Unlike the classical viral marketing problem [13], our revenue maximization problem, under both MCIC and K-LT models, is *neither monotonic*, *nor sub-modular*. Therefore, an iterative greedy algorithm [13] that maximally increases the marginal gain at every iteration, and which has been widely-used to derive a solution with theoretical approximation bounds for the conventional viral marketing problem, can no longer be employed in our case for deriving similar approximation guarantees. Hence, we first design our novel approximated solutions in Sections IV and V, where we provide theoretical performance guarantees in the presence of some additional constraints. Finally, in Section VI, we also provide more efficient greedy solutions for our problem. However, prior to introducing our solution techniques, we demonstrate below the non-monotonicity and non-sub-modularity of the revenue maximization problem using counter-examples.

Non-Monotonicity. In Figure 4, we assume that node v is already assigned to S_2 (i.e., seed set of campaigner C_2) and that we still need to assign the seed set to campaigner C_1. Under both the MCIC and the K-LT model, node u will be activated by node v with probability 1. Therefore, the host's revenue is $0.5 + 0.9 = 1.4$ with only node v assigned to the seed set S_2 of campaigner C_2. If we now assign node u to the seed set S_1 of campaigner C_1, the host's revenue is reduced to $0.3 + 0.9 = 1.2$. If we assign the seed sets in the other order (i.e., first S_1 to C_1 and then S_2 to C_2), then the revenue would have increased from $0.3 + 0.8 = 0.11$ to $0.3 + 0.9 = 1.2$. In general, the revenue maximization problem is non-monotonic with respect to addition of seed sets.

Non-Sub-Modularity. Let us denote by $F(S) = \sum_{i=1}^{m} \sum_{u \in V} [A_{iu} \cdot Pr(u, i, S)]$. Here, $S = \{S_1, S_2, \ldots, S_m\}$ is the collection of m seed sets corresponding to m different campaigners. In order to illustrate non-sub-modularity, we need to show that the following inequality does not always hold: $F(S \cup S_i) - F(S) \geq F(S' \cup S_i) - F(S')$, where $S' \supseteq S$, $S_i \notin S'$. In Figure 5, we assume that S : $\{u_1$ assigned to $C_1\}$; and S' : $\{u_1$ assigned to C_1, u_2 assigned to $C_2\}$. Under the MCIC model, node v is always activated, if either node u_1 or u_2 is activated. Under the K-LT model, assume we use an activation threshold of 0.4 for node v. Therefore, node v is always activated under the K-LT model as well. Now, we assign v to S_3 (i.e., seed set of campaigner C_3) and check the sub-modularity criteria for S and S'. We get $F(S \cup S_3) - F(S) = (0.7 + 0.1) - (0.7 + 0.5) = -0.4$ and $F(S' \cup S_3) - F(S') = (0.7 + 0.9 + 0.1) - (0.7 + 0.9 + (0.5 \times \frac{1}{2}) + (0.1 \times \frac{1}{2})) = -0.2$. Therefore, the sub-modularity property is not satisfied.

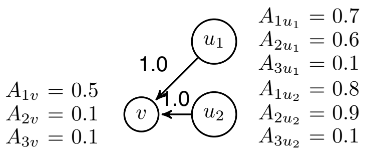


Fig. 5: Counter-example of sub-modularity

IV. Solution with Independent Cascade Model

In this section, we consider the revenue maximization problem under the MCIC influence cascading model. For simplicity, we assume that each campaigner has a seed set of the same size k.

Overview. Although our revenue maximization problem is **NP**-hard over graphs, we shall illustrate that the problem is solvable in polynomial time in a tree dataset. Therefore, we consider a two step heuristic approach: first, given a graph dataset, we extract the *most influential tree*, which will be formally defined in Section IV-C. Intuitively, the most influential tree approximates a social network by preserving the *most influential path* between every pair of nodes as much as possible [15]. A path between a source and a destination node is called the most influential path if the probability of influence cascading along that path is maximal in comparison with all other paths between these two nodes. While most-influential-path-based approaches were used in the past, e.g. [7], to solve the classical viral marketing problem, our current problem is a different one. We design a polynomial-time exact algorithm to solve the host's revenue maximization problem over a tree dataset (Section IV-A). We describe our algorithm with a simpler binary tree in the following section, and later in Section IV-B, we show how to convert a tree to an equivalent binary tree suitable for our method.

A. Exact Solution over Directed Binary Trees

On a directed tree, a node v can only be activated by its closest ancestor u, including the node itself, such that u belongs to one of the seed sets S_1, S_2, …, S_m. This is simply because u blocks the path from any farther ancestor u' to v, where u' is also a seed node. Therefore, $Pr(v, i, S)$, the probability that node v will be influenced by C_i's campaign, has the following expression in case of a directed tree.

$$Pr(v, i, S) = \begin{cases} 0 & \text{if } u \notin S_i; \\ \prod_{(u',v') \in Path(u,v)} p_{u'v'} & \text{otherwise.} \end{cases}$$

Here, u denotes the closest ancestor of v, such that u is a seed node. First, we compute for every node v in the tree dataset, the probability that v gets activated by each of its ancestors u, which is simply $\prod_{(u',v') \in Path(u,v)} p_{u'v'}$. We store these activation probabilities in a table $\mathcal{B}$ of size $\mathcal{O}(nd)$, where n is the total number nodes in the tree and d is the depth of the tree. Let $\mathcal{B}(u, v)$ denote the activation probability that node v is activated by its ancestor u. Clearly, $\mathcal{B}(v, v) = 1$. The computation of table $\mathcal{B}$ requires $\mathcal{O}(nd)$ time, if we re-use the activation probabilities from a parent node in order to compute the activation probabilities for its children nodes.

Next, we apply a dynamic-programming-based algorithm to find the optimal seed sets for all campaigners over a directed binary tree. For this purpose, we introduce another table OPT

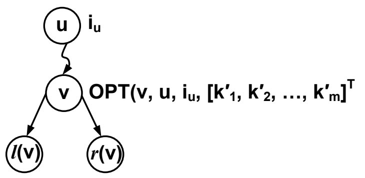


Fig. 6: Exact solution over binary tree: MCIC model

of the form $\mathsf{OPT}(v, u, j, [k'_1, k'_2, \ldots, k'_m]^T)$, where: **(a)** v is any node in the tree, **(b)** node u denotes the nearest ancestor of node v such that u is a seed node, **(c)** $j \in (1, m)$ denotes the campaigner C_j such that u is a seed node of campaigner C_j, and **(d)** each $k'_i \leq k$ denotes the number of seed nodes already assigned to campaigner C_i in the subtree rooted at v. An entry in the OPT table, e.g., $\mathsf{OPT}(v, u, j, [k'_1, k'_2, \ldots, k'_m]^T)$ represents the host's expected revenue for the optimal assignment of all seed sets S_i, $i \in (1, m)$, $|S_i| = k'_i$ over the subtree rooted at node v: given u, which is v's nearest ancestor that is a seed node, is rather assigned as a seed node to campaigner C_j. It can be noted that the size of OPT table is $\Theta(ndmk^m)$. The entries in OPT are computed by performing a post-order traversal over the tree dataset as given in Equations 2, 3, 4.

In our dynamic programming, $\mathsf{OPT}(v, u, j, [k'_1, k'_2, \ldots, k'_m]^T)$ is computed as the maximum over two cases. **(a)** Case$_1$: the first case considers the scenario when v is not selected as a seed node, and **(b)** Case$_2$: the second case considers the situation when v is assigned to some campaigner as a seed node. Here, $l(v)$ and $r(v)$ denote the left and right subtrees of v, respectively, as illustrated in Figure 6. For simplicity of description, we assumed that the budget of seed-set size for each of the m campaigners is k. Then, to fill one entry in the OPT table, we need $\mathcal{O}(mk^m)$ time. Therefore, the time complexity of our dynamic programming is $\mathcal{O}(ndm^2k^{2m})$.

It is important to note that the dynamic programming terminates by computing the OPT entries for the root node v_r, that is, $\mathsf{OPT}(v_r, -1, i, [k, \ldots, k]^T)$ for all $i \in (1, m)$. The value -1 at the second index indicates that the root node v_r does not have an ancestor. Therefore, $\mathsf{OPT}(v_r, -1, i, [k, \ldots, k]^T)$ is invariant of i. Once we terminate our dynamic programming, we determine whether a node will be a seed node (and if so, that node will be assigned to which campaigner) by *backtracking* using the OPT entries of its children nodes. The backtracking process requires another $\mathcal{O}(mnk^m)$ time. Therefore, the overall time complexity of finding the optimal seed nodes over a directed binary tree is $\mathcal{O}(ndm^2k^{2m})$. We note that our dynamic-programming-based exact solution over directed binary trees is polynomial-time in the number of tree nodes; however, it has exponential time-complexity in the number of client campaigners.

Space and Time Complexity. In this section, we summarize the space and time complexity of our dynamic-programming-based solution. The space complexity is $\mathcal{O}(ndmk^m)$: table $\mathcal{B}$ has size $\mathcal{O}(nd)$ and table OPT has size $\Theta(ndmk^m)$. The time complexity of our dynamic programming is $\mathcal{O}(ndm^2k^{2m})$.

B. Directed Trees to Binary Trees Conversion

Our dynamic-programming-based exact solution for the revenue maximization problem (Problem 1) can be applied over non-binary trees. In fact, given a directed tree, we first convert it to an equivalent directed binary tree. We use the conversion technique in [16]. For each non-leaf node v with children $v_1, v_2, \ldots, v_\Delta$, where $\Delta > 2$ in the original tree, we replace v with a binary tree of depth at most $\log \Delta$ and leaves $v_1, v_2, \ldots, v_\Delta$. For each newly introduced node u, we assign the revenue $A_{iu} = 0$, for all campaigners $i \in (1, m)$. While applying our dynamic programming (Section IV-A), we also incur an additional constraint that no such newly introduced node can be selected as a seed node. Finally, for each newly introduced edge, the direction is always from the root towards the leaves, and each of them has probability 1. The incoming edges to the leaves $v_1, v_2, \ldots, v_\Delta$ have the same probability as that of the previous incoming edges to nodes $v_1, v_2, \ldots, v_\Delta$, respectively. This conversion process ensures that the newly introduced edges and nodes will not affect the MCIC propagation model as in the original directed tree.

As shown in [16], for the aforementioned tree-to-binary-tree conversion method, the number of nodes in the equivalent binary tree is at most twice the number of nodes in the directed input tree, and the depth of the binary tree is at most a factor of $\log \Delta^*$ larger than the depth of the original tree, where Δ^* is the maximum out-degree of any node in the input tree.

$$\mathsf{OPT}\left(v, u, j, \begin{bmatrix} k'_1 \\ \vdots \\ k'_m \end{bmatrix}\right) = \max\{\text{Case}_1, \text{Case}_2\} \tag{2}$$

$$\text{Case}_1 = \max_{k''_1=0}^{k'_1} \overset{\cdots}{\underset{\cdots}{\max}} \max_{k''_m=0}^{k'_m} \left\{ \mathsf{OPT}\left(l(v), u, j, \begin{bmatrix} k''_1 \\ \vdots \\ k''_m \end{bmatrix}\right) \right.$$
$$\left. + \mathsf{OPT}\left(r(v), u, j, \begin{bmatrix} k'_1 - k''_1 \\ \vdots \\ k'_m - k''_m \end{bmatrix}\right) + A_{j,v} \times \mathcal{B}(u, v) \right\} \tag{3}$$

$$\text{Case}_2 = \max_{i=1}^{m} \left\{ \max_{k''_1=0}^{k'_1} .. \max_{k''_i=0}^{k'_i-1} .. \max_{k''_m=0}^{k'_m} \left\{ \mathsf{OPT}\left(l(v), v, i, \begin{bmatrix} k''_1 \\ \vdots k''_i \\ \vdots \\ k''_m \end{bmatrix}\right) \right.\right.$$
$$\left.\left. + \mathsf{OPT}\left(r(v), v, i, \begin{bmatrix} k'_1 - k''_1 \\ \vdots \\ k'_1 - k''_i - 1 \\ \vdots \\ k'_m - k''_m \end{bmatrix}\right) + A_{i,v} \right\}\right. \tag{4}$$

C. Graphs to Most Influential Directed Tree Extraction

The revenue maximization problem is $\mathbf{NP}$-hard in directed graphs (Theorem 2). Therefore, given a directed and connected graph G, we first extract the *most influential* tree T^*, which is a directed spanning tree of G, and formally defined below.

Definition 1 (Most Influential Tree): Given a connected graph $G = (V, E, P)$ with a root node v_r, the most influential tree $T^* = (V, E_{T^*}, P)$ with $E_{T^*} \subseteq E$ is a directed spanning tree of G, with the same root node v_r, such that the product of the edge probabilities in T^* is maximized. Formally,

$$T^* = \underset{T \in \text{SpanningTrees}(G)}{\arg\max} \prod_{(u,v) \in E_T} p_{u,v} \tag{5}$$

SpanningTrees(G) denotes all directed spanning trees of G. Intuitively, the most influential tree aims at preserving the most influential path between every pair of nodes as much as possible. These most influential paths play an important role in influence cascade over real-world social networks [15].

The problem of finding the most influential tree can be converted to the problem of finding the minimum-cost *directed* spanning tree by minimizing the sum of negative logarithms to the edge probabilities in T^* as given in Equation 6.

$$T^* = \underset{T \in \text{SpanningTrees}(G)}{\arg\min} \sum_{(u,v)\in E_T} -\log(p_{u,v}) \tag{6}$$

Thus, one can find the most influential directed tree in time $\mathcal{O}(e+n\log n)$ due to Gabow et al. [10]. It is important to note that [10] requires some root node v_r to be present in the input graph G such that all other nodes in G are reachable from v_r. Therefore, we first add a dummy root node v_r and then connect all nodes in G to v_r, with edges directed towards the nodes in G. Each of these newly-introduced edges is assigned a very low edge-probability. For the dummy root node v_r, we also assign revenue $A_{iv_r} = 0$ for all campaigners $i \in (1, m)$; and then, we further incur an additional constraint that v_r cannot be selected as a seed node during our dynamic-programming-based exact solution over the most influential tree T^*.

V. Solution with Linear Threshold Model

In this section, we consider the revenue maximization problem (Problem 1) under the $\mathsf{K\text{-}LT}$ influence cascade model. We recall that our problem is $\mathbf{NP}$-hard under the $\mathsf{K\text{-}LT}$ model (Theorem 2). However, we shall later illustrate that given an *already-selected* set of seed nodes, one can optimally partition these seed nodes among m campaigners in polynomial time such that the host's expected revenue is maximized. Using this result, we design a two-step approximation algorithm to solve our original revenue maximization problem with a *theoretical performance guarantee* of $\frac{1}{m}(1-\frac{1}{e})$.

Overview. In the first phase (Section V-A), the host *optimistically* assumes that it is possible to influence each user by a campaign that gives the maximum revenue to the host for that user. This is equivalent to assigning, for each user u, a revenue A_u which is the maximum of A_{iu} values over all campaigners i. Thus, the host identifies mk seed nodes assuming there is only *one campaigner* and with the objective that her expected revenue is maximized under this optimistic assumption. However, in reality, there are m campaigners, each with a seed-set of size k. Therefore, in the second step (Section V-B), the host partitions these *already-selected* mk seeds among m campaigners with the objective that her expected revenue is maximized under the *multi-campaigner* setting and considering the original revenue matrix. Below, we describe both these steps in details.

A. Optimistic Seed Set Selection

In the first phase, the host optimistically assumes that each user in the network can be influenced by a campaign such that the corresponding campaigner gives the maximum amount of money for that particular user. In other words, for each user u in the network, the host optimistically assigns a revenue A_u which is the maximum of A_{iu} values over all campaigners i. Formally, $A_u = \max_{i\in(1,m)} \{A_{iu}\}$. Therefore, the host solves the following problem in the first step.

Problem 2 (Optimistic Seed Set Selection): Assuming there is only one campaigner and given a revenue A_u for each user u in the network, find the seed set of size mk such that the expected revenue of the host is maximized. Formally,

$$\begin{aligned} &\underset{S}{\arg\max} \sum_{u\in V} [A_u \cdot Pr_{\mathsf{LT}}(u,S)] \\ &\text{such that} \quad |S| = mk \end{aligned} \tag{7}$$

Here, $Pr_{\mathsf{LT}}(u,S)$ denotes the expected spread of an influence from the seed set S to node u following LT model with one campaigner. Unfortunately, Problem 2 is also $\mathbf{NP}$-hard following [13]; nevertheless, the objective function is monotonic and sub-modular as shown in Theorem 3.

Theorem 3: The objective function of Problem 2 is sub-modular. Formally, let $F(S) = \sum_{u\in V}[A_u \cdot Pr_{\mathsf{LT}}(u,S)]$. Then,

$$F(S\cup\{v\}) - F(S) \geq F(S_1\cup\{v\}) - F(S_1) \tag{8}$$

Here, $S_1 \supseteq S$ and $v \notin S_1$.

Proof: The proof follows by considering the *live-edge* model, which is shown to be equivalent to the LT model in [13]. In the live-edge model, each node v picks at most one of its incoming edges at random, that is, it selects the incoming edge from u with probability $p_{u,v}$, and it does not select any incoming edge with probability $1-\sum_{u\in in(v)} p_{u,v}$. Let X be one possible world with probability $Prob(X)$ under the live-edge model, and $R_X(S)$ be the host's revenue due to nodes that are reachable from the seed set S in that possible world X. One may verify that $R_X(S)$ is sub-modular with respect to S. Now, our objective function $F(S)$ is given by:

$$F(S) = \sum_{\text{all possible world } X} [Prob(X)\cdot R_X(S)] \tag{9}$$

As the non-negative linear combination of sub-modular functions is also sub-modular, $F(S)$ is sub-modular. ■

Thus, we apply an iterative hill-climbing algorithm (Algorithm 1) that finds the seed set with an approximation guarantee $(1-\frac{1}{e})$ of the optimal solution [13]. The hill-climbing algorithm works in mk iterative steps. At each iteration, the algorithm selects a non-seed node u as a seed node, such that the expected revenue due to u and the previously selected seed nodes is maximized. Our hill-climbing-based iterative solution for the optimistic seed selection problem (Problem 2) is similar to state-of-the-art viral marketing techniques that identify the top-k seed nodes for a single campaigner such that its expected influence spread in the network is maximized [13]. Although we optimize the host's expected revenue instead of the expected influence spread, due to the single-campaigner and sub-modular nature of Problem 2, one can easily apply an existing viral marketing algorithm [7], [11], [13] (with some modification in the objective function) as the underlying technique to solve Problem 2.

We shall later show in Theorem 4 that the iterative hill-climbing algorithm for the optimistic seed selection, coupled with an optimal partition of those seed sets among m campaigners, generates a solution to the original revenue maximization problem with the approximation guarantee $\frac{1}{m}(1-\frac{1}{e})$.

Algorithm 1 Hill-Climbing for Optimistic Seed Set Selection

```
Require: Graph $G = (V, E, P)$, $A_u = \max_i\{A_{iu}\}\ \forall u \in V$
Ensure: Seed set $S$ of size $mk$
1: $S = \phi$
2: for $i = 1$ to $mk$ do
3:     $v = \arg\max_{v \in V \setminus S} F(S \bigcup \{v\})$    // $F()$ is defined in Th. 3
4:     $S = S \bigcup \{v\}$
5: end for
6: Output $S$
```

Time Complexity. The time complexity of our iterative hill climbing algorithm is $\mathcal{O}(mkn(n+e)t)$, where mk is the total number of seed nodes identified, and t is the number of Monte-Carlo samples performed over the entire graph in order to find one seed node.

B. Partition of Seed Set

In the second phase, the host optimally partitions the previously selected mk seed nodes among m campaigners, such that her expected revenue is maximized under the actual *multi-campaigner* K-LT model and considering the original revenue matrix. We formally define our problem statement for the second step as follows.

Problem 3 (Optimal Seed-Set-Partition): Given already selected seed set S of size mk and the revenue matrix $(A_{iu})_{m\times n}$, partition S into m subsets $S_1, S_2, \dots, S_m$, such that each S_i has size k, and the expected revenue of the host is maximized following the multi-campaigner K-LT model.

We show that Problem 3 can be solved optimally in polynomial time using a dynamic-programming-based approach. For this purpose, we introduce the notion of *individual revenue* of the host from every seed node.

Definition 2 (Individual Revenue): The *individual revenue* $\mathcal{R}_i(u)$ represents the expected revenue of the host from a seed node $u \in S$ when u is assigned to the i-th campaigner C_i.

Individual Revenue Computation. We now describe our method to compute individual revenues. We start by randomly assigning a distinct number from 1 to mk to every seed node in S. Let us denote by $I(u)$ the number assigned to seed node u. We also associate a list $\mathcal{L}$ of size mk with each node v in the network. The j-th entry of list $\mathcal{L}(v)$, denoted as $\mathcal{L}_j(v)$, represents the spread that some seed node $u \in S$ can achieve at node v following the K-LT model, where $I(u) = j$. For a seed node $u \in S$, we initialize:

$$\mathcal{L}_j(u) = \begin{cases} 1, & \text{if } I(u) = j; \\ 0, & \text{otherwise.} \end{cases}$$

Next, we simulate K-LT model starting from seeds in S. At any discrete step of K-LT model, if some node v becomes active, we consider all its in-neighbors $v' \in in(v)$ that were activated in the previous step. We compute $\mathcal{L}_j(v)$ as follows:

$$\mathcal{L}_j(v) = \frac{\displaystyle\sum_{\substack{v' \in in(v) \\ v' \text{activated in prev. step}}} [p_{v'v} \times \mathcal{L}_j(v')]}{\displaystyle\sum_{\substack{v' \in in(v) \\ v' \text{activated in prev. step}}} p_{v'v}} \tag{10}$$

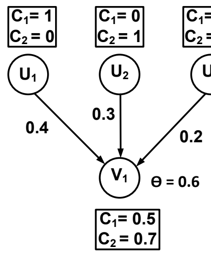


Fig. 7: Example of individual revenue computation

Finally, we compute individual revenue $\mathcal{R}_i(u)$ for every seed $u \in S$ and for every campaigner C_i as given below.

$$\mathcal{R}_i(u) = \sum_{v \in V} [A_{i,v} \times \mathcal{L}_{I(u)}(v)] \tag{11}$$

We refer to $\mathcal{R}_i(u)$ as the individual revenue of the host due to seed u, when u is assigned to C_i. We demonstrate the computation of individual revenues with an example.

Example 4: In Figure 7, we assume there are three seed nodes: u_1, u_2, and u_3, and also two campaigners: C_1 and C_2. The seed nodes are not assigned to any specific campaigners yet. We show the revenue vectors corresponding to each node inside the rectangular boxes. In the beginning, all seed nodes are activated, and in the next round, v_1 gets activated, since its activation threshold $\theta_{v_1} = 0.6 < p_{u_1,v_1} + p_{u_2,v_1} + p_{u_3,v_1} = 0.9$. Following Equation 10, we get: $\mathcal{L}_{I(u_1)}(v_1) = \frac{0.4}{0.4+0.3+0.2} = \frac{4}{9}$. Similarly, $\mathcal{L}_{I(u_2)}(v_1) = \frac{3}{9}$, and $\mathcal{L}_{I(u_3)}(v_1) = \frac{2}{9}$. Finally, we compute the individual revenues by following Equation 11. For example, $\mathcal{R}_1(u_1) = \sum_{v=u_1,u_2,u_3,v_1} A_{1,v} \times \mathcal{L}_{I(u_1)}(v) = 1 \times 1 + 0 + 0 + 0.5 \times \frac{4}{9} = 1.22$. Similarly, we have: $\mathcal{R}_1(u_2) = 0 + 0 + 0 + 0.5 \times \frac{3}{9} = 0.17$.

Properties of Individual Revenue. The individual revenue $\mathcal{R}_i(u)$ satisfies several interesting properties which are critical for our dynamic-programming-based exact solution.

Proposition 1: For a given seed node u and a given campaigner C_i, $1 \le i \le m$, $\mathcal{R}_i(u)$ is invariant to how other seed nodes are assigned to the various campaigners.

Proposition 2: If some seed nodes $u_1, u_2, \dots, u_j \in S$ are assigned to a campaigner C_i, $1 \le i \le m$, then the expected revenue of the host due to $u_1, u_2, \dots, u_j$ is simply the sum of $\mathcal{R}_i(u_1), \mathcal{R}_i(u_2), \dots, \mathcal{R}_i(u_j)$; and this is invariant to how the remaining seeds are assigned to other campaigners.

We omit the proofs due to limitation of space. The first proposition follows from the definition of the K-LT model — given a pre-defined seed set S, the activation of other nodes in the network is determined by the first phase of the K-LT model, that is, the classical LT model assuming all the campaigners cascading the same information. Whether a node in the network will be activated or not is independent of how S is partitioned among multiple campaigners [19]. The partition of S only influences the following: among the active nodes in the network, which one will adopt what campaign and with how much probability. The second proposition follows from the linearity property of $\mathcal{L}_j(v)$ in Equation 10, that is, $\mathcal{L}_{\{u_1,u_2\}}(v) = \mathcal{L}_{u_1}(v) + \mathcal{L}_{u_2}(v)$, for any two distinct seed nodes $u_1, u_2 \in S$ and for any node v in the graph.

Example 5: In Figure 7, $\mathcal{R}_1(u_1) = 1.22$ and $\mathcal{R}_1(u_2) = 0.17$. Note that, $\mathcal{R}_1(u_1) + \mathcal{R}_1(u_2) = 1.39$, and this is exactly same as $\mathcal{R}_1(\{u_1, u_2\})$, that is, the expected revenue of the host when both u_1 and u_2 are assigned to the campaigner C_1.

Dynamic Programming Based Exact Solution. We are now ready to describe our dynamic-programming-based exact algorithm to solve the optimal seed-set-partitioning problem (Problem 3). Our algorithm processes the seed nodes $u \in S$ in the ascending order of their assigned $I(u)$ numbers. We recall that $I(u) \in (1, mk)$. The dynamic programming maintains a table $\mathsf{EXACT}(j, [k'_1, k'_2, \ldots, k'_m]^T)$, that stores the optimal expected revenue of the host when we have already partitioned the seed nodes with number from 1 to j into m subsets $\{S_1, S_2, \ldots, S_m\}$, such that $|S_i| = k'_i \le k$, and we have also assigned them to the respective campaigners. Clearly, $1 \le j \le mk$ and $j = \sum_{i=1}^{m} k'_i$. The dynamic programming proceeds as given in Equation 12.

$$\mathsf{EXACT}\left(j, \begin{bmatrix} k'_1 \\ k'_2 \\ \vdots \\ k'_m \end{bmatrix}\right) = \max_{i \in (1,m)} \left\{ \mathsf{EXACT}\left(j-1, \begin{bmatrix} k'_1 \\ \vdots \\ k'_i - 1 \\ \vdots \\ k'_m \end{bmatrix}\right) + \mathcal{R}_i(u) \right\} \qquad (12)$$

In Equation 12, u denotes the node with $I(u) = j$. The optimal assignment of the last seed node is determined by $\mathsf{EXACT}(mk, [k, k, \ldots, k]^T)$. The optimal assignment of previous seed nodes are determined by *backtracking* with the usage of EXACT values. The correctness of our dynamic-programming-based solution follows from Propositions 1, 2.

Space and Time Complexity. Our algorithm has space complexity $\mathcal{O}(k^m)$ due to the EXACT table. The time complexity of our dynamic programming is $\mathcal{O}(mk^m)$. This is because we need to fill the table of size k^m; and in order to fill each entry in the table, we compute the maximum of m values. The backtracking requires another $\mathcal{O}(m^2k)$ time; since there are mk seed nodes that we need to assign to different campaigners; and for each seed node, we compare m values to find the best assignment. In addition, one needs to compute the expected revenue vectors for all seed nodes by running a breadth-first-search from each of these mk seed nodes. Hence, the time required to compute the expected revenue vectors for all seed nodes is $\mathcal{O}(mk(n+e))$, where n and e are the number of nodes and edges in the graph, respectively. Therefore, the time complexity of our exact solution is $\mathcal{O}(mkn + mke + m^2k + mk^m)$ — which is polynomial in the size of the graph.

Performance Guarantee. Theorem 4 provides the overall *approximation guarantee* of our method for the host's revenue maximization problem under the K-LT model. We provide the proof in our extended version [14].

Theorem 4: The iterative hill-climbing solution of the optimistic seed selection (Problem 2), coupled with the *optimal* partition of those seed sets (Problem 3), guarantees $\frac{1}{m}(1-\frac{1}{e})$ approximation to the original revenue maximization problem (Problem 1) under the K-LT model, and with the assumption that each campaigner has the same number of seed nodes. Here, m is the number of campaigners.

Algorithm 2 RevMax-Separate: Greedy Seed Set Selection

```
Require: Graph $G = (V, E, P)$, rev. matrix $(A_{iu})$, $m$ campaigners
Ensure: Seed sets $S_1, \ldots, S_m$, each of size $k$
 1: Sort and process campaigners in descending order of $\sum_{u \in V} A_{iu}$
    for each campaigner $C_i$; $1 \le i \le m$
 2: for $i = 1$ to $m$ do
 3:     $S_i = \phi$
 4:     for $j = 1$ to $k$ do
 5:         $v = \arg\max_{v \in V \setminus S_i} F_i(S_i \bigcup \{v\})$ [2]
 6:         $S_i = S_i \bigcup \{v\}$
 7:     end for
 8:     $V = V \setminus S_i$
 9: end for
10: Output $S_1, \ldots, S_m$
```

VI. Greedy Solutions

Our proposed techniques in the previous sections are polynomial-time to the graph size, and they also provide theoretical performance guarantees under additional constraints, e.g., exact solution over tree datasets for MCIC model, and $\frac{1}{m}(1-\frac{1}{e})$-optimal solution over any graph under K-LT model. Nevertheless, the running time of our algorithms increases exponentially with the number of seed nodes. Therefore, in this section, we propose more efficient greedy techniques for the host's revenue maximization problem. For ease of discussion, we refer to our earlier solution techniques in Sections IV and V as RevMax-Combined (RevMax-C), while we call our greedy solutions as RevMax-Separate (RevMax-S).

RevMax-Separate. This is a greedy method as given in Algorithm 2. We first sort the campaigners in descending order of $\sum_{u \in V} A_{iu}$, that is, the aggregated money that each campaigner C_i is willing to provide to the host if all the users in the network adopt her product. Next, we process the campaigners in that sorted order. For each campaigner, we identify the top-k seed nodes such that the host's revenue is maximized by considering only that campaigner (and disregarding the existence of other campaigners). Nevertheless, in order to eliminate the influence-cascading effect of already-selected seed nodes of previous campaigners, we delete these already-selected seed nodes from the graph before identifying the top-k seed nodes for the next campaigner (Line 8, Algorithm 2).

Time Complexity. The time complexity of our greedy algorithm is $\mathcal{O}(mkn(n+e)t)$, where m is the number of campaigners, k the number of seed nodes per campaigner, n and e are the number of nodes and edges in the graph, respectively, and t is the number of Monte-Carlo samples performed to find one seed node. We note that unlike our approximated algorithms in Sections IV and V, our greedy approach is very scalable — the running time increases linearly with the number of campaigners, number of seeds per campaigner, and polynomially with the size of the graph.

VII. Experimental Results

We present experimental results to illustrate effectiveness, efficiency, and scalability of our algorithms. The code is

[2] $F_i(S_i) = \sum_{u \in V}[A_{iu} \cdot Pr(u, S_i)]$, i.e., host's revenue considering only campaigner C_i, with seed set S_i

implemented in C++ and the experiments were performed on a single core of a 132GB, 2.26GHz Xeon server.

A. Experiment Setup

☐ **Datasets:** We summarize our data sets in Table I. Additional results over tree datasets can be found in the extended version.

TABLE I: Dataset Characteristics

Dataset	# Nodes	# Edges	Edge Prob: Mean, SD, Quartiles
Flickr	78 322	20 343 018	0.09 ± 0.06, $\{0.06, 0.07, 0.09\}$
DBLP	684 911	4 569 982	0.08 ± 0.07, $\{0.05, 0.05, 0.10\}$
NetHEPT	15 229	62 752	0.28 ± 0.28, $\{0.0006, 0.27, 0.53\}$

Flickr. Flickr is an online community, where users share photos, and participate in common-interest groups. The probability of an edge between any two users is computed assuming *homophily* [20]; in particular, the Jaccard coefficient of the interest groups that the two users belong to.

DBLP. The dataset is a subset of the popular co-authorship network used in [20]. The edge probabilities express the strength of the collaboration between the two incident authors. Particularly, if two authors collaborated c times, we assign the corresponding probability as $1 - \exp^{-c/10}$.

NetHEPT. This graph is created from the "High Energy Physics - Theory" papers of arXiv from 1991 to 2003 [7]. Since there is no edge probabilities on this graph, we synthetically assign probabilities on edges that simulates the community structure in a social network. We identify 60 non-overlapping communities from this graph dataset, each with 170 nodes. If an edge is completely inside a community, we uniformly assign a probability between 0.2 to 0.8; all other edges are assigned probabilities uniformly from 0 to 0.001. Such an edge probability assignment reflects the fact that users inside the same community have higher influence on each other than on someone else outside that community. The edge probabilities are assigned differently in both directions, i.e., $p_{uv} \neq p_{vu}$.

For K-LT model, if the sum of probabilities of incoming edges to a node is more than 1, we normalize those edge probabilities by their aggregate, such that the sum of probabilities for in-edges to every node is no more than 1 [20]. We also limit the number of Monte-Carlo samples to 1 000 in all our experiments [20].

☐ **Number of Campaigners and Seed Nodes:** We vary the number of campaigners from 2 to 10, while the number of seed nodes per campaigner is varied from 5 to 100.

☐ **Revenue Distribution:** We consider three categories of revenue distribution to simulate various real-world scenarios.

Uniform (U)**.** In this setting, each campaigner selects its target users uniformly over the network and independent of other campaigners. Thus, we assign every revenue-matrix-element $A_{iu} = 1$ monetary unit, with probability $\frac{1}{m}$; and $A_{iu} = 0.1$ monetary unit, with probability $(1 - \frac{1}{m})$. Here, m is the number of campaigners. One may note that we have normalized the amount of money that a campaigner gives to the host for one user on a scale from 0.1 to 1 monetary units.

Clustering with Low Competition (CLC)**.** In this setting, we assume that each campaigner's target users form certain clusters in the network. In addition, we also assume that there are some users who belong to target sets of all the campaigners. We call this model "clustering with low competition" as we limit the ratio of such mutually overlapping target users to a relatively small percentage. We simulate this setting as follows. We first partition the graph into 15 non-overlapping and highly-connected clusters, each cluster having equal number of nodes. For the first 5 clusters, all A_{iu} values are set to 1 monetary unit, i.e., 33% of the nodes belong to the target users of all campaigners. The remaining clusters are assigned to the campaigners in a round-robin manner. If a cluster is assigned to campaigner C_j as its target set, we then assign each $A_{ju} = 0.5$ monetary unit, and the remaining $A_{iu} = 0.1$ monetary unit, for all $j \neq i$, inside that cluster.

Clustering with High Competition (CHC)**.** This setting is similar to the previous CLC setting — the only difference is that there is a relatively large number of users who belong to the target sets of all campaigners. We simulate this setting as before; however, we assign the first 10 out of the 15 clusters as the target sets for all campaigners. This implies that 66% of the nodes belong to the target users of all campaigners.

U, CLC, and CHC models ensure almost equal host's revenue from each of her client campaigners. Due to limitation of space, we provide in our extended version [14] the results with some additional *unequal* revenue distributions, as well as for the case when *the campaigners allow different number of seed nodes*. We find that those results are similar to other results with U, CLC, and CHC distributions as shown below.

☐ **Comparing Methods:** We compare our approximated algorithms RevMax-C (Sections IV and V) and greedy RevMax-S (Section VI) with a randomized seed selection approach.

Random. We randomly select a distinct seed set for each campaigner. In our experiments, we did 10 runs of the Random method, and selected the best one that results in the maximum revenue out of all these 10 runs.

We compare the three aforementioned techniques — RevMax-C, RevMax-S, and Random under both IC and LT models. As the underlying viral marketing method in RevMax-S and RevMax-C, we use the CELF++ algorithm [11] due to its efficiency. We use the publicly-available source code of CELF++ provided by the respective authors [11].

☐ **Evaluation Metrics:** We compare host's revenues obtained from RevMax-S and RevMax-C with that of Random.

Revenue Improvement Rate (RIR). This is defined as the ratio of the host's expected revenue from the seed sets identified by RevMax-C (or, RevMax-S) with respect to the host's revenue obtained from a random selection of seed sets.

B. Performance: Effectiveness & Efficiency

We first demonstrate our results over the MCIC model (Section VII-B1), and the K-LT model (Section VII-B2). Since RevMax-C does not scale well with many campaigners and with a large number of seed nodes, we consider at most 5 campaigners and up to 20 seeds per campaigner in these experiments. The scalability of RevMax-S with more campaigners and seed nodes is illustrated later in Section VII-C.

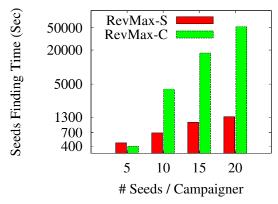
1) Performance with MCIC *Model:* We present revenue improvement rates with the MCIC model in Tables II and III. The host's revenue from RevMax-C technique almost always outperforms that from the RevMax-S approach by a margin of 5%∼10%. We show the corresponding efficiency results for the MCIC model in Figures 8(a) and 8(b). We find that

TABLE II: Revenue Improvement Rate (RIR), MCIC Model with 2 Campaigners, *NetHEPT* Dataset

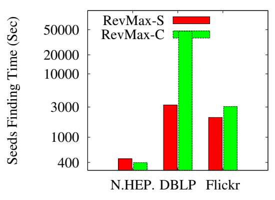
Revenue Distribution	#Seed Nodes per Camp.	RIR RevMax-S	RIR RevMax-C
CRH	5	2.52	**3.14**
	10	2.92	**3.28**
	15	2.68	**3.07**
	20	1.94	**2.23**
CRL	5	3.30	**3.33**
	10	2.91	**3.20**
	15	2.48	**2.94**
	20	2.09	**2.37**
U	5	3.23	**3.52**
	10	**2.12**	2.04
	15	2.72	**2.80**
	20	2.34	**2.52**

TABLE III: Revenue Improvement Rate (RIR), MCIC Model with 2 Campaigners, 5 Seeds/Campaigner

Dataset	Revenue Distribution	RIR RevMax-S	RIR RevMax-C
NetHEPT	CRH	2.52	**3.14**
	CHL	3.30	**3.33**
	U	3.23	**3.52**
DBLP	CRH	1.04	**1.04**
	CRL	1.02	**1.03**
	U	1.03	**1.03**
Flickr	CRH	1.43	**1.66**
	CHL	**1.20**	1.15
	U	1.01	**1.11**



(a) *NetHEPT* dataset



(b) 5 Seeds/Campaigner

Fig. 8: Seed sets finding time, MCIC model, 2 campaigners

RevMax-C requires less amount of time to identify the top-k seed nodes as compared to that of RevMax-S, over the smaller *NetHEPT* dataset and for 5 seeds per campaigner. However, as we consider larger datasets and more seeds per campaigner, RevMax-C requires more time. This is because RevMax-C identifies the seed sets over the most influential tree of the corresponding graph dataset in an *exact* manner, and this process requires time $\mathcal{O}(ndm^2k^{2m})$. Clearly, the running time of RevMax-C increases at a higher rate as one increases the number of nodes n, and the number of seeds k per campaigner.

2) Performance with K-LT *Model:* We first illustrate in Tables IV and V the performance over the *NetHEPT* dataset by varying the number of campaigners from 2 to 5, number of seed nodes per campaigner from 5 to 15, and with three different revenue distributions: uniform, clustering with low competition, and clustering with high competition. We observe the following for the K-LT model. With more campaigners as well with more seed nodes, our greedy method RevMax-S often outperforms our approximated technique RevMax-C. This is because the performance guarantee provided by RevMax-C is $\frac{1}{m}(1-\frac{1}{e})$, which decreases with m.

We show the efficiency results over the *NetHept* dataset and with the K-LT model in Figure 9. We find that up to 5 seed nodes per campaigner, along with 2, 3, or even 5 campaigners, RevMax-C requires smaller running time as compared to that of RevMax-S. This is due to how the underlying viral marketing algorithm (i.e, CELF++ [11]) is applied differently in both these methods. For RevMax-C, CELF++ is applied only once to identify all the mk seed nodes; while for RevMax-S, CELF++ is applied m times — each time it identifies k seed nodes for one campaigner. However, CELF++ itself is an iterative algorithm; more specifically, it requires k iterations to identify the top-k seed nodes. In one run of the CELF++ algorithm, the first iteration is the most expensive, and the subsequent iterations are significantly faster. In RevMax-S, the first iteration of CELF++ runs for m times; while in RevMax-C, the first iteration of CELF++ runs only once. This explains why our proposed method RevMax-C is faster compared to the RevMax-S. Nevertheless, RevMax-C requires more time as one increases the number of seed nodes per campaigner. This is because the pruning technique in the CELF++ algorithm starts deteriorating with increasing number of seed nodes. Since RevMax-C directly identifies mk seed nodes, whereas RevMax-S iterates for k times and in each iteration, it identifies m seed nodes; RevMax-C takes more time for higher values of m or k.

We show the performance of RevMax-C and RevMax-S with K-LT model over *DBLP* in Table VI. We find very similar

TABLE IV: Revenue Improvement Rate (RIR), K-LT Model, CRH Revenue Distribution, *NetHEPT*

# Camp.	#Seed Nodes per Camp.	RIR RevMax-S	RIR RevMax-C
2	5	8.99	**9.20**
	10	7.73	**7.97**
	15	**6.89**	6.82
3	5	5.86	**5.94**
	10	7.01	**7.29**
	15	**6.10**	5.73
5	5	5.70	**5.85**
	10	**5.00**	4.83
	15	**5.04**	4.77

TABLE V: Revenue Improvement Rate (RIR), K-LT Model with 2 Campaigners, *NetHEPT* Dataset

Revenue Distribution	#Seed Nodes per Camp.	RIR RevMax-S	RIR RevMax-C
CRH	5	8.99	**9.20**
	10	7.73	**7.97**
	15	**6.89**	6.82
CRL	5	8.62	**8.67**
	10	9.52	**9.53**
	15	**7.06**	6.98
U	5	5.06	**5.29**
	10	7.93	**8.38**
	15	5.12	**5.17**

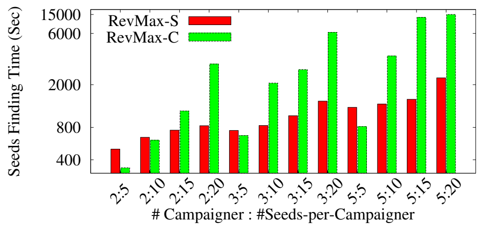


Fig. 9: Seed sets finding time, K-LT model, *NetHEPT* dataset

TABLE VI: Revenue Improvement Rate (RIR), K-LT Model, CRL Revenue Distribution, *DBLP* Dataset

	# Camp.=2		# Camp.=3	
#Seed Nodes per Camp.	RIR RevMax-S	RIR RevMax-C	RIR RevMax-S	RIR RevMax-C
5	77.27	**77.73**	27.41	**27.42**
10	42.27	**46.78**	40.80	**41.28**
15	**39.19**	37.04	36.74	**37.05**
20	**42.03**	42.01	**40.14**	34.25
25	31.01	**31.62**	**39.78**	34.90

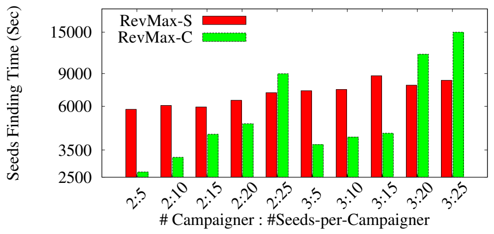
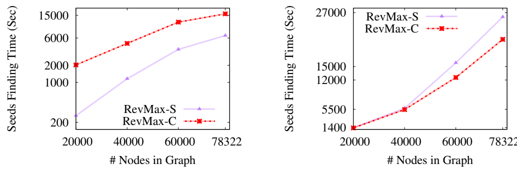


Fig. 10: Seed sets finding time, K-LT model, *DBLP* dataset



(a) MCIC model (b) K-LT model

Fig. 11: Scalability: Seed sets finding time vs. varying graph sizes, *Flickr* dataset, 2 campaigners with 5 seeds/campaigner

characteristics as before. With small number of seed nodes, RevMax-C almost always outperforms RevMax-S both in terms of revenue improvement rate as well as in terms of the running time to identify the seed sets. However, RevMax-S starts performing well with more campaigners and more seeds.

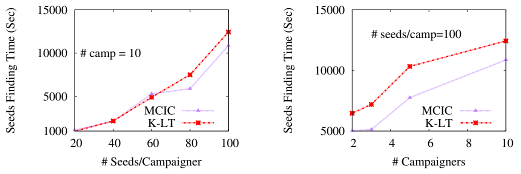
C. Scalability

In Figure 11, we analyze the variation of running times of RevMax-C and RevMax-S with different graph sizes. In particular, we consider varying sizes of the *Flickr* dataset, while keeping the number of campaigners and number of seeds per campaigner fixed at 2 and 5, respectively. We find that the running time of RevMax-C increases log-linearly with increasing graph sizes under the MCIC model, while it increases almost linearly for the K-LT model.

In Figure 12, we illustrate the scalability of our greedy method RevMax-S with the number of seed nodes per campaigner (up to 100) and also with the number of campaigners (up to 10). Our results show that RevMax-S scales linearly with the number of campaigners and also with the number of seed nodes per campaigner.

VIII. Conclusions

In this paper, we formulate and investigate the novel problem of revenue maximization of a social network host that sells viral marketing campaigns to multiple client campaigners. While our problem under both IC and LT models of influence cascading is **NP**-hard, and neither monotonic, nor sub-modular; we develop approximated algorithms with theoretical performance guarantees. For scalability reasons, we also design efficient greedy methods. Our experimental evaluation conducted on various real-world graph datasets and with diverse settings of revenue distributions attest that our approximated algorithms usually outperform our greedy methods by a margin of 5%$\sim$10%. All our algorithms are scalable with respect to the input graph size. While our approximated techniques suffer from scalability issues with increasing number of seed nodes and with many campaigners, our greedy methods are scalable even with a large number of campaigners and many seed nodes.



(a) Varying #seeds (b) Varying #campaigners

Fig. 12: Scalability of RevMax-S: seed sets finding time vs. varying seed set size and campaigners, *NetHept* dataset

References

[1] J. L. Aaker, A. M. Brumbaugh, and S. A. Grier. Nontarget Markets and Viewer Distinctiveness: The Impact of Target Marketing on Advertising Attitudes. *Consumer Psychology*, 9(3):127–140, 2000.

[2] S. Bharathi, D. Kempe, and M. Salek. Competitive Influence Maximization in Social Networks. In *WINE*, 2007.

[3] A. Borodin, Y. Filmus, and J. Oren. Threshold Models for Competitive Influence in Social Networks. In *WINE*, 2010.

[4] C. Budak, D. Agrawal, and A. E. Abbadi. Limiting the Spread of Misinformation in Social Networks. In *WWW*, 2011.

[5] T. Carnes, C. Nagarajan, S. M. Wild, and A. v. Zuylen. Maximizing Influence in a Competitive Social Network: A Follower's Perspective. In *ICEC*, 2007.

[6] W. Chen, A. Colin, R. Cumming, T. Ke, Z. Liu, D. Rincon, X. Sun, Y. Wang, W. Wei, and Y. Yuan. Influence Maximization in Social Networks when Negative Opinions May Emerge and Propagate. In *SDM*, 2011.

[7] W. Chen, C. Wang, and Y. Wang. Scalable Influence Maximization for Prevalent Viral Marketing in Large-Scale Social Networks. In *KDD*, 2010.

[8] W. Chen, Y. Wang, and S. Yang. Efficient Influence Maximization in Social Nerworks. In *KDD*, 2009.

[9] P. Domingos and M. Richardson. Mining the Network Value Customers. In *KDD*, 2001.

[10] H. N. Gabow, Z. Galil, T. Spencer, and R. E. Tarjan. Efficient Algorithms for Finding Minimum Spanning Trees in Undirected and Directed Graphs. *Combinatorica*, 6(2), 1986.

[11] A. Goyal, W. Lu, and L. V. S. Lakshmanan. CELF++: Optimizing the Greedy Algorithm for Influence Maximization in Social Networks. In *WWW*, 2011.

[12] S. Goyal and M. Kearns. Competitive Contagion in Networks. In *STOC*, 2012.

[13] D. Kempe, J. Kleinberg, and E. Tardos. Maximizing the Spread of Influence through Social Network. In *KDD*, 2003.

[14] A. Khan, B. Zehnder, and D. Kossmann. Extended version. http://www.ntu.edu.sg/home/arijit.khan/Papers/revmax_ext.pdf, 2016.

[15] M. Kimura and K. Saito. Tractable Models for Information Diffusion in Social Networks. In *PKDD*, 2006.

[16] T. Lappas, E. Terzi, D. Gunopulos, and H. Mannila. Finding Effectors in Social Networks. In *KDD*, 2010.

[17] F.-H. Li, C.-T. Li, and M.-K. Shan. Labeled Influence Maximization in Social Networks for Target Marketing. In *SocialCom*, 2011.

[18] H. Li, S. S. Bhowmick, J. Cui, Y. Gao, and J. Ma. GETREAL: Towards Realistic Selection of Influence Maximization Strategies in Competitive Networks. In *SIGMOD*, 2015.

[19] W. Lu, F. Bonchi, A. Goyal, and L. V. S. Lakshmanan. The Bang for the Buck: Fair Competitive Viral Marketing from the Host Perspective. In *KDD*, 2013.

[20] M. Potamias, F. Bonchi, A. Gionis, and G. Kollios. k-Nearest Neighbors in Uncertain Graphs. *PVLDB*, 2010.

[21] Y. Tang, X. Xiao, and Y. Shi. Influence Maximization: Near-Optimal Time Complexity Meets Practical Efficiency. In *SIGMOD*, 2014.

[22] V. Tzoumas, C. Amanatidis, and E. Markakis. A Game-Theoretic Analysis of a Competitive Diffusion Process over Social Networks. In *WINE*, 2012.